A novel fixed point theorem, towards a replacement for replacement

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Sadly often, we see "successor" and "limit" constructions recited as the construction of a fixed point, as if this were a proof. Missing are citations to John von Neumann (1928) for deriving recursion from induction and to Friedrich Hartogs (1917) for providing a suitable ordinal. Even then, we merely have two ordinals with the same value, so some other argument is needed to deduce a fixed point.

Casimir (Kasimierz) Kuratowski (1922) explained how many theorems in set theory, topology and measure theory that had been proved using transfinite recursion could be replaced with closure conditions.

Bourbaki (1949) and Ernst Witt (1951) showed how the problem provides its own ordinal, although Ernst Zermelo (1908) already had their proof and it was re-discovered repeatedly during the 20th century. It should have been fundamental to the curriculum, but only as an afterthought in Serge Lang's *Algebra* (1965) does it appear in a textbook. Wikipedia wrongly states that it is proved by transfinite recursion.

The huge clunking transfinite machine depends on excluded middle at every step. In the 1990s two intuitionistic approaches were developed, by André Joyal and Ieke Moerdijk (*Algebraic Set Theory*, 1994) and by me (1996). The key messages were that there are many kinds of ordinals, with distinct universal properties (forms of transfinite recursion) and the relations $\in$ and $\subset$ should be treated independently.

However, Hartogs' Lemma cannot be recovered, so there was no proof of the fixed point.

Then along came Dito Pataraia (1996), who threw off all of this set-theoretic baggage and used functions instead, to give an intuitionistic proof. It is breathtakingly simple and, unlike Bourbaki–Witt, could easily be reconstructed by a student in an exam. The key observation, to which domain theorists like me were somehow blind, is that composition makes the poset of inflationary monotone endofunctions *directed*, whilst also being directed *complete*.

In 2019, I returned to my 1990s work on well founded coalgebras, in order to meet the challenge of functors that preserve monos and not inverse images. I knew that I had to use Pataraia's theorem, but I tied myself in knots trying to deduce the result that I needed as a *corollary* of Pataraia: it was much more natural to do it the other way round.

So my version is this: Let $s : X \to X$ be an endofunction of a poset such that

- X has a least element $\bot$;
- X has joins ($\bigvee^{\uparrow}$) of directed subsets;
- s is monotone: $\forall xy.\ x \leq y \Rightarrow sx \leq sy$;
- s is inflationary: $\forall x.\ x \leq sx$;
- $\forall xy.\ x = sx \leq y = sy \Rightarrow x = y$ (the *Special Condition*).

Then

- X has a greatest element $\top$;
- $\top$ is the unique fixed point of s;

- if $\bot$ satisfies some predicate and it is preserved by s and directed joins then it holds for $\top$.

When I asked on MathOverflow whether anyone had seen my *Special Condition*, I was told that it was unnatural and given lectures on ordinal recursion.

How is the Special Condition to be achieved, just given $s : Y \to Y$ satisfying the other conditions? The Zermelo–Bourbaki–Witt theorem uses the subset $X \subset Y$ *generated* by $\bot$, s and $\bigvee^{\uparrow}$. But this already invokes second order logic or recursion as a preliminary to what should be the key tool for recursion.

Much more simply, if we take X to be

$$\{x : Y \mid x \le sx \ \wedge\ \forall a : Y.\ sa \le a \Rightarrow x \le a\}$$

or, if Y also has meets,

$$\{x : Y \mid x \le sx \ \wedge\ \forall u : Y.\ su \wedge x \le u \Rightarrow x \le u\},$$

then the Special Condition holds. These subsets are defined using the poset versions of the categorical notions of recursive and well founded coalgebras, so if x satisfies the second we call it a *well founded element* of Y with respect to the operation s.

As categorists we know that we have a small toolbox of very powerful tools: whenever we apply a general tool (unpack its definition) in some mathematical setting, this often turns out to be an important concept there.

The same seems to be true of well founded *elements*: in the natural structures, both well founded *relations* and well founded *coalgebras* are examples. Then most of the work of von Neumann's recursion theorem has been done in these settings.

In pure category theory, constructing coequalisers of algebras is perceived to be difficult and require transfinite recursion, but this is a simple direct application of the special condition itself.

This was meant to be a Lemma in a much larger programme. It warrants its own publicity because it is simple tool that any mathematician can take away and apply to their own subject, but also because of the regressive propaganda for transfinite methods that has been going on for over a century.

The larger programme, which I will not have time to discuss, generalises recursion and the "Mostowski" extensional quotient for well founded coalgebras well beyond their 1990s setting by replacing monos with factorisation systems. Applying this to posets and lower subsets instead of sets and subsets will give a much clearer explanation of the *plump* ordinals.

Based on that, we have a characterisation of transfinite iteration of functors that is another example of the generalised notion of extensionality.

It is a *characterisation* and not a *construction* because it cannot be done within the logic of an elementary topos. The set theorists will come in and tell us that we must use the Axiom-Scheme of Replacement in ZF to do it.

However, I am not prepared to allow them to dictate to us how to do our subject. I acknowledge that there are mathematical constructions that require transfinite iteration of functors, but this can be stated *within the native language of category theory.*

As Bill Lawvere taught us, it uses *Adjointness in Foundations.*

The numerous bibliographical details above, my past and present papers on this subject and slides for some recent seminars may be found on my website at

`www.paultaylor.eu/ordinals/`