Local Compactness and Bases in various formulations of Topology

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Abstract

A basis for a locally compact space is a family of pairs of subspaces, one open and the other compact, where containment of the compact subspace indicates whether the open one contributes to the union expressing a general open subspace. This is captured abstractly by saying which finite sets of basic opens cover a basic compact subspace. We identify the complete axiomatisation of this "way below" relation without assuming that the system is closed under unions or intersections, so balls in a metric space provide an example. We show how to reconstruct a space from an abstract basis in Point–Set Topology, locale theory, formal topology and abstract Stone duality. We also characterise continuous functions by means of relations called matrices that generalise the way-below relation. Hence our category defined using relations is weakly equivalent to that of locally compact spaces in each of these four formulations of topology, according to its appropriate logical foundations. Subsequent work will develop abstract bases towards computation.

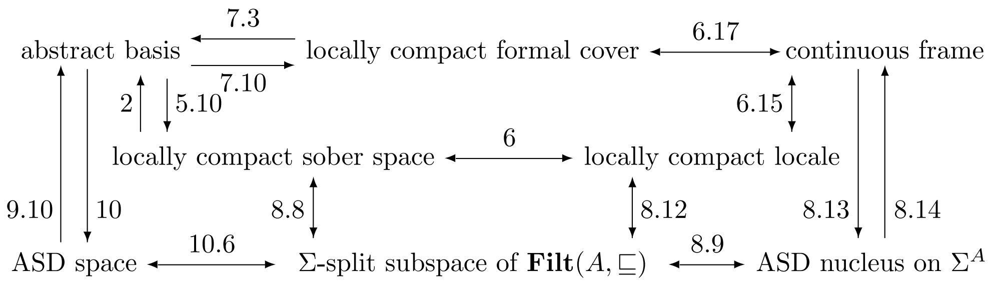
Note on the length of this paper:

The principal objective is to establish definitively the axioms for an abstract basis so that future work can build on them. Everything up to Section 7 is needed to show that they are sound and complete in Point–Set Topology, since it turns out to be necessary to go *via* locale theory and formal topology. Sections 8–10 are about my own subject (ASD) and for me Section 10 contains the core result. For technical reasons, the next paper, which will show that bases and matrices provide a model of ASD, must restrict to overt spaces (Section 12) with bases using compact subspaces (Section 11). Finally, we need some examples (Section 13) and to sum up the complicated argument as equivalences of categories (Section 14).

Arguably, however, I tend to include too much detail in my proofs, so I am open to opinions that particular results are obvious.

1 Introduction

In the old debate about the nature of mathematical research, maybe we can say that isolated definitions are *human inventions*, whereas it is a *natural discovery* when two or more different concepts turn out to be equivalent. We shall see that the notion of *local compactness* (unlike more general kinds of topology) has rather more than two formulations; moreover we can talk about it using four different strengths of logical foundations. The weakest of these, which we shall call an *abstract basis*, requires only relations between discrete sets and either lists or finite subsets, offering a way in which to do *computation* with locally compact spaces.



Definition 1.1 A ***concrete basis*** for a (not necessarily locally compact) topological space X indexed by a preorder $(A, \sqsubseteq)$ is a family of open subspaces $U_a \subset X$ such that
(a) if $a \sqsubseteq b$ then $U_a \subset U_b$;
(b) if $x \in U_a$ and $x \in U_b$ then $x \in U_c$ for some $c \in A$ with $a \sqsupseteq c \sqsubseteq b$; and
(c) if $x \in U \subset X$ with U open then $\exists a.\, x \in U_a \subset U$.

Definition 1.2 A space X is ***locally compact*** if it has the ***interpolation property*** that, given $x \in V \subset X$ with V open, there are $x \in U \subset K \subset V \subset X$ with U open and K compact. This definition is suitable for non-Hausdorff (but sober) spaces and was given by Karl Hofmann and Michael Mislove [HM81].

When there are plenty of compact subspaces like this, they may be used like the ***dual basis*** in linear algebra to specify *which* basic open subspaces U_a should contribute to the union in the last axiom above:

Definition 1.3 A ***concrete basis using compact subspaces*** for a locally compact space X is a family of pairs (U_a, K_a) of subspaces of X indexed by a preorder $(A, \sqsubseteq)$ such that
(a) each U_a is open and K_a is compact;
(b) if $a \sqsubseteq b$ then $U_a \subset U_b$ and $K_b \subset U \Rightarrow K_a \subset U$ for any open $U \subset X$;;
(c) $x \in U_a \ \wedge\ x \in U_b \Longrightarrow \exists c.\, x \in U_c \ \wedge\ (a \sqsupseteq c \sqsubseteq b)$ and
(d) $x \in V \iff \exists a.\, x \in U_a \ \wedge\ K_a \subset V$,

where the last part is called the ***basis expansion***. The constructions in Section 8 show that it is convenient *not* to require $U_a \subset K_a$, but if $K_a \subset U$ with U open then $U_a \subset U$ because it contributes to the basis expansion.

From the point of view of the information content of the equivalences that are summarised in the diagram above, we shall consider a *locally compact space* to be equipped with a *specified* concrete basis, in particular the family (K_a) or its equivalents. On the other hand, Corollary 6.15 will introduce another property called *continuity* that depends only on the open subspaces and so contains less information.

Remark 1.4 The elements of the set A are intended to be codes that we can use for computation, such as the rational endpoints of real intervals or the centres and radii of balls in Euclidean space.

We choose to write the order relation $\sqsubseteq$ in the topological direction (as above and in Section 7) rather than the domain-theoretic one (*cf.* Proposition 8.7), but exponentiation reverses it (Proposition 8.3). Note, however, that we do *not* require $U_a \subset U_b \Longrightarrow a \sqsubseteq b$. Also, we may have both $a \sqsubseteq b$ and $b \sqsubseteq a$ without requiring $a = b$. Lemma 3.6 shows that it can be eliminated altogether, but we feel that there are conceptual reasons for keeping it.

As an application of the traditional ("finite open sub-cover") definition of compactness, we may replace the point x in the interpolation property by a compact subspace L with $L \subset V$, obtaining $L \subset U \subset K \subset V \subset X$. The open subspace U can be then expressed as a union of basic ones, which form an open cover of the compact space L, so finitely many of them suffice.

Notation 1.5 We therefore need to use unions of *finite* sets or lists ℓ of basic open–compact pairs. Everything that we do will be consistent with interpreting such ℓ *either* as a list *or* as a subset of A and there are computational advantages in maintaining this ambiguity. We adopt the convention that the early letters $(a, \ldots, e)$ of the alphabet denote *individual* members of the indexing set A, those (h, k, ℓ) in the middle are *lists* or *finite subsets* of A and the later ones $(p, \ldots, w)$ are *possibly infinite* subsets.

Then we write

$$U_\ell \equiv \bigcup_{a\in\ell} U_a \qquad \text{and} \qquad K_\ell \equiv \bigcup_{a\in\ell} K_a$$

and define the ***way-below*** relation

$$a \prec\!\!\!\prec \ell \qquad \text{as} \qquad K_a \subset U_\ell \equiv \bigcup_{b\in\ell} U_b.$$

Our goal is the complete axiomatisation of this relation. Then the set A, the preorder $\sqsubseteq$ and way below relation $\prec\!\!\!\prec$ will be enough to describe a locally compact space.

Example 1.6 The real line $\mathbb{R}$ has a familiar basis consisting of open and closed intervals, whose endpoints we may perhaps choose to be dyadic rationals. A typical instance of $a \prec\!\!\!\prec \ell$ in this basis is

$$[d, u] \subset (e_1, t_1) \cup \cdots \cup (e_n, t_n).$$

We can characterise this arithmetically, without considering the intervals as sets or quantifying over the real numbers inside them: up to permutation of the indices, the condition is

$$e_1 < d \;\wedge\; e_2 < t_1 \;\wedge\; e_3 < t_2 \;\wedge\; \cdots \;\wedge\; e_n < t_{n-1} \;\wedge\; u < t_n.$$

Admittedly, this formula is awkward and its analogues for balls in $\mathbb{R}^{\mathsf{n}}$ would be quite unwieldy, but that is a fact of life in geometry. In practice, we get to *choose* how to divide up a region for computation, so it might be better in later work to formulate systems that *generate* bases. However, we are concerned here with establishing the axiomatisation, so we prefer to assume closure under conditions such as transitivity of $\prec\!\!\!\prec$.

Remark 1.7 Complete axiomatisations of the $\prec\!\!\!\prec$ relation were given in [JS96] and [G]. However, these accounts relied on using bases that are indexed by *lattices*, *i.e.* there are operations $\sqcap$ and $\sqcup$ on *indices* such that

$$U_{a\sqcup b} = U_a \cup U_b \qquad\qquad K_{a\sqcup b} = K_a \cup K_b$$

$$U_{a\sqcap b} = U_a \cap U_b \qquad \text{and (NB)} \qquad K_{a\sqcap b} \subset K_a \cap K_b,$$

where we also define $U_\circ \equiv K_\circ \equiv \emptyset$, $U_\bullet \equiv X$ and (if X is compact) $K_\bullet \equiv X$. In [JS96] the lattice structure is exploited to illustrate *Lawson duality* between the open subspaces of one stably locally compact space and the compact saturated subspaces of another.

The innovation in the present work is to use "individual" basis elements, so that the basis need not be closed under these lattice operations. *Some of the most interesting features of our new approach arise from making this distinction.* It separates the notions of *roundedness* and *locatedness* and [future work] will show how this leads to some of the key features of *interval computation*.

Our axioms are rather simpler than the earlier ones because they do not have to take account of the lattice operations:

Definition 1.8 An ***abstract basis*** is a structure $(A, \sqsubseteq, \prec\!\!\!\prec)$ that satisfies the ***primary axioms***

		$a \sqsubseteq a$	***reflexivity***
$a \sqsubseteq b \sqsubseteq c$	$\Longrightarrow$	$a \sqsubseteq c$	***transitivity***
$a \sqsubseteq b \prec\!\!\!\prec k \sqsubseteq \ell$	$\Longrightarrow$	$a \prec\!\!\!\prec \ell$	***co-*** & ***contravariance***
$(a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1) \wedge (k \prec\!\!\!\prec \ell_2)$	$\Longrightarrow$	$a \prec\!\!\!\prec \ell_1 \sqcap \ell_2$	***weak intersection***
$a \prec\!\!\!\prec \ell$	$\Longrightarrow$	$\exists k.\, a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell,$	***Wilker***

where the additional symbols will be defined in Notation 1.12.

Remark 1.9 In the case of real intervals above, the Wilker property says that we may shrink the (e_i, t_i) slightly but maintain the "way below" property amongst them. On the other hand, the single interpolation rule below says that we may also enlarge $[d, u]$.

The Wilker rule allows ***interpolation*** of some k between given $a \prec\!\!\!\prec \ell$. However, Wilker is stronger because it says that each $b \in k$ is covered by a *single* c with $b \prec\!\!\!\prec c \in \ell$, whereas interpolation only says that the list ℓ covers *collectively*, $b \prec\!\!\!\prec \ell$.

Conversely, the special case of the weak intersection rule with $\ell_1 \equiv \ell_2$ is ***transitivity***:

$$a \prec\!\!\!\prec k \prec\!\!\!\prec \ell \;\equiv\; a \prec\!\!\!\prec k \;\wedge\; \forall b \in k.\, b \prec\!\!\!\prec \ell \;\Longrightarrow\; a \prec\!\!\!\prec \ell.$$

We motivate these primary axioms and prove that concrete bases obey them in the next section. However, whilst they provide the complete axiomatisation for concrete bases in general, when we come to *use* bases for applications such as computation (or even to characterise continuous functions, *cf.* Lemma 4.3), we find ourselves wanting to assume that there are enough *individual* basis elements for certain purposes, instead of using *unions* of them.

Definition 1.10 We will also assume that concrete bases satisfy the following (***secondary*** or) ***roundedness*** axioms:

$$K_a \subset U_\ell \;\Longrightarrow\; \exists b.\, K_a \subset U_b \;\wedge\; K_b \subset U_\ell$$

$$(K_{b_1} \subset U_a) \;\wedge\; (K_{b_2} \subset U_a) \;\Longrightarrow\; \exists b.\, (K_{b_1} \subset U_b) \;\wedge\; (K_{b_2} \subset U_b) \;\wedge\; (K_b \subset U_a)$$

$$\exists b.\, K_a \subset U_b \qquad \text{and} \qquad \exists b.\, K_b \subset U_a,$$

which we call ***single interpolation***, ***rounded union*** and ***boundedness above*** and ***below***. Equivalently, the abstract bases satisfy

$$a \prec\!\!\!\prec \ell \;\Longrightarrow\; \exists b.\, a \prec\!\!\!\prec b \prec\!\!\!\prec \ell$$

$$(b_1 \prec\!\!\!\prec a) \;\wedge\; (b_2 \prec\!\!\!\prec a) \;\Longrightarrow\; \exists b.\, (b_1 \prec\!\!\!\prec b \prec\!\!\!\prec a) \;\wedge\; (b_2 \prec\!\!\!\prec b)$$

$$\exists b.\, a \prec\!\!\!\prec b \qquad \text{and} \qquad \exists b.\, b \prec\!\!\!\prec a.$$

We may "assume without loss of generality" that our bases have these properties because of manipulations with bases that we describe in Section 3.

If we fail to assert these extra axioms, it is easy to find ourselves using them without noticing it. For example, if single interpolation holds then the list k in the Wilker rule may be taken to be bijective with ℓ, but otherwise k may have to be longer. Even in the simple case of $a \prec\!\!\!\prec b$, we would need to interpolate a *list* in $a \prec\!\!\!\prec k \prec\!\!\!\prec b$, rather than a single member of the basis.

Definition 1.11 In fact, any basis that uses compact subspaces (Definition 1.3) actually satisfies the ***strong intersection*** rule,

$$\exists \ell.\, a \prec\!\!\!\prec \ell \qquad \text{and} \qquad (a \prec\!\!\!\prec \ell_1) \wedge (a \prec\!\!\!\prec \ell_2) \implies a \prec\!\!\!\prec \ell_1 \sqcap \ell_2,$$

which is equivalent to the weak rule above together with ***rounded intersection***,

$$(K_a \subset U_{b_1}) \wedge (K_a \subset U_{b_2}) \implies \exists b.\, (K_a \subset U_b) \wedge (K_b \subset U_{b_1}) \wedge (K_b \subset U_{b_2})$$

or

$$(a \prec\!\!\!\prec b_1) \wedge (a \prec\!\!\!\prec b_2) \implies \exists b.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec b_1) \wedge (b \prec\!\!\!\prec b_2).$$

As with the other roundedness rules, it is much more convenient *in applications* to work with bases that have this property. However, for several reasons we avoid it in this paper in our discussion of the *formulation* and completeness of the axioms.

One reason for this is that we fully embrace non-Hausdorff spaces. In a Hausdorff space, the intersection of two compact spaces is closed in either of them and therefore compact. This need no longer be the case in a non-Hausdorff space, so the space is called ***stably locally compact*** if it is.

Another is that, in the passage from Point–Set Topology to the formulations in weaker logics that we consider, it is easier to make the analogy amongst them by considering the neighbourhood filter $\mathcal{K}_a$ instead of the compact subspace K_a. We then find that the filter requirement is not really necessary.

We will explain these issues in the next section.

Finally, whilst it is possible to turn a basis with the weak intersection property into one obeying the strong rule, the construction (Theorem 11.6) requires the Axiom of Dependent Choice, which may be undesirable in certain foundational settings.

Note that, although the *weak* intersection rule implies transitivity, the latter must be stated explicitly alongside the *strong* intersection rule.

Notation 1.12 In these axioms, we extend $\sqsubseteq$ and $\prec\!\!\!\prec$ to lists or finite subsets by writing

$$\begin{array}{lcl}
a \sqsubseteq \ell & \equiv & \exists b \in \ell.\, a \sqsubseteq b \\
a \sqsubseteq \ell_1 \sqcap \ell_2 & \equiv & a \sqsubseteq \ell_1 \wedge a \sqsubseteq \ell_2 \\
 & \equiv & \exists b_1 \in \ell_1.\, \exists b_2 \in \ell_2.\, b_1 \sqsupseteq a \sqsubseteq b_2 \\
k \sqsubseteq \ell & \equiv & \forall a \in k.\, a \sqsubseteq \ell \;\equiv\; \forall a \in k.\, \exists b \in \ell.\, a \sqsubseteq b \\
a \prec\!\!\!\prec b & \equiv & a \prec\!\!\!\prec \{b\} \\
k \prec\!\!\!\prec \ell & \equiv & \forall a \in k.\, a \prec\!\!\!\prec \ell \\
a \prec\!\!\!\prec \ell_1 \sqcap \ell_2 & \equiv & \exists k.\, a \prec\!\!\!\prec k \wedge \forall b \in k.\, b \sqsubseteq \ell_1 \sqcap \ell_2 \\
 & \equiv & \exists k.\, a \prec\!\!\!\prec k \wedge \forall b \in k.\, \exists c_1 \in \ell_1.\, \exists c_2 \in \ell_2.\, c_1 \sqsupseteq b \sqsubseteq c_2 \\
a \prec\!\!\!\prec^1 \ell & \equiv & \exists b \in \ell.\, a \prec\!\!\!\prec b \\
k \prec\!\!\!\prec^1 \ell & \equiv & \forall a \in k.\, a \prec\!\!\!\prec^1 \ell \;\equiv\; \forall a \in k.\, \exists b \in \ell.\, a \prec\!\!\!\prec b.
\end{array}$$

We write $\mathsf{Fin}\,(A)$ for either the set of lists or of finite subsets of A, $\circ$ for the empty list, $k \sqcup \ell \;\equiv\; k \cup \ell$ for the union of two lists and $\bigsqcup L$ for the union of a list of lists. This structure makes $\mathsf{Fin}\,(A)$ the free join semilattice on the preorder $(A, \sqsubseteq)$ and as such the preorder on $\mathsf{Fin}\,(A)$ is $k \sqsubseteq \ell$ above. In constructive settings, we mean finite in the sense of Kazimierz Kuratowski [Kur20], *i.e.* finitely enumerable, so arbitrary subsets of finite sets need not necessarily be finite.

After completing our introduction to the axiomatisation of bases for topological spaces in Point–Set Topology, we carry out the same task for continuous functions in Section 4:

Definition 1.13 Let $f : X \to Y$ be a continuous function between locally compact sober spaces X and Y with concrete bases $\{(U_a, K_a) \mid a \in A\}$ and $\{(V_b, L_b) \mid b \in B\}$ respectively. We define a binary relation between the indices of the bases,

$$\langle\, a \,|\, f \,|\, b \,\rangle \qquad \text{by} \qquad K_a \subset f^{-1}V_b \qquad \text{or equivalently} \qquad fK_a \subset V_b.$$

In particular, $$\langle\, a \,|\, \mathsf{id} \,|\, a' \,\rangle \iff (a \prec\!\!\!\prec a').$$

We call $\langle\, a \,|\, f \,|\, b \,\rangle$ the ***concrete matrix*** of f, following the loose analogy between bases in topology and in linear algebra that we have already made in Definition 1.3. The notation was inspired by that of Paul Dirac in Quantum Mechanics, whereas [G] used the notation $\widehat{H}^b_a$ from Albert Einstein's General Relativity.

The matrix ***represents*** f in the sense that

$$fx \in V_b \iff \exists a.\, (x \in U_a) \;\wedge\; \langle\, a \,|\, f \,|\, b \,\rangle,$$

using the basis expansion of $f^{-1}V_b$.

Such matrices are characterised as follows:

Definition 1.14 An ***abstract matrix*** between bases $(A, \sqsubseteq, \prec\!\!\!\prec)$ and $(B, \sqsubseteq, \prec\!\!\!\prec)$, which we write as

$$\langle\, a \,|\, M \,|\, b \,\rangle \qquad \text{or} \qquad M : (A, \sqsubseteq, \prec\!\!\!\prec) \longrightarrow (B, \sqsubseteq, \prec\!\!\!\prec),$$

is a binary relation between the sets A and B that is ***contravariant*** and ***rounded*** in a,

$$(a \sqsubseteq a') \;\wedge\; \langle\, a' \,|\, M \,|\, b \,\rangle \implies \langle\, a \,|\, M \,|\, b \,\rangle \iff \exists a'.\, (a \prec\!\!\!\prec a') \;\wedge\; \langle\, a' \,|\, M \,|\, b \,\rangle$$

and ***covariant*** and ***rounded*** in b,

$$\langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; (b' \sqsubseteq b) \implies \langle\, a \,|\, M \,|\, b \,\rangle \iff \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; (b' \prec\!\!\!\prec b).$$

The next axioms correspond to the preservation of arbitrary joins and finite meets by inverse image maps: M has the ***partition property*** if

$$\langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; (b \prec\!\!\!\prec \ell) \implies \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b' \in \ell.\, \langle\, a' \,|\, M \,|\, b' \,\rangle,$$

it is ***bounded*** if $\quad \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \,|\, M \,|\, b \,\rangle$

it is ***weakly filtered*** if

$$(a \prec\!\!\!\prec a') \;\wedge\; \langle\, a' \,|\, M \,|\, b_1 \,\rangle \;\wedge\; \langle\, a' \,|\, M \,|\, b_2 \,\rangle \implies$$

$$\exists k\ell.\, (a \prec\!\!\!\prec k) \;\wedge\; \big(\forall a' \in k.\, \exists b \in \ell.\, \langle\, a' \,|\, M \,|\, b \,\rangle\big) \;\wedge\; \big(\forall b \in \ell.\, b_1 \sqsupseteq b \sqsubseteq b_2\big).$$

and ***strongly*** so if the same holds without $(a \prec\!\!\!\prec a')$. Finally, M is ***saturated*** if

$$(a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \langle\, a' \,|\, M \,|\, b \,\rangle \implies \langle\, a \,|\, M \,|\, b \,\rangle,$$

but beware that we also use the word *saturated* in an unrelated sense in Definition 3.15.

We show that the category of locally compact sober spaces and continuous functions is equivalent to the category of bases and matrices that have all of the above properties.

More explanation is needed here to summarise the rest of the paper.

That is only the beginning of the story, since we still have to demonstrate *completeness* or *sufficiency* of our axioms for bases by constructing a space with a concrete basis that corresponds to any given abstract one.

In Point–Set Topology of course we need the Axiom of Choice to construct a space and concrete basis from a given abstract one. However, even though an explicit description is available, it seems only to be possible to prove its correctness if the basis is countable.

The general case only seems to be provable via Locale Theory together with either Formal Topology or Abstract Stone Duality.

Moreover, for both of the latter subjects, abstract bases provide a considerably simpler definition of local compactness than those that currently exist in the literature. Future work will use matrices to develop computation.

Abstract bases therefore provide a unifying framework across these four formulations of topology. Categorical equivalence is provable using the strength of logic that each customarily requires: set theory with Choice, topos logic, Martin-Löf type theory and an arithmetic universe respectively. We can therefore say logically that a space or continuous function exists in each subject iff it is definable in the appropriate logic.

2 Point–Set Topology

We begin by showing that any concrete basis for a locally compact space in traditional Point–Set Topology gives rise to an abstract basis that satisfies the principal axioms. We also introduce a more general form of concrete basis that identifies more precisely the criterion whereby a basic open subspace should contribute to the basis expansion.

Lemma 2.1 Any basis using compact subspaces (Definition 1.3) satisfies the ***boundedness*** and ***strong intersection rules*** (Definition 1.11),

$$\exists \ell.\, a \prec\!\!\!\prec \ell \qquad \text{and} \qquad a \prec\!\!\!\prec \ell_1 \wedge a \prec\!\!\!\prec \ell_2 \implies a \prec\!\!\!\prec \ell_1 \sqcap \ell_2,$$

where $\quad a \prec\!\!\!\prec \ell_1 \sqcap \ell_2 \quad$ means $\quad \exists h.\, a \prec\!\!\!\prec h \wedge \forall b \in h.\, \exists c_1 \in \ell_1.\, \exists c_2 \in \ell_2.\, c_1 \sqsupseteq b \sqsubseteq c_2.$

Proof For boundedness, consider the basis expansion of the whole space *quâ* open subspace. This covers the given basic compact subspace K_a, but a finite subset ℓ of this cover suffices.

The hypotheses $a \prec\!\!\!\prec \ell_1$ and $a \prec\!\!\!\prec \ell_2$ of the intersection rule say that

$$K_a \subset U_{\ell_1} \cap U_{\ell_2} \equiv \bigcup\{U_{b_1} \mid b_1 \in \ell_1\} \cap \bigcup\{U_{b_1} \mid b_1 \in \ell_1\}.$$

Using distributivity and part (c) of Definition 1.3, this union is

$$\bigcup\{U_{b_1} \cap U_{b_2} \mid b_1 \in \ell_1,\ b_2 \in \ell_2\} = \bigcup\{U_c \mid \exists b_1 \in \ell_1.\, \exists b_2 \in \ell_2.\, b_1 \sqsupseteq c \sqsubseteq b_2\}.$$

Since K_a is compact, a finite set h of such c suffices to cover it, so

$$K_a \subset U_h \equiv a \prec\!\!\!\prec h \qquad \text{and} \qquad \forall c \in h.\, \exists b_1 \in \ell_1.\, \exists b_2 \in \ell_2.\, b_1 \sqsupseteq c \sqsubseteq b_2,$$

which is the definition of $a \prec\!\!\!\prec \ell_1 \sqcap \ell_2$. □

Definition 2.2 Definition 1.3 and this lemma would have been simpler if the preorder $\sqsubseteq$ had had a formal intersection *operation*, $\sqcap$, satisfying

$$a \sqsupseteq a \sqcap b \sqsubseteq b \qquad \text{and} \qquad a \sqsupseteq c \sqsubseteq b \implies c \sqsubseteq a \sqcap b,$$

whilst the basic subspaces would satisfy

$$U_{a\sqcap b} \;=\; U_a \;\cap\; U_b \qquad \text{but} \qquad K_{a\sqcap b} \;\subset\; K_a \;\cap\; K_b,$$

where the containment of compact subspaces need not be an equality.

We call the space ***stably locally compact*** when such (binary) intersections do exist.

A ***stable abstract basis*** is one that has such a $\sqcap$ operation and satisfies the boundedness and strong intersection rules.

Examples 2.3 Many important examples do have such an operation:
(a) intervals in $\mathbb{R}$ and *cuboids* in $\mathbb{R}^{\mathsf{n}}$, with geometric intersection for $\sqcap$; and
(b) lists of constraints on data, with conjunction or concatenation for $\sqcap$.

On the other hand,
(c) it is more common to use *balls* as bases for $\mathbb{R}^{\mathsf{n}}$ and other metric spaces, but they do not intersect in balls; but
(d) more fundamentally, the intersection of two compact subspaces in a non-Hausdorff space need not be compact. Consider, for example, an interval $[0,1]$ together with an extra $1'$, or more formally the cokernel of $[0,1) \hookrightarrow [0,1]$.

Besides this, the subspaces need not overlap at all, so we would need a name for the empty subspace. Keeping track of empty subspaces creates some quite absurd difficulties. For example, in the Tychonov basis for the product of two spaces,

$$(a,b) \prec\!\!\!\prec (a',b') \iff (a \prec\!\!\!\prec a') \;\wedge\; (b \prec\!\!\!\prec b') \;\vee\; (a \prec\!\!\!\prec \circ) \;\vee\; (b \prec\!\!\!\prec \circ)$$

since $K \times L \subset U \times \emptyset$ for *any* compact K and L and open U. In order to avoid this complication when we construct the Tychonov product of two abstract bases, in [work in progress] we shall restrict to the case where $a \prec\!\!\!\prec \circ$ is forbidden, *cf.* Section 12.

Remark 2.4 There are two ways of proceeding without assuming stable local compactness:
(a) in applications we generally prefer to use compact subspaces for the dual basis, but not intersections of them; whilst
(b) in proving the equivalence of various notions in this paper, we replace compact subspaces by something weaker, which would allow us to use intersections, although we usually do not.

Lemma 2.5 For any compact space K, the family $\mathcal{K} \equiv \{V \mid K \subset V\}$ is a ***Scott-open filter***:
(a) if $\mathcal{K} \ni V \subset W$ then $\mathcal{K} \ni W$;
(b) if $\mathcal{K} \ni \bigcup_{i\in I} V_i$ then there is some finite subset $\ell \subset I$ for which $\mathcal{K} \ni \bigcup_{i\in\ell} V_i$; and
(c) $\mathcal{K} \ni X$, and $\mathcal{K} \ni V, W \Longrightarrow \mathcal{K} \ni V \cap W$. □

In fact, so long as the space is sober, *every* Scott-open filter of open subspaces arises in this way (Lemma 3.14). We adopt the habit of writing $\mathcal{K} \ni U$ rather than $U \in \mathcal{K}$ because we will be able to interchange $K \subset$ with $\mathcal{K} \ni$ in most of our arguments.

The difficulty in the non-stable case is that there is a conflict between the two uses of intersections: of compact subspaces and of open ones in the definition of a filter. However, for many purposes, it is unnecessary to use filters. So we can sacrifice the *subspaces* but retain the essence of *compactness*. ***Scott-open families*** satisfy parts (a) and (b) and we can rewrite the Definition of a basis using them:

Definition 2.6 A ***basis using Scott-open families*** consists of
(a) for each $a \in A$, an open subspace U_a and a Scott-open family $\mathcal{K}_a$ of open subspaces;
(b) if $a \sqsubseteq b$ then $U_a \subset U_b$ and $\mathcal{K}_a \supset \mathcal{K}_b$;
(c) $U_a \cap U_b = \bigcup\{U_c \mid a \sqsupseteq c \sqsubseteq b\}$; and
(d) $V = \bigcup\{U_a \mid \mathcal{K}_a \ni V\}$.
We amend Notation 1.5 by writing

$$a \prec\!\!\!\prec \ell \quad \text{for} \quad \mathcal{K}_a \ni U_\ell \qquad \text{and} \qquad \mathcal{K}_\ell \;\equiv\; \bigcap\{\mathcal{K}_b \mid b \in \ell\}.$$

Remark 2.7 It is easy to add intersections ($\sqcap$) to a basis, but at the cost of using Scott-open families instead of compact subspaces. If $(U_a, \mathcal{K}_a)$ is a basis of either kind then

$$U_{(\ell)} \;\equiv\; \bigcap\{U_a \mid a \in \ell\} \qquad \text{and} \qquad \mathcal{K}_{(\ell)} \;\equiv\; \bigcup\{\mathcal{K}_a \mid a \in \ell\}$$

define another one for the same space such that $\sqcap$ is given by union of lists, where the the parentheses on the subscripts distinguish this construction from the notation that we have just defined.

The other direction, replacing Scott-open families with compact subspaces, possibly at the expense of intersections, is rather more difficult, so we defer it to Section 11.

Lemma 2.8 A basis using Scott-open families obeys the ***weak intersection rule***,

$$a \prec\!\!\!\prec k \quad \wedge \quad k \prec\!\!\!\prec \ell_1 \quad \wedge \quad k \prec\!\!\!\prec \ell_2 \quad \Longrightarrow \quad a \prec\!\!\!\prec \ell_1 \sqcap \ell_2.$$

Proof The hypothesis $k \prec\!\!\!\prec \ell_1$ says that, for each $b \in k$,

$$\mathcal{K}_b \;\ni\; U_{\ell_1} \;\equiv\; \bigcup\{U_c \mid c \in \ell_1\},$$

so U_b contributes to the basis expansion of U_ℓ and $U_b \subset U_{\ell_1}$. Since $a \prec\!\!\!\prec k$, it follows that

$$\mathcal{K}_a \;\ni\; U_k \;\equiv\; \bigcup\{U_b \mid b \in k\} \;\subset\; U_{\ell_1} \cap U_{\ell_2},$$

but a Scott-open family $\mathcal{K}_a$ must be closed upwards, so $\mathcal{K}_a \ni U_{\ell_1} \cap U_{\ell_2}$ too. By a similar argument as in Lemma 2.1, but using Scott-openness of $\mathcal{K}_a$ in place of compactness of K_a, there is some finite set h with

$$(\mathcal{K}_a \ni U_h) \;\equiv\; (a \prec\!\!\!\prec h) \qquad \text{and} \qquad \forall c \in h.\, \exists b_1 \in \ell_1.\, \exists b_2 \in \ell_2.\, (b_1 \sqsupseteq c \sqsubseteq b_2),$$

which is the definition of $a \prec\!\!\!\prec \ell_1 \sqcap \ell_2$. □

We also need a rule to govern unions. The frequency with which this property appears in print without attribution indicates its importance. This condition was identified by Peter Wilker [Wil70] as part of the study of topological function-spaces, *cf.* Proposition 8.3, anticipating many of the ideas of locale theory and continuous lattices that we used in Section 6.

Lemma 2.9 If a compact subspace is covered by two open ones, $K \subset U_1 \cup U_2$, then there are compact L_1 and L_2 and open V_1, V_2 with $K \subset V_1 \cup V_2$, $V_1 \subset L_1 \subset U_1$ and $V_2 \subset L_2 \subset U_2$. □

Lemma 2.10 A basis of either kind also obeys the ***Wilker rule*** that

$$a \prec\!\!\!\prec \ell \implies \exists k.\; a \prec\!\!\!\prec k \;\wedge\; \forall b \in k.\, \exists c \in \ell.\, b \prec\!\!\!\prec c.$$

Proof Given $\mathcal{K}_a \ni U_\ell \equiv \bigcup\{U_b \mid b \in \ell\}$, the basis expansion of U_b for each $b \in \ell$ yields

$$\mathcal{K}_a \ \ni \ U_\ell \ = \ \bigcup_{b\in\ell} U_b \ = \ \bigcup_{b\in\ell}\bigcup\{U_c \mid \mathcal{K}_c \ni U_b\} \ = \ \bigcup\{U_c \mid \exists b \in \ell.\, \mathcal{K}_c \ni U_b\}.$$

Since $\mathcal{K}_a$ is a Scott-open family, there is some finite set k of such c for which we still have

$$\mathcal{K}_a \ \ni \ \bigcup\{U_c \mid c \in k\} \ \equiv \ U_k \qquad \text{and} \qquad \forall c \in k.\, \exists b \in \ell.\, (\mathcal{K}_c \ni U_b),$$

which is what the conclusion says. □

In the rest of the paper we will make heavy use of Scott-open families and it will not surprise you to learn that they are part of something bigger:

Proposition 2.11 The Scott-open subsets of any complete lattice form a topology, called the ***Scott topology***. A function $M^* : \Omega_2 \to \Omega_1$ between complete lattices is ***Scott-continuous***, *i.e.* with respect to this topology, iff it preserves ***directed joins***, written $\bigvee^{\uparrow}$ or $\bigcup^{\uparrow}$. These are joins of families $\{U_i \mid i \in I\}$ for which

$$\exists i.\, i \in I \qquad \text{and} \qquad i_1, i_2 \in I \implies \exists i \in I.\ U_{i_1} \leqslant U_i \geqslant U_{i_1}. \qquad □$$

In fact, we shall see in Proposition 8.3 that this is the topology on the topology on a locally compact space X that defines the exponential Σ^X.

3 Manipulating bases

Having established the primary axioms for a basis (Definition 1.8), we can "improve" it to one with additional properties that are useful for applications (Definition 1.10). This section is a toolbox for later use and need not be read sequentially as part of the narrative of this paper.

Remark 3.1 In particular, it is very difficult to make progress — and very easy to make errors — in this subject without the ***single interpolation*** property

$$(a \prec\!\!\prec k) \implies \exists b.\, (a \prec\!\!\prec b \prec\!\!\prec k),$$

where, according to the convention in Notation 1.5, b denotes a *single* basis element rather than a list.

For example, the concrete basis for $\mathbb{R}$ using (open and closed) intervals of length < 1 has this property and is bounded above, but that using intervals of length ≤ 1 does not. We leave it to the interested reader to find a similar counterexample for rounded unions. □

Whilst bases need not have this property in general, we assume it in the first two lemmas, which are variations on the rules for intersections, and then show how to turn any given concrete basis into one that has this and some similar properties.

Lemma 3.2 $(a \prec\!\!\prec b \prec\!\!\prec \ell) \implies \exists k.\, (a \prec\!\!\prec k \prec\!\!\prec b) \wedge (k \prec\!\!\prec^1 \ell).$

Proof By the Wilker and single interpolation rules (twice), there are a', b' and ℓ' with

$$a \prec\!\!\prec a' \prec\!\!\prec b' \prec\!\!\prec b \prec\!\!\prec \ell' \prec\!\!\prec^1 \ell, \qquad \text{so} \qquad a' \prec\!\!\prec \ell'.$$

Then $a \prec\!\!\prec b' \sqcap \ell'$ by the weak intersection rule, *i.e.* there is k such that

$$a \prec\!\!\prec k \sqsubseteq b' \prec\!\!\prec b \qquad \text{and} \qquad k \sqsubseteq \ell' \prec\!\!\prec^1 \ell'.$$

Then $k \prec\!\!\!\prec b$ and $k \prec\!\!\!\prec^1 \ell$ as required. $\square$

Lemma 3.3 Suppose that $(A, \sqsubseteq, \prec\!\!\!\prec)$ satisfies the covariance, transitivity and single interpolation rules. Then it obeys the strong intersection rule,

$$(a \prec\!\!\!\prec \ell_1) \ \wedge\ (a \prec\!\!\!\prec \ell_2) \implies a \prec\!\!\!\prec \ell_1 \sqcap \ell_2$$

iff it obeys both the weak intersection rule

$$(a \prec\!\!\!\prec b \prec\!\!\!\prec \ell_1) \ \wedge\ (b \prec\!\!\!\prec \ell_2) \implies a \prec\!\!\!\prec \ell_1 \sqcap \ell_2$$

(with a singleton b instead of a set k) and the rounded intersection rule

$$(a \prec\!\!\!\prec c_1) \ \wedge\ (a \prec\!\!\!\prec c_2) \implies \exists b.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec c_1) \ \wedge\ (b \prec\!\!\!\prec c_2).$$

Proof The weak rule follows from the strong one by transitivity. The strong rule, single interpolation and covariance give

$$(a \prec\!\!\!\prec c_1) \ \wedge\ (a \prec\!\!\!\prec c_2) \implies \exists kb.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec k \sqsubseteq c_1 \sqcap c_2) \implies \exists b.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec c_1) \ \wedge\ (b \prec\!\!\!\prec c_2).$$

Conversely, single interpolation, rounded intersection, transitivity and weak intersection give

$$\begin{aligned}
(a \prec\!\!\!\prec \ell_1) \ \wedge\ (a \prec\!\!\!\prec \ell_2) \ &\Rightarrow\ \exists c_1 c_2.\, (a \prec\!\!\!\prec c_1 \prec\!\!\!\prec \ell_1) \ \wedge\ (a \prec\!\!\!\prec c_2 \prec\!\!\!\prec \ell_2) \\
&\Rightarrow\ \exists b c_1 c_2.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec c_1 \prec\!\!\!\prec \ell_1) \ \wedge\ (b \prec\!\!\!\prec c_2 \prec\!\!\!\prec \ell_2) \\
&\Rightarrow\ a \prec\!\!\!\prec \ell_1 \sqcap \ell_2.
\end{aligned}$$

$\square$

We have already seen that the strong intersection rule holds for concrete bases using compact subspaces and we will prove the converse of this in Section 11.

The next construction adds unions to the basis, because some of the results that we shall discuss do need this. On the other hand, it is a major goal of this paper to develop bases that need *not* be closed under unions or intersections. Nevertheless, this result shows that there *some* richer basis that has the extra properties that it is convenient to assume. Unfortunately, there seems to be no canonical way of doing this: in selecting a basis for a space, we have to choose *enough* unions to satisfy them.

Lemma 3.4 If $(U_a, \mathcal{K}_a)$ is a basis for a space X then its ***directed basis*** has

$$U_\ell \ \equiv\ \bigcup\{U_a \mid a \in \ell\} \qquad \text{and} \qquad \mathcal{K}_\ell \ \equiv\ \bigcap\{\mathcal{K}_a \mid a \in \ell\}.$$

This has the ***single interpolation*** and ***rounded union*** properties. If the given basis has the strong intersection property then so does the directed basis.

Proof For the filtered condition on basic opens,

$$\begin{aligned}
x \in U_k \ \wedge\ x \in U_\ell \ &\equiv\ \exists a \in k.\, \exists b \in \ell.\, x \in U_a \ \wedge\ x \in U_b \\
&\Rightarrow\ \exists abc.\, x \in U_c \ \wedge\ k \ni a \sqsupseteq c \sqsubseteq b \in \ell \\
&\Rightarrow\ \exists h.\, x \in U_h \ \wedge\ k \sqsupseteq h \sqsubseteq \ell,
\end{aligned}$$

where $h \equiv \{c\}$. The basis expansion is

$$\begin{aligned}
x \in V \ &\Leftrightarrow\ \exists a.\, x \in U_a \ \wedge\ \mathcal{K}_a \ni V \implies \exists \ell.\, x \in U_\ell \ \wedge\ \mathcal{K}_\ell \ni V \\
&\equiv\ \exists \ell a.\, a \in \ell \ \wedge\ x \in U_a \ \wedge\ \forall b \in \ell.\, \mathcal{K}_b \ni V \implies \exists a.\, x \in U_a \ \wedge\ \mathcal{K}_a \ni V.
\end{aligned}$$

The way below relation is

$$\begin{aligned} k \prec\!\!\!\prec_{\mathsf{Fin}\,(A)} L \quad &\equiv \quad \mathcal{K}_k \ni \bigcup \{U_a \mid \exists \ell.\, a \in \ell \in L\} \\ &\Leftrightarrow \quad \forall b \in k.\, \mathcal{K}_b \ni \bigcup \{U_a \mid a \in \bigsqcup L\} \;\equiv\; k \prec\!\!\!\prec_A \bigsqcup L. \end{aligned}$$

This inherits co- and contravariance, the Wilker and intersection rules, essentially as they stand. The interpolation property for $(U_a, \mathcal{K}_a)$ gives *single* interpolation for $(U_\ell, \mathcal{K}_\ell)$,

$$k \prec\!\!\!\prec_{\mathsf{Fin}\,(A)} L \;\equiv\; k \prec\!\!\!\prec_A \bigsqcup L \implies \exists h.\, k \prec\!\!\!\prec_A h \prec\!\!\!\prec_A \bigsqcup L \;\equiv\; \exists h.\, k \prec\!\!\!\prec_{\mathsf{Fin}\,(A)} \{h\} \prec\!\!\!\prec_{\mathsf{Fin}\,(A)} L.$$

For rounded binary unions,

$$\begin{aligned} \{\ell_1, \ell_2\} \prec\!\!\!\prec_{\mathsf{Fin}\,(A)} k \quad &\equiv \quad \ell_1 \sqcup \ell_2 \prec\!\!\!\prec_A k \\ &\Rightarrow \quad \ell_1 \sqcup \ell_2 \prec\!\!\!\prec_A h \prec\!\!\!\prec_A k \\ &\equiv \quad \{\ell_1, \ell_2\} \prec\!\!\!\prec_{\mathsf{Fin}\,(A)} h \prec\!\!\!\prec_{\mathsf{Fin}\,(A)} k \end{aligned}$$

using the interpolation property that we already have.

Finally, the rounded intersection property for the directed basis,

$$h \prec\!\!\!\prec \ell_1 \;\wedge\; h \prec\!\!\!\prec \ell_2 \implies \exists k.\, h \prec\!\!\!\prec k \;\wedge\; k \sqsubseteq \ell_1 \sqcap \ell_2,$$

is the same as the strong intersection property for the given one and we deduce strong intersection for the directed basis using Lemma 3.3. □

We can make a basis bounded below by adding $\circ$, but this is undesirable (Section 12) and there is another way:

Lemma 3.5 If $(A, \sqsubseteq, \prec\!\!\!\prec)$ has single interpolation then $(A^+, \sqsubseteq, \prec\!\!\!\prec)$ is also bounded below, where $A^+ \equiv \{b \mid \exists a.\, a \prec\!\!\!\prec b\}$. □

Another simple transformation is to eliminate the preorder $\sqsubseteq$, more or less just by replacing it with $\prec\!\!\!\prec$:

Lemma 3.6 Any abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ with single interpolation satisfies

$$a \prec\!\!\!\prec k \prec\!\!\!\prec \ell \implies a \prec\!\!\!\prec \ell \implies \exists b.\, a \prec\!\!\!\prec b \prec\!\!\!\prec \ell$$

and

$$a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1, \ell_2 \implies \exists k'.\, a \prec\!\!\!\prec k' \prec\!\!\!\prec^1 \ell_1, \ell_2.$$

If $a \prec\!\!\!\prec b$ then $U_a \subset U_b$ and $\mathcal{K}_a \supset \mathcal{K}_b$ in the concrete basis, where the filter property is

$$x \in U_a \;\wedge\; x \in U_b \implies \exists d.\, x \in U_d \;\wedge\; (a \succ\!\!\!\succ d \prec\!\!\!\prec b).$$

Conversely, any relation $\prec\!\!\!\prec$ with these properties defines an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ by

$$a \sqsubseteq b \;\equiv\; a \prec\!\!\!\prec b \;\vee\; a = b.$$

Proof We deduce the second property from the weak intersection, Wilker and covariance rules:

$$a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1, \ell_2 \implies \exists k' \ell.\, a \prec\!\!\!\prec k' \prec\!\!\!\prec^1 \ell \sqsubseteq \ell_1, \ell_2 \implies \exists k'.\, a \prec\!\!\!\prec k' \prec\!\!\!\prec^1 \ell_1, \ell_2.$$

In a concrete basis, $a \prec\!\!\!\prec b \;\equiv\; \mathcal{K}_a \ni U \implies U_a \subset U_b$ since U_a contributes to the basis expansion of U_b. Similarly, $a \prec\!\!\!\prec b \;\wedge\; \mathcal{K}_b \ni U \implies \mathcal{K}_a \ni U_b \subset U \implies \mathcal{K}_a \ni U$ since $\mathcal{K}_a$ is upper.

If $x \in U_a$ and $x \in U_b$ then $x \in U_c$ for some $c \in A$ with $a \sqsupseteq c \sqsubseteq b$, then the basis expansion of U_c gives some $d \in A$ with $x \in U_d$ and $\mathcal{K}_d \ni U_c$, so $d \prec\!\!\!\prec c$ and $a \succ\!\!\!\succ d \prec\!\!\!\prec b$.

For the converse, we prove transitivity of $\sqsubseteq$ by an easy case analysis, the extension of which to (Kuratowski) finite sets or lists gives covariance of $\prec\!\!\!\prec$ with respect to $\sqsubseteq$:

$$\begin{aligned} b \prec\!\!\!\prec k \sqsubseteq \ell \quad &\Rightarrow \quad \exists k_1 k_2 k'.\, (b \prec\!\!\!\prec k' \prec\!\!\!\prec^1 k = k_1 \sqcup k_2) \ \wedge\ (k_1 \prec\!\!\!\prec^1 \ell) \ \wedge\ (k_2 \subset \ell) \\ &\Rightarrow \quad \exists k'.\, (b \prec\!\!\!\prec k' \prec\!\!\!\prec^1 \ell). \end{aligned}$$ □

Now we consider bases for open and closed subspaces.

Lemma 3.7 A concrete basis for an open subspace $V \subset X$ is given by

$$U_a^V \ \equiv\ U_a \cap V \qquad \text{and} \qquad \mathcal{K}_a^V \ \equiv\ \mathcal{K}_a \cap {\downarrow} V.$$

If the given basis for X uses compact subspaces then that for V has

$$a \prec\!\!\!\prec^V \ell \iff a \prec\!\!\!\prec^X \ell \ \wedge\ (K_a \subset V).$$

Proof The basis expansion of $x \in U \subset V$ is

$$\begin{aligned} x \in U \quad &\Leftrightarrow \quad \exists a.\, x \in U_a \ \wedge\ \mathcal{K}_a \ni U \\ &\Leftrightarrow \quad \exists a.\, x \in (U_a \cap V) \ \wedge\ (\mathcal{K}_a \ni U \subset V). \end{aligned}$$

The filter property is

$$\begin{aligned} x \in U_a^V \ \wedge\ x \in U_b^V \quad &\Leftrightarrow \quad x \in U_a \ \wedge\ x \in U_b \ \wedge\ x \in V \\ &\Leftrightarrow \quad \exists c.\, x \in (U_c \cap V) \ \wedge\ (a \sqsupseteq c \sqsubseteq b). \end{aligned}$$ □

This fails the boundedness property, but we can (re)impose it:

Lemma 3.8 For any basis $(A, \sqsubseteq, \prec\!\!\!\prec)$, the subset $A' \equiv \{a \mid \exists \ell.\, a \prec\!\!\!\prec \ell\}$ defines another basis for the same space but also obeys ***boundedness***.

Proof First observe that

$$\mathcal{K}_a \ni U = \bigcup\nolimits^{\uparrow} \{U_\ell \mid U_\ell \subset U\} \implies \exists \ell.\, \mathcal{K}_a \ni U_\ell \subset U \implies \exists \ell.\, a \prec\!\!\!\prec \ell$$

since the family (U_ℓ) provides a directed basis and $\mathcal{K}_a$ is Scott continuous. Hence the basis expansion is

$$\begin{aligned} x \in U \quad &\Leftrightarrow \quad \exists a.\, (x \in U_a) \ \wedge\ (\mathcal{K}_a \ni U) \\ &\Leftrightarrow \quad \exists a.\, (x \in U_a) \ \wedge\ (\mathcal{K}_a \ni U) \ \wedge\ (\exists \ell.\, a \prec\!\!\!\prec \ell) \\ &\equiv \quad \exists a \in A'.\, (x \in U_a) \ \wedge\ (\mathcal{K}_a \ni U). \end{aligned}$$

The subset A' is downwards-closed with respect to $\sqsubseteq$ and $\prec\!\!\!\prec$ because of contravariance and transitivity of $\prec\!\!\!\prec$. Hence the concrete basis still has the filtered property and the abstract one still obeys the Wilker and intersection rules:

$$a \prec\!\!\!\prec \ell \subset A' \implies \exists k.\, a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell \ \wedge\ k \subset A'$$

$$a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1 \subset A' \ \wedge\ k \prec\!\!\!\prec \ell_2 \subset A' \implies \exists \ell'.\, a \prec\!\!\!\prec \ell' \sqsubseteq \ell_1 \ \wedge\ \ell' \sqsubseteq \ell_2 \ \wedge\ \ell' \subset A'.$$ □

Lemma 3.9 A basis for an closed subspace $C \subset X$ is obtained from a basis $(U_a, \mathcal{K}_a)$ for X by

$$U_a^C \;\equiv\; U_a \cup V \qquad \text{and} \qquad \mathcal{K}_a^C \;\equiv\; \mathcal{K}_a,$$

where V is the complementary open subspace to C (in the sense of Proposition 8.1). Hence

$$a \prec\!\!\!\prec^C \ell \iff \exists k.\ (a \prec\!\!\!\prec k \sqcup \ell) \;\wedge\; (\mathcal{K}_k \ni V).$$

Proof If $x \in C$, so $x \notin V$, then

$$x \in (U_a \cup V) \;\wedge\; x \in (U_b \cup V) \iff \exists c.\, x \in (U_b \cup V) \;\wedge\; (a \sqsupseteq c \sqsubseteq b)$$

and $$x \in W \iff \exists a.\, x \in (U_a \cup V) \;\wedge\; \mathcal{K}_a \ni W.$$

Notice in particular that $(a \prec\!\!\!\prec^C \circ)$ if $\mathcal{K}_a \ni V$. In Section 12 we investigate when it is possible to eliminate such empty covers. □

The topological ideas that we shall use in the course of this paper rely, naturally enough, on the *open subspaces* of a space. We will come to learn that the *points* are secondary, so the rest of this section shows how to express points and compact subspaces in terms of open ones. The first result is due to Jimmie Lawson [GHK+80, §I 3.3].

Lemma 3.10 Let $a \in r \subset A$ where r is ***rounded***,

$$r \ni b \iff \exists c.\, r \ni c \prec\!\!\!\prec b.$$

Then there is a ***$\prec\!\!\!\prec$-filter*** s with $a \in s \subset r$, *i.e.*

$$\exists a.\, a \in s, \qquad (a \in s) \;\wedge\; (b \in s) \iff \exists c \in s.\, (c \prec\!\!\!\prec a) \;\wedge\; (c \prec\!\!\!\prec b).$$

Proof Let $a_0 \equiv a$. By repeated use of single interpolation, roundedness of r and Dependent Choice, let $r \ni a_{i+1} \prec\!\!\!\prec a_i$. Having constructed such a sequence, let $s \equiv \{b \mid \exists i.\, a_i \prec\!\!\!\prec b\}$.

Then $a \in s$ because $a_1 \prec\!\!\!\prec a_0 \equiv a$.

Also s is upper because if $b \in s$ with $b \prec\!\!\!\prec b'$ or $b \sqsubseteq b'$ then $a_i \prec\!\!\!\prec b \prec\!\!\!\prec b'$ and $b' \in s$.

Also s is a $\prec\!\!\!\prec$-filter because if $a_{i_1} \prec\!\!\!\prec b_1$ and $a_{i_2} \prec\!\!\!\prec b_2$ then with $i = \mathsf{max}(i_1, i_2)$, $a_i \prec\!\!\!\prec a_{i_1} \prec\!\!\!\prec b_1$ and $a_i \prec\!\!\!\prec a_{i_2} \prec\!\!\!\prec b_2$. □

Lemma 3.11 Let $(U_a, \mathcal{K}_a)$ be a basis using Scott-open families and $s \subset A$ a $\prec\!\!\!\prec$-filter. Then $\mathcal{K} \;\equiv\; \{U \mid \exists a \in s.\, \mathcal{K}_a \ni U\}$ is a Scott-open filter of open subspaces.

Proof It is Scott-open since the $\mathcal{K}_a$ are and we have $\mathcal{K} \ni X$ because s is inhabited. If $U \in \mathcal{K}_a \subset \mathcal{K}$ and $V \in \mathcal{K}_b \subset \mathcal{K}$ with $d \prec\!\!\!\prec c \prec\!\!\!\prec a, b$ in s then by the weak intersection rule there is some k with $d \prec\!\!\!\prec k \sqsubseteq a, b$, so

$$\mathcal{K}_d \;\ni\; U_k \;\subset\; U_a \cap U_b \;\subset\; U \cap V$$

and $U \cap V \in \mathcal{K}_d \subset \mathcal{K}$ since it's upper. □

We will see in Corollary 8.4 that, if the basis is directed, then every Scott-open filter may be expressed in this way.

Lemma 3.12 Let $\mathcal{K} \subset \Omega$ be a Scott-filter with $V \notin \mathcal{K}$. Then there is a maximal Scott-open filter $\mathcal{P}$ with $\mathcal{K} \subset \mathcal{P} \subset \Omega$ and $V \notin \mathcal{P}$ and then $\mathcal{P}$ is completely coprime.

Proof This is based on a well known argument for commutative rings, using Zorn's Lemma, but see [Joh82, Lemma VII 4.3] for a proof. □

In order to deduce information about the points, we need an additional property to say that a space has exactly the points that are dictated by its open subspaces.

Definition 3.13 A ***completely co-prime filter*** or ***formal point*** for the topology on X is a family $\mathcal{P}$ of open subspaces of X such that

$$X \in \mathcal{P}, \qquad U, V \in \mathcal{P} \iff U \cap V \in \mathcal{P} \qquad \text{and} \qquad \bigcup U_i \in \mathcal{P} \iff \exists i.\, U_i \in \mathcal{P}.$$

In particular, for every ordinary point $x \in X$, the ***neighbourhood filter*** $\mathcal{P}_x \equiv \{U \mid x \in U\}$ is a formal point.

Then a space X is ***sober*** if every formal point is of this form for some unique ordinary point $x \in X$. Sobriety is often stated as requiring that every irreducible closed subspace C is the closure of a unique point p. This is equivalent to our definition, with

$$\mathcal{P} \;\equiv\; \{U \mid U \cap C = \emptyset\} \qquad \text{and} \qquad C \;\equiv\; X \setminus \bigcup \{U \mid U \notin \mathcal{P}\},$$

so that $U \cap C = \emptyset \iff U \in \mathcal{P} \iff x \in U$.

Containment, $\mathcal{P}_1 \subset \mathcal{P}_2$, of one formal point in another is called the ***specialisation order***, as is the corresponding relation between ordinary points.

Now we can give the topological characterisation of compact subspaces that is due to Karl Hofmann and Michael Mislove [HM81]. Beware that it requires the space to be sober, though not necessarily locally compact.

Proposition 3.14 Any Scott-open filter $\mathcal{K}$ of open subspaces of a sober space satisfies

$$\mathcal{K} \ni U \iff K \subset U \qquad \text{where} \qquad K \;\equiv\; \bigcap \mathcal{K} \quad \text{is compact.}$$

Proof If $\mathcal{K} \ni U$ then $K \subset U$ by definition of $\bigcap \mathcal{K}$. Conversely, by Lemma 3.12, if $U \notin \mathcal{K}$ then there is a formal point $\mathcal{P}$ with $U \notin P \supset \mathcal{K}$, so by sobriety (Definition 3.13) there is a (concrete) point p with $p \in V \iff V \in \mathcal{P}$. Hence $p \in K$ but $p \notin U$, as required. The subspace K is compact because its neighbourhood filter $\mathcal{K}$ is Scott-open. □

Definition 3.15 We therefore call any Scott-open filter $\mathcal{K}$ a ***formal compact subspace***. However, the constructions in Section 8 illustrate that not every (concrete) compact subspace is the intersection of its neighbourhoods like this; one that does so is called ***saturated***, although this use of the word is unrelated to that in Definition 1.14.

Proposition 3.16 If the abstract basis satisfies the boundedness and strong intersection rules then each Scott-open family $\mathcal{K}_a$ is a filter and $K_a \equiv \bigcap \mathcal{K}_a$ is a compact subspace with $K_a \subset U \iff \mathcal{K}_a \ni U$. Then the basis expansion is

$$p \in U \iff \exists a.\, p \in U_a \,\wedge\, K_a \subset U \qquad \text{or} \qquad U \;=\; \bigcup \{U_a \mid K_a \subset U\}.$$

Proof Each $a \in A$ has some $a \prec\!\!\!\prec b$ by boundedness and $U_b \subset X$, so $\mathcal{K}_a \ni X$.

If $\mathcal{K}_a \ni U, V$ then $U_k \subset U$ and $U_\ell \subset V$ with $a \prec\!\!\!\prec k, \ell$, so $a \prec\!\!\!\prec h \sqsubseteq k, \ell$ for some h by the strong intersection rule, but then $U_h \subset U_k \subset U$ and similarly $U_h \subset V$ and $U_h \subset U \cap V$, making $\mathcal{K}_a \ni U \cap V$. So $\mathcal{K}_a \ni U \iff K_a \subset U$ by Proposition 3.14 and the basis expansion follows. □

We will describe K_a more explicitly in terms of the abstract basis in Theorem 5.10. In Section 11 we will show how to replace a concrete basis that uses Scott-open families or an abstract one that obeys the weak intersection rule with another that has compact subspaces or the strong rule.

4 Continuous maps

Having described concrete and abstract *bases* for locally compact *spaces*, we now undertake a similar task for *continuous functions*, which we shall also characterise using binary relations. Since they are induced by bases and our notion of a dual base of compact subspaces already alludes to an analogy with linear algebra, we shall call these relations *matrices*.

Most of the rest of this paper is about equivalent formulations of locally compact spaces and does not make much use of the results in this section about continuous functions. Indeed, we discuss the minutiae of the axioms for matrices more for the benefit of subsequent work than for our present requirements.

Remark 4.1 Given a continuous function $f : X \to Y$ between locally compact sober spaces that have bases (U_a, K_a) and (V_b, L_b) respectively using compact subspaces, we study the binary relation, which we call a ***concrete matrix***, that is defined by

$$\langle\, a \,|\, f \,|\, b \,\rangle \;\equiv\; (fK_a \subset L_b) \;\equiv\; (K_a \subset f^{-1}L_b),$$

so in particular $\langle\, a \,|\, \mathsf{id} \,|\, b \,\rangle \equiv (a \prec\!\!\!\prec b)$. As usual, we can replace the second form of the definition by $\mathcal{K}_a \ni f^{-1}L_b$ and use Scott-open families instead of compact subspaces. We will characterise matrices for continuous functions by the axioms in Definition 1.13.

In fact we can set up a lot of the correspondence for Scott-continuous operators $M^* : \Omega Y \to \Omega X$ (Proposition 2.11), although it only works properly when they preserve all unions.

Lemma 4.2 For any such Scott-continuous operator M^*, the concrete matrix $\langle\, a \,|\, M \,|\, b \,\rangle \equiv (K_a \subset M^*V_b)$ is contravariant and saturated in a and covariant in b. It also satisfies

$$M^*V_b \;=\; \bigcup_a \{U_a \mid \langle\, a \,|\, M \,|\, b \,\rangle\} \;=\; \bigcup_k{}^{\uparrow} \{U_k \mid \forall a \in k.\, \langle\, a \,|\, M \,|\, b \,\rangle\}.$$

Proof The variance properties follow from those of K_a and V_b (Definition 1.3(b)) and monotonicity of M^*. The last part is the basis expansion of M^*V_b, from which we deduce

$$K_a \subset M^*V_b \iff \exists k.\, K_a \subset U_k \;\wedge\; \forall a' \in k.\, K_{a'} \subset M^*V_b$$

since K_a is compact. Hence the matrix is ***saturated*** in a:

$$\langle\, a \,|\, M \,|\, b \,\rangle \iff \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \langle\, a' \,|\, M \,|\, b \,\rangle. \qquad \square$$

We can improve on this using the ideas of the previous section:

Lemma 4.3 If the bases obey the single interpolation, rounded union and boundedness below properties (Definition 1.10 and Lemma 3.4) then the matrix is rounded on both sides.

Proof By single interpolation within the saturation property of the previous result,

$$\begin{aligned} \langle\, a \,|\, M \,|\, b \,\rangle \;&\Leftrightarrow\; \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a'' \in k.\, \langle\, a'' \,|\, M \,|\, b \,\rangle \\ &\Leftrightarrow\; \exists a'k.\, (a \prec\!\!\!\prec a' \prec\!\!\!\prec k) \;\wedge\; \forall a'' \in k.\, \langle\, a'' \,|\, M \,|\, b \,\rangle \\ &\Leftrightarrow\; \exists a'.\, (a \prec\!\!\!\prec a') \;\wedge\; \langle\, a' \,|\, M \,|\, b \,\rangle, \end{aligned}$$

we deduce roundedness in a.

The expansion of V_b with respect to the directed basis (Lemma 3.4) is

$$V_b \;=\; \bigcup_\ell{}^{\uparrow} \{V_\ell \mid \forall b' \in \ell.\, L_{b'} \subset V_b\} \;\equiv\; \bigcup_\ell{}^{\uparrow} \{V_\ell \mid \ell \prec\!\!\!\prec b\},$$

so, since M^* is Scott-continuous and K_a is compact,

$$\begin{aligned}\langle\, a \,|\, M \,|\, b \,\rangle \quad &\equiv \quad K_a \subset M^* V_b \;=\; \bigcup\nolimits^{\uparrow}_{\ell} \{ M^* V_\ell \mid \ell \prec\!\!\!\prec b \} \\ &\Leftrightarrow \quad \exists \ell.\, K_a \subset M^* V_\ell \;\wedge\; (\ell \prec\!\!\!\prec b) \\ &\Leftrightarrow \quad \exists \ell b'.\, K_a \subset M^* V_\ell \;\wedge\; (\ell \prec\!\!\!\prec b' \prec\!\!\!\prec b) \\ &\equiv \quad \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; (b' \prec\!\!\!\prec b),\end{aligned}$$

where b' comes from the rounded union property. Hence the matrix is rounded in b. □

Lemma 4.4 For any matrix $\langle\ |\, M \,|\ \rangle$ that is rounded in b, the operator $M^\dagger$ defined by

$$\begin{aligned} M^\dagger V \quad &\equiv \quad \bigcup_a \{ U_a \mid \exists b.\, \langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; L_b \subset V \} \\ &= \quad \bigcup\nolimits^{\uparrow}_k \{ U_k \mid \forall a \in k.\, \exists b.\, \langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; L_b \subset V \}\end{aligned}$$

is Scott-continuous and

$$M^\dagger V_b \;=\; \bigcup_a \{ U_a \mid \langle\, a \,|\, M \,|\, b \,\rangle \} \;=\; \bigcup\nolimits^{\uparrow}_k \{ U_k \mid \forall a \in k.\, \langle\, a \,|\, M \,|\, b \,\rangle \}.$$

Hence if the matrix $\langle\ |\, M \,|\ \rangle$ was defined from an operator M^* then

$$M^\dagger V \;\subset\; M^* V \qquad \text{and} \qquad M^\dagger V_b \;=\; M^* V_b.$$

Proof Scott continuity is immediate from compactness of L_b, whilst roundedness gives

$$\begin{aligned} M^\dagger V_b \quad &\equiv \quad \bigcup_a \{ U_a \mid \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; L_{b'} \subset V_b \} \\ &\equiv \quad \bigcup_a \{ U_a \mid \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; (b' \prec\!\!\!\prec b) \} \\ &\Leftrightarrow \quad \bigcup_a \{ U_a \mid \langle\, a \,|\, M \,|\, b \,\rangle \}.\end{aligned}$$

To show that $M^\dagger V \subset M^* V$ it suffices to observe that

$$\langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; L_b \subset V \;\Longrightarrow\; \mathcal{K}_a \ni M^* V_b \;\wedge\; L_b \subset V \;\Longrightarrow\; U_a \subset M^* V_b,$$

by the basis expansion of $M^* V_b$. Equality in the case $V \equiv V_b$ follows from Lemma 4.2. □

Lemma 4.5 If the matrix $\langle\ |\, M \,|\ \rangle$ is co- and contravariant, rounded on both sides and saturated in its input then it is recovered from the operator $M^\dagger$.

Proof By the previous lemma, the derived matrix is

$$K_a \subset M^\dagger V_b \iff \exists k.\, K_a \subset U_k \;\wedge\; \forall a' \in k.\, \langle\, a' \,|\, M \,|\, b \,\rangle,$$

but the right hand side of this is just $\langle\, a \,|\, M \,|\, b \,\rangle$ because this is saturated by hypothesis. □

Notation 4.6 Given matrices $\langle\ |\, M \,|\ \rangle$ and $\langle\ |\, N \,|\ \rangle$,

$$\begin{aligned} M^\dagger(N^\dagger W) \quad &= \quad \bigcup\nolimits^{\uparrow} \{ U_k \mid \forall a \in k.\, \exists b.\, \langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; L_b \subset N^\dagger W \}, \\ \text{so} \qquad \langle\, a \,|\, M \,;\, N \,|\, c \,\rangle \quad &\equiv \quad K_a \subset M^\dagger(N^\dagger W_c) \\ &= \quad \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; \langle\, b \,|\, N \,|\, c \,\rangle,\end{aligned}$$

which we call the ***saturated composite***. However, this definition is not yet safe to use:

Example 4.7 Even when Scott-continuous operators M^* and N^* are representable by matrices, their composite $P^* \equiv M^* \cdot N^*$ not not be.

Proof Let $X \equiv \mathbf{1} \equiv \{\bullet\}$ with prime basis $A \equiv \{\bullet\}$, $Y \equiv \mathbf{2} \equiv \{0,1\}$ with directed basis $B \equiv \{0,1,\bullet\}$ and $Z \equiv \mathbf{2} \times \mathbf{2}$ with prime basis $C \equiv \{(0,0),(0,1),(1,0),(1,1)\}$.

Let $M^* : \Sigma^Y \to \Sigma^X$ be conjunction, so its only true matrix element is $\langle \bullet \,|\, M \,|\, \bullet \rangle$.

Let $N^* : \Sigma^Z \to \Sigma^Y$ be disjunction on the second component, its true matrix elements being $\langle 0 \,|\, N \,|\, (0,0) \rangle$, $\langle 0 \,|\, N \,|\, (0,1) \rangle$, $\langle 1 \,|\, N \,|\, (1,0) \rangle$ and $\langle 1 \,|\, N \,|\, (1,1) \rangle$.

Then $M^*N^*\{(0,1),(1,0)\} = \{\bullet\}$ but $M^*N^*\{(z_1,z_2)\} = \emptyset$ for any of the four singletons. Therefore, since these singletons provide the basis for Z, the matrix $\langle \bullet \,|\, P \,|\, c \rangle$ for $P^* \equiv M^* \cdot N^*$ is everywhere false and $P^\dagger V = \emptyset$. The relational and saturated composite matrices are also everywhere false. □

It is not this failure that will surprise you but that we ever suggested that we could define matrices using *singletons* instead of *lists*, when we needed to use lists in bases to capture the way below relation for locally compact spaces other than domains. We will see that it does work for inverse images operators for continuous functions. For Scott-continuous operators, we have

Lemma 4.8 If the basis (V_b, L_b) is directed then every Scott-continuous operator is represented by its matrix.

Proof By hypothesis, the basis expansion $V = \bigcup\{V_b \mid L_b \subset V\}$ is directed and M^* preserves it. By Lemma 4.4, the operator $M^\dagger$ that is derived from the matrix $\langle a \,|\, M \,|\, b \rangle$ that was obtained from M^* also preserves this union, whilst $M^\dagger V_b = M^* V_b$. Hence $M^\dagger V = M^* V$ for any V. □

In this case, Lemmas 4.2–4.4 define a bijection between these operators and matrices that are co- and contravariant, rounded and saturated. It follows that the category of locally compact spaces and Scott-continuous operators is equivalent to one of bases and matrices with saturated composition, although it is not obvious from its abstract definition that this composition is associative.

Without this assumption, however, rounded saturated matrices just correspond to *some* of the Scott-continuous operators between open-set lattices, but unfortunately not even to a subcategory of them. In order to improve on this result, we require better control of finite unions, so we must restrict attention to operators that preserve them.

First we want to extend the definition of the matrix to unions in the output:

Lemma 4.9 If M^* preserves all unions then

$$K_a \subset M^* V_\ell \iff \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b \in \ell.\, \langle a' \,|\, M \,|\, b \rangle.$$

Proof Using the directed basis expansion of $M^* V_b$,

$$\begin{aligned} M^* V_\ell \;\equiv\; M^* \bigcup_{b \in \ell} V_b &= \bigcup_{b \in \ell} M^* V_b \;=\; \bigcup_{b \in \ell} \bigcup_{a'} \{U_{a'} \mid K_{a'} \subset M^* V_b\} \\ &= \bigcup_{a'} \{U_{a'} \mid \exists b \in \ell.\, K_{a'} \subset M^* V_b\} \\ &= \bigcup_{k}^{\uparrow} \{U_k \mid \forall a' \in k.\, \exists b \in \ell.\, K_{a'} \subset M^* V_b\}. \end{aligned}$$

Then, since K_a is compact,

$$K_a \subset M^* V_\ell \iff \exists k.\, K_a \subset U_k \;\wedge\; \forall a' \in k.\, \exists b \in \ell.\, K_{a'} \subset M^* V_b,$$

whence the result follows by the definitions of $(a \prec\!\!\!\prec k)$ and $\langle\, a' \,|\, M \,|\, b \,\rangle$. $\square$

This brings us to the matrix characterisation of operators that preserve arbitrary unions:

Lemma 4.10 If M^* preserves unions then $\langle\ |\,M\,|\ \rangle$ has the ***partition property***,

$$\langle\, a \,|\, M \,|\, b \,\rangle \ \wedge\ (b \prec\!\!\!\prec \ell) \implies \exists k.\, (a \prec\!\!\!\prec k)\ \wedge\ \forall a' \in k.\, \exists b \in \ell.\, \langle\, a' \,|\, M \,|\, b \,\rangle.$$

Proof Since $(b \prec\!\!\!\prec \ell) \equiv (L_b \subset V_\ell) \Rightarrow (V_b \subset V_\ell) \Rightarrow (M^* V_b \subset M^* V_\ell)$, the previous result gives

$$\begin{aligned} \langle\, a \,|\, M \,|\, b \,\rangle \ \wedge\ (b \prec\!\!\!\prec \ell) \quad &\Rightarrow \quad K_a \subset M^* V_b \subset M^* V_\ell \\ &\Rightarrow \quad \exists k.\, (a \prec\!\!\!\prec k)\ \wedge\ \forall a' \in k.\, \exists b' \in \ell.\, \langle\, a' \,|\, M \,|\, b' \,\rangle. \end{aligned}$$ $\square$

Lemma 4.11 For any predicate ϕ on the indexing set of the basis,

$$\bigcup \{U_a \mid \exists k.\, (a \prec\!\!\!\prec k)\ \wedge\ \forall a' \in k.\, \phi a'\} \ \subset\ \bigcup \{U_{a'} \mid \phi a'\}.$$

Proof If $a \prec\!\!\!\prec k$ then $K_a \subset U_k$, so $U_a \subset U_k \equiv \bigcup\{U_{a'} \mid a' \in k\}$. Hence if also $\forall a' \in k.\, \phi a'$ then $U_a \subset \{U_{a'} \mid \phi a'\}$ and the result follows. $\square$

Lemma 4.12 If the matrix $\langle\ |\,M\,|\ \rangle$ has the partition property then $M^\dagger$ preserves unions.

Proof If $b \in \ell$ then $V_b \subset V_\ell$ and $M^\dagger V_b \subset M^\dagger V_\ell$, so $\bigcup\{M^\dagger V_b \mid b \in \ell\} \ \subset\ M^\dagger V_\ell$.

For the reverse inclusion, by the partition property and Lemma 4.11,

$$\begin{aligned} M^\dagger V_\ell \quad &= \quad \bigcup \{U_a \mid \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \ \wedge\ (b' \prec\!\!\!\prec \ell)\} \\ &\subset \quad \bigcup \{U_a \mid \exists k.\, (a \prec\!\!\!\prec k)\ \wedge\ \forall a' \in k.\, \exists b \in \ell.\, \langle\, a' \,|\, M \,|\, b \,\rangle\} \\ &\subset \quad \bigcup \{U_{a'} \mid \exists b \in \ell.\, \langle\, a' \,|\, M \,|\, b \,\rangle\} \\ &= \quad \bigcup \{M^\dagger V_b \mid b \in \ell\}. \end{aligned}$$

Then, since $M^\dagger$ also preserves directed unions, it preserves all of them. $\square$

Proposition 4.13 If the bases obey the single interpolation, rounded union and boundedness below properties then the correspondence above defines a bijection between union-preserving operators and matrices that are co- and contravariant, rounded and saturated and have the partition property.

Proof As in Lemma 4.8, but with arbitrary unions instead of directed ones, where the previous two lemmas establish the connection between finite unions and the partition property. $\square$

Example 4.14 For $f : \mathbb{R} \to \mathbb{R}$ with the usual interval basis, the partition property expresses uniform ϵ–δ continuity *à la* Weierstrass: If ℓ is a list of intervals each of width ϵ that together cover the range of a function, there is a list k of intervals of width δ covering its argument. Then if x_1 and x_2 belong to the same δ-interval, fx_1 and fx_2 will belong to the same ϵ-interval. $\square$

It remains to find the properties of matrices that correspond to the fact that inverse images maps preserve the whole space and intersections. We say that a matrix is ***bounded*** and ***filtered*** respectively if it has the relevant properties. Unfortunately, we cannot do this independently of the unions: we must assume either that the bases are directed or that the matrices have the partition property.

Remark 4.15 Suppose first that the bases are stable (Definition 2.2). Then the matrix for a continuous function satisfies

$$\langle\, a \mid f \mid \bullet \,\rangle \;\equiv\; (fK_a \subset Y) \;\Leftrightarrow\; \top$$

and $$(fK_a \subset V_{b_1}) \;\wedge\; (fK_a \subset V_{b_2}) \iff (fK_a \subset V_{b_1 \sqcap b_2}),$$

which is $$\langle\, a \mid f \mid b_1 \,\rangle \;\wedge\; \langle\, a \mid f \mid b_2 \,\rangle \iff \langle\, a \mid f \mid b_1 \sqcap b_2 \,\rangle.$$

However, as we discussed in Examples 2.3, we do not want to assume that our bases carry this semilattice structure. In some cases we may replace the *actual* top element or intersection above with an existentially quantified variable b:

Definition 4.16 A matrix is ***uniformly*** bounded and filtered respectively if

$$\exists b.\, \langle\, a \mid f \mid b \,\rangle$$

and $$\langle\, a \mid f \mid b_1 \,\rangle \;\wedge\; \langle\, a \mid f \mid b_2 \,\rangle \implies \exists b.\, \langle\, a \mid f \mid b \,\rangle \;\wedge\; (b \prec\!\!\!\prec b_1) \;\wedge\; (b \prec\!\!\!\prec b_2).$$

However, in the leading examples we do not want to assume a union operation for our bases any more than we did an intersection. We really need to use $\langle\, a \mid f \mid \ell \,\rangle \equiv (K_a \subset f^{-1}V_\ell)$, but this is not defined in Notation 1.13, although Lemma 4.9 gave a formula for it that is related to saturation. So, instead of requiring *uniform* boundedness and filteredness as above, we ask that these properties hold *after they have been saturated.*

Lemma 4.17 If M^* preserves unions and $M^*Y = X$ then $\langle\, \mid M \mid \,\rangle$ is ***bounded***:

$$\exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \mid M \mid b \,\rangle.$$

Conversely, if $\langle\, \mid M \mid \,\rangle$ is bounded then $M^\dagger Y = X$.

Proof We have $K_a \subset M^*Y$ for each basic compact subspace and so $\exists \ell.\, K_a \subset M^*V_\ell$. Using Lemma 4.9, this amounts to the given formula for boundedness. Conversely, by Lemma 4.11,

$$\begin{aligned} M^\dagger Y &= \bigcup \{U_{a'} \mid \exists b.\, \langle\, a' \mid M \mid b \,\rangle\} \\ &\supset \bigcup \{U_a \mid \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \mid M \mid b \,\rangle\} \\ &\supset \bigcup \{U_a \mid \top\} \;=\; X. \end{aligned}$$

□

This complicated property reduces to the simpler ones if we have the relevant structure:

Lemma 4.18 Let the abstract matrix $\langle\, \mid M \mid \,\rangle$ be covariant, rounded, bounded and saturated. Then

(a) if the basis B has a top element $\bullet$ with respect to $\sqsubseteq$ then $\langle\, a \mid M \mid \bullet \,\rangle \Leftrightarrow \top$;

(b) if B is directed then $\exists b.\, \langle\, a \mid M \mid b \,\rangle$; and

(c) if A is prime ($a \prec\!\!\!\prec k \Rightarrow \exists b.\, a \prec\!\!\!\prec b \in k$, Example 13.2) then $\exists b.\, \langle\, a \mid M \mid b \,\rangle$. □

Example 4.19 Let $X \equiv \mathbf{2}$ with the singleton basis, $Y \equiv \mathbf{2}$ with the directed basis and consider the identity map between them. Then the matrix $\langle\, a \mid \mathsf{id} \mid \ell \,\rangle \equiv (a \in \ell)$ for $\mathsf{id} : X \to Y$ is uniformly bounded but its inverse $\langle\, \ell \mid \mathsf{id} \mid a \,\rangle \equiv (\ell \subset \{a\})$ is not. □

Turning to binary intersections, we have different results for bases that use compact subspaces or Scott-open families:

Lemma 4.20 Let $M^* : \Omega Y \to \Omega X$ be an operator that preserves all unions and binary intersections. If the basis for X uses compact subspaces then the matrix $\langle\ | M |\ \rangle$ is ***strongly filtered***:

$$\langle a \,|\, M \,|\, b_1 \rangle \;\wedge\; \langle a \,|\, M \,|\, b_2 \rangle \;\Longrightarrow$$

$$\exists k\ell.\,(a \prec\!\!\!\prec k) \;\wedge\; \big(\forall a' \in k.\,\exists b \in \ell.\,\langle a' \,|\, M \,|\, b \rangle\big) \;\wedge\; \big(\forall b \in \ell.\, b_1 \sqsupseteq b \sqsubseteq b_2\big).$$

If instead it uses Scott-open families then $\langle\ | M |\ \rangle$ is ***weakly filtered***:

$$(a \prec\!\!\!\prec a') \;\wedge\; \langle a' \,|\, M \,|\, b_1 \rangle \;\wedge\; \langle a' \,|\, M \,|\, b_2 \rangle \;\Longrightarrow$$

$$\exists k\ell.\,(a \prec\!\!\!\prec k) \;\wedge\; \big(\forall a' \in k.\,\exists b \in \ell.\,\langle a' \,|\, M \,|\, b \rangle\big) \;\wedge\; \big(\forall b \in \ell.\, b_1 \sqsupseteq b \sqsubseteq b_2\big).$$

Proof The hypotheses for the strong rule are $K_a \subset M^*V_{b_1}$ and $K_a \subset M^*V_{b_1}$. Then

$$K_a \subset M^*V_{b_1} \cap M^*V_{b_2} = M^*(V_{b_1} \cap V_{b_2}) \qquad \text{and so} \qquad K_a \subset M^*V_\ell$$

for some ℓ with $\ell \sqsubseteq b_1$ and $\ell \sqsubseteq b_2$. By Lemma 4.9, this is the stated conclusion.

In the weak case, we are given $\mathcal{K}_a \ni U_{a'}$, $K_{a'} \ni M^*V_{b_1}$ and $K_{a'} \ni M^*V_{b_1}$. Then we deduce $K_a \ni M^*V_{b_1} \cap M^*V_{b_2}$ as in Lemma 2.8 and the rest of the argument is the same as in the strong case. □

Lemma 4.21 If the matrix $\langle\ | M |\ \rangle$ is weakly or strongly filtered and has the partition property then $M^\dagger$ preserves binary intersections.

Proof By Lemma 4.4, the filter property of a concrete basis, contravariance, the basis expansion of U_a (for roundedness), the weak intersection rule and Lemma 4.11,

$$\begin{array}{ll}
& M^\dagger V_{b_1} \cap M^\dagger V_{b_2} \\
\subset & \bigcup\{U_a \mid \exists a_1 a_2.\,(a_1 \sqsupseteq a \sqsubseteq a_2) \;\wedge\; \langle a_1 \,|\, M \,|\, b_1 \rangle \;\wedge\; \langle a_2 \,|\, M \,|\, b_2 \rangle\} \\
\subset & \bigcup\{U_a \mid \langle a \,|\, M \,|\, b_1 \rangle \;\wedge\; \langle a \,|\, M \,|\, b_2 \rangle\} \\
\subset & \bigcup\{U_{a'} \mid \exists a.\,(a' \prec\!\!\!\prec a) \;\wedge\; \langle a \,|\, M \,|\, b_1 \rangle \;\wedge\; \langle a \,|\, M \,|\, b_2 \rangle\} \\
\subset & \bigcup\{U_{a'} \mid \exists k.\,(a' \prec\!\!\!\prec k) \;\wedge\; \forall a'' \in k.\,\exists b.\,\langle a'' \,|\, M \,|\, b \rangle \;\wedge\; (b_1 \sqsupseteq b \sqsubseteq b_2)\} \\
\subset & \bigcup\{U_{a''} \mid \exists b.\,\langle a'' \,|\, M \,|\, b \rangle \;\wedge\; (V_{b_1} \supset V_b \subset V_{b_2})\} \\
\subset & M^\dagger(V_{b_1} \cap V_{b_2}),
\end{array}$$

where the fourth line is not needed if $\langle\ | M |\ \rangle$ is strongly filtered. Then $M^\dagger V_1 \cap M^\dagger V_2 = M^\dagger(V_1 \cap V_2)$ since $M^\dagger$ also preserves arbitrary unions by Lemma 4.12. □

Finally we use sobriety (Definition 3.13) to complete the characterisation of matrices for continuous functions:

Theorem 4.22 Let X and Y be locally compact sober spaces with concrete bases (U_a, K_a) and (V_b, L_b) that have single interpolation and rounded unions. Then the formulae

$$\langle a \,|\, f \,|\, b \rangle \;\equiv\; K_a \subset f^{-1}V_b \;\equiv\; fK_a \subset V_b \quad \text{and} \quad fx \in V_b \iff \exists a.\, x \in U_a \;\wedge\; \langle a \,|\, f \,|\, b \rangle$$

define bijections amongst
(a) a continuous function $f : X \to Y$;
(b) an operator f^{-1} that takes open subspaces of Y to open subspaces of X preserving finite intersections and arbitrary unions; and
(c) a matrix $\langle\, a \,|\, f \,|\, b \,\rangle$ that is co- and contravariant, rounded, saturated, bounded and filtered and has the partition property.

Proof The correspondence between (a) and (b) is the definition of sobriety and that between (b) and (c) was the subject of this section. In detail, for each ordinary point $x \in X$,

$$\mathcal{P}_x \;\equiv\; \{V \in \Omega Y \mid x \in f^* V\} \;\equiv\; \{V \mid \exists a.\, x \in U_a \;\wedge\; \langle\, a \,|\, f \,|\, b \,\rangle\}$$

is a formal point of Y, because $\bigcup V_i \in \mathcal{P}_x \Longrightarrow \exists i.\, V_i \in \mathcal{P}_x$ by Lemma 4.12, $X \in \mathcal{P}_x$ by Lemma 4.17 and $V_1, V_2 \in \mathcal{P}_x \Longrightarrow V_1 \cap V_2 \in \mathcal{P}_x$ by Lemma 4.21. Then by sobriety $V \in \mathcal{P}_x \iff y \in V$ for some unique $y \in Y$ and we put $fx \equiv y$. This defines a continuous function because $f^{-1}V = f^*V \subset X$ and this is open by construction, for any open $V \subset Y$. $\square$

Remark 4.23 In order to check that you understand the axioms for bases and matrices and how to use them, you should verify that $\langle\, a \,|\, \mathsf{id} \,|\, b \,\rangle \equiv (a \prec\!\!\!\prec b)$ has all of the properties of a matrix and is a unit for saturated composition. By making modifications to this, show that the “improved” abstract bases in the previous section are isomorphic to the given ones.

5 Classical completeness

In order to prove that the category of locally compact sober spaces and continuous functions is equivalent to one of bases and matrices, it remains to construct a space from any given abstract basis.

As is customary, we begin with the points, which are the same as continuous functions from the singleton. The latter has just one basis element $\bullet$, with $\bullet \prec\!\!\!\prec \bullet$, so points correspond to matrices of the form $\langle\, \bullet \,|\, f \,|\, b \,\rangle$. This means that, in the axioms in the previous section, contravariance, saturation and roundedness in the argument are trivial, whilst by Lemma 4.18(c) boundedness and filteredness are uniform. The partition property is also simplified, to one that we call locatedness by analogy with Example 13.7 for the (Dedekind) real line. Writing $p \equiv \{b \mid \langle\, \bullet \,|\, f \,|\, b \,\rangle\}$, we have

Definition 5.1 A ***formal point*** for an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ is a (typically infinite) subset $p \subset A$ such that

$a \sqsupseteq b \in p$	$\Rightarrow$	$a \in p$	upper
$a \in p$	$\Leftrightarrow$	$\exists b.\, (b \prec\!\!\!\prec a) \;\wedge\; b \in p$	rounded
		$\exists a.\, a \in p$	bounded
$(a \in p) \;\wedge\; (b \in p)$	$\Rightarrow$	$\exists c.\, (a \sqsupseteq c \sqsubseteq b) \;\wedge\; c \in p$	filtered
$(a \in p) \;\wedge\; (a \prec\!\!\!\prec k)$	$\Rightarrow$	$k \between p \;\equiv\; \exists b.\, (b \in k) \;\wedge\; (b \in p)$.	located

We write X for the set of formal points. Beware that this notion of formal point is related to the abstract basis, whereas the one in Definition 3.13 is defined by the topology, which we now describe. We will show that the two notions are isomorphic in Lemma 5.9. The specialisation order is given by inclusion.

Definition 5.2 For each $a \in A$ and $k \in \mathsf{Fin}\,(A)$, the basic open subsets of X are

$$U_a \;\equiv\; \{p \mid a \in p\} \qquad \text{and} \qquad U_k \;\equiv\; \{p \mid k \between p \;\equiv\; \exists a.\, a \in k \;\wedge\; a \in p\}.$$

General open subsets are unions of these, so they are of the form U_u, as in the second expression, but with a possibly infinite set $u \subset A$ in place of k. It is convenient to write

$$u \between v \;\equiv\; \exists a.\, (a \in u) \;\wedge\; (a \in v)$$

for any two such subsets $u, v \subset A$.

Lemma 5.3 If $a \sqsubseteq b$ or $a \prec\!\!\!\prec b$ then $U_a \subset U_b$. The whole set X of formal points is open, *i.e.* it is expressible as a union of basic open subsets, as is the intersection of any two subsets that are so expressible.

Proof The first three parts follow from the requirements that formal points be upper, rounded and bounded respectively, whilst the filteredness property of formal points says that

$$U_a \cap U_b \;=\; \bigcup \{U_c \mid a \sqsupseteq c \sqsubseteq b\}$$

and the property for intersections of general unions follows from this. $\square$

Lemma 5.4 The family of open subspaces given by

$$\mathcal{K}_a \;\equiv\; \{U \mid \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; U_k \subset U\}$$

is Scott-open. If $a \prec\!\!\!\prec k$ then $\mathcal{K}_a \ni U_k$ and if $a \sqsubseteq b$ then $\mathcal{K}_a \supset \mathcal{K}_b$.

Proof By the Wilker property of $\prec\!\!\!\prec$ and the previous lemma,

$$\begin{aligned}
\mathcal{K}_a \ni U \quad &\equiv \quad \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; U_k \subset U \\
&\Rightarrow \quad \exists k\ell.\, (a \prec\!\!\!\prec \ell \prec\!\!\!\prec^1 k) \;\wedge\; U_k \subset U \\
&\equiv \quad \exists k\ell.\, (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\, \exists c \in k.\, (b \prec\!\!\!\prec c) \;\wedge\; U_k \subset U \\
&\Rightarrow \quad \exists k\ell.\, (a \prec\!\!\!\prec \ell) \;\wedge\; U_\ell \subset U_k \subset U \\
&\equiv \quad \exists k.\, \mathcal{K}_a \ni U_k \subset U.
\end{aligned}$$

Contravariance of $\mathcal{K}_{(-)}$ follows from that of $\prec\!\!\!\prec$. $\square$

Lemma 5.5 The system $(U_a, \mathcal{K}_a)$ satisfies the basis expansion

$$p \in U \iff \exists a.\, p \in U_a \;\wedge\; \mathcal{K}_a \ni U \qquad \text{or} \qquad U \;=\; \bigcup \{U_a \mid K_a \ni U\}$$

and is therefore a concrete basis for X using Scott-open families.

Proof [$\Rightarrow$] Since general open subsets are unions of basic ones, $p \in U_b \subset U$ for some b. Then $b \in p$ and by roundedness of p there is some $a \in p$ with $a \prec\!\!\!\prec b$. Hence $p \in U_a$ and $\mathcal{K}_a \ni U$ because $k \equiv \{b\}$ gives $a \prec\!\!\!\prec k$ and $U_k \subset U$.

[$\Leftarrow$] For some a and k, we have $a \in p$ and $a \prec\!\!\!\prec k$ with $U_k \subset U$, so by locatedness of p there is some $b \in k \cap p$ and $p \in U_b \subset U_k \subset U$. $\square$

This is all very well, but the problem was to find a space with a concrete basis that induces the *given* abstract basis, *i.e.* such that $a \prec\!\!\!\prec k \iff \mathcal{K}_a \ni U_k$. Proving such things in *point*–set topology involves *finding* points with specific properties. In particular, if $\mathcal{K}_a$ is of the form $\{U \mid K_a \subset U\}$ but $a \not\prec\!\!\!\prec k$ then we need to find a point that is in K_a but not in U_k.

For us, a "point" is a certain kind of subset of A (Definition 5.1) and we need one that includes some elements of A but excludes others. Lawson's Lemma 3.10 provides a $\prec\!\!\!\prec$-filter, so we need a way of obtaining rounded located subsets of the basis.

Lemma 5.6 For any subset $r \subset A$, we obtain a rounded located subset $\underline{r} \subset r$ by

$$\underline{r} \;\equiv\; \{a \in A \mid \exists a'.\, (a' \prec\!\!\!\prec a) \;\wedge\; a' \bullet r\}$$

where
$$a' \bullet r \;\equiv\; (\forall k.\, a' \prec\!\!\!\prec k \implies k \between r).$$

Indeed, $r \mapsto \underline{r}$ is coclosure operation for which $r = \underline{r}$ iff r is rounded and located.

Proof The operation is decreasing ($\underline{r} \subset r$), by putting $k \equiv \{a\}$, so $a \in k \cap r$.

It also preserves order: if $r \subset r'$ then $a' \bullet r \Rightarrow a' \bullet r'$ and so $\underline{r} \subset \underline{r}'$.

If r is already rounded and located then $\underline{r} = r$: given $a \in r$, by roundedness there is some $a' \in r$ with $a' \prec\!\!\!\prec a$ and if $a' \prec\!\!\!\prec k$ then $k \between r$ by locatedness.

For general r, the subset $\underline{r}$ is rounded: if $a \in \underline{r}$ then by the definition of $\underline{r}$ and single interpolation there are $a'' \prec\!\!\!\prec a' \prec\!\!\!\prec a$ with $a'' \bullet r$, so $a' \in \underline{r}$.

The difficult part is locatedness of $\underline{r}$. Let $a \in \underline{r}$ with $a \prec\!\!\!\prec \ell$, so there are a' and k with $a' \prec\!\!\!\prec k \prec\!\!\!\prec a$ and $k \prec\!\!\!\prec^1 \ell$ by Lemma 3.2. We need to find $b \in k$ with $b \bullet r$, from which we obtain c with $b \prec\!\!\!\prec c \in \ell$ since $k \prec\!\!\!\prec^1 \ell$ and then $c \in \ell \cap \underline{r}$.

Suppose that there is no such $b \in k$, so

$$\forall b \in k.\, \neg(b \bullet r) \quad\equiv\quad \forall b \in k.\, \exists h_b.\, (b \prec\!\!\!\prec h_b) \wedge (h_b \cap r = \emptyset).$$

Then
$$a \prec\!\!\!\prec k \prec\!\!\!\prec h \equiv \bigcup \{h_b \mid b \in k\} \qquad \text{with} \qquad h \cap r = \emptyset,$$

which contradicts $a \bullet r$. Hence there is some $b \in k$ with $b \bullet r$ as required

□

Now we want to find a point p such that $s \subset p \subset r \subset A$, where s is a $\prec\!\!\!\prec$-filter and r a rounded located subset. One way of making a formal point from a filter is to incorporate instances of locatedness into the proof of Lemma 3.10, which we can do if the basis is countable:

Lemma 5.7 Let $(A, \sqsubseteq, \prec\!\!\!\prec)$ be a countable abstract basis and $a \in r \subset A$, where r is rounded and located. Then there is a point p with $a \in p \subset r$.

Proof Let k_i be an enumeration of $\mathsf{Fin}\,(A)$ such each finite set k occurs infinitely often, so for any $k \in \mathsf{Fin}\,(A)$ and $i \in \mathbb{N}$ there is some $j > i$ with $k = k_j$.

As in Lemma 3.10, we put $a_0 \equiv a$ and define a descending sequence with $a_{i+1} \prec\!\!\!\prec a_i$, but we use locatedness to modify the choice of the terms.

As before, at each stage $i \in \mathbb{N}$, we first let $a' \prec\!\!\!\prec a_i$ with $a' \in r$ since r is rounded. If $a_i \not\prec\!\!\!\prec k_i$ then just let $a_{i+1} \equiv a'$.

If $a' \prec\!\!\!\prec a_i \prec\!\!\!\prec k_i$ then by Lemma 3.2 there is some k' with $a' \prec\!\!\!\prec k' \prec\!\!\!\prec^1 a_i, k_i$. Since $a' \in r$ and r is located, there is some $a'' \in r \cap k'$, so $a'' \prec\!\!\!\prec a_i$ and $a'' \prec\!\!\!\prec b \in k_i$, so $b \in r$ since r is upper. We put $a_{i+1} \equiv a''$.

Again as before, the subset $p \equiv \{b \mid \exists i.\, a_i \prec\!\!\!\prec b\}$ is a $\prec\!\!\!\prec$-filter with $a \in p \subset r$.

But p is also located. If $a_i \prec\!\!\!\prec a' \prec\!\!\!\prec k$ then, by assumption on the enumeration of $\mathsf{Fin}\,(A)$, $k \equiv k_j$ for some j with $i < j$. By construction, $a_j \prec\!\!\!\prec a_i \prec\!\!\!\prec a' \prec\!\!\!\prec k \equiv k_j$ and then $a_{j+1} \prec\!\!\!\prec b \in k_j$, so $b \in k \cap p$ as required.

Then p is a filter with respect to $\sqsubseteq$ as well as $\prec\!\!\!\prec$: If $a \in p \ni b$ then there is $d \in p$ with $a \succ\!\!\!\succ d \prec\!\!\!\prec b$ and a further $e \in p$ with $e \prec\!\!\!\prec d$. Then by the weak intersection rule there is some k with $e \prec\!\!\!\prec k \sqsubseteq a, b$. Since p is located, there is some $c \in k \cap p$, so $a \sqsupseteq c \sqsubseteq b$.

Hence p has all the properties of a formal point. □

The statement of this result is very similar to Lemma 3.12, so with some ingenuity you may be able to adapt that to the uncountable case. In fact, we will see how to do this in the next two sections, with the benefit of the point-free view of topology. But for the moment we accept the countability restriction and use the result that we have to recover $a \prec\!\!\!\prec k$:

Lemma 5.8 If the basis is countable and $\mathcal{K}_a \ni U_k$ then $a \prec\!\!\!\prec k$.

Proof We claim first that

$$(b \prec\!\!\!\prec c) \ \wedge\ (U_c \subset U_k) \ \equiv\ (b \prec\!\!\!\prec c) \ \wedge\ (\forall p.\, c \in p \Rightarrow p \between k) \implies (b \prec\!\!\!\prec k).$$

Otherwise, by Lemma 5.6, there is a rounded located subset $r \subset A$ with $c \in r \subset A \setminus k$. Then by Lemma 5.7 there is a point p with $c \in p \subset r$. This means that $p \in U_c \subset U_k$, so $p \between k$, contradicting $p \cap k = \emptyset$ from the construction.

We generalise this to covers by lists using the Wilker and transitivity properties for $\prec\!\!\!\prec$:

$$\begin{aligned}
\mathcal{K}_a \ni U_k &\Rightarrow \exists \ell\ell'.\, (a \prec\!\!\!\prec \ell' \prec\!\!\!\prec^1 \ell) \ \wedge\ \forall c \in \ell.\, (U_c \subset U_k) \\
&\Rightarrow \exists \ell'.\, (a \prec\!\!\!\prec \ell') \ \wedge\ \forall b \in \ell'.\, \exists c.\, (b \prec\!\!\!\prec c) \ \wedge\ (\forall p.\, c \in p \Rightarrow p \between k) \\
&\Rightarrow \exists \ell'.\, (a \prec\!\!\!\prec \ell') \ \wedge\ \forall b \in \ell'.\, (b \prec\!\!\!\prec k) \quad \implies \quad a \prec\!\!\!\prec k.
\end{aligned}$$ □

Now we can at last return to the topological ideas.

Lemma 5.9 If the basis is countable then the space X is sober.

Proof Let $\mathcal{P}$ be a formal point in the sense of Definition 3.13, *i.e.* a family of open subspaces of X such that

$$X \in \mathcal{P}, \qquad U, V \in \mathcal{P} \iff U \cap V \in \mathcal{P} \qquad \text{and} \qquad \bigcup U_i \in \mathcal{P} \iff \exists i.\, U_i \in \mathcal{P}.$$

We claim that $p \equiv \{a \mid U_a \in \mathcal{P}\}$ is a formal point in the sense of Definition 5.1 and satisfies $\mathcal{P} = \{U \mid p \in U\}$. Indeed, $p \in U_a \iff a \in p \iff U_a \in \mathcal{P}$ and this extends to $p \in U \equiv U_u \iff U_u \in \mathcal{P}$ by the third property of $\mathcal{P}$.

We leave it to the reader to show that $\mathcal{P}$ is a filter, *i.e.* bounded, filtered and upper.

It is located: if $a \in p$ and $a \prec\!\!\!\prec \ell$ then $U_a \in \mathcal{P}$ and $\mathcal{K}_a \ni U_\ell$, so $U_a \subset U_\ell \in \mathcal{P}$ from the basis expansion, but then $U_b \in \mathcal{P}$ by the third property of $\mathcal{P}$, for some $b \in \ell$, for which $b \in p$.

Finally, using Lemma 5.8, the basis expansion $U_a = \bigcup \{U_b \mid \mathcal{K}_b \ni U_a\}$ gives the roundedness property $a \in p \iff \exists b.\, p \in b \wedge b \prec\!\!\!\prec a$.

Alternatively, $q \equiv \{a \mid \exists b.\, U_b \in \mathcal{P} \wedge b \prec\!\!\!\prec a\}$ is easily seen to be rounded and upper, whilst the proof that p is filtered and located can be adapted to q, but then showing that $q \in U \iff U \in \mathcal{P}$ depends on Lemma 5.8. □

Theorem 5.10 Every countable abstract basis arises from some concrete basis using Scott-open families for some locally compact sober topological space. If the abstract basis satisfies the boundedness and strong intersection rules then it arises from some basis using compact subspaces, where

$$K_a \ \equiv\ \bigcap \mathcal{K}_a \ \equiv\ \{p \mid \forall k.\, (a \prec\!\!\!\prec k) \Longrightarrow p \between k\}.$$

Proof We have already completed the proof for Scott-open families, so it only remains to identify the points of the compact subspace in the strong case, using Proposition 3.16:

$$\begin{aligned}
p \in \bigcap \mathcal{K}_a &\equiv \forall U \in \mathcal{K}_a.\, p \in U \\
&\equiv \forall k.\, \forall U.\, (a \prec\!\!\!\prec k) \ \wedge\ U_k \subset U \implies p \in U \\
&\Leftrightarrow \forall k.\, (a \prec\!\!\!\prec k) \implies p \in U_k \\
&\equiv \forall k.\, (a \prec\!\!\!\prec k) \implies (p \between k).
\end{aligned}$$ □

Remark 5.11 If $a \prec\!\!\!\prec c$ but $a \not\prec\!\!\!\prec k$ then $K_a \subset U_c$ but $K_a \not\subset U_k$, so there is a point p with $p \in K_a \subset U_c$ but $p \notin U_k$, so $c \in p$ but $p \cap k = \emptyset$. However, this begs the question, because we used this property to prove sobriety and so to characterise compact subspaces.

Examining the place where we needed to use the partial result (Lemma 5.7), we notice first that the topology on X is not actually being used: the arguments just concern the relationship between the abstract basis and its formal points. In fact the difficulty was in translating the containment of subspaces $U_c \subset U_k$ in Lemma 5.8 and the basis expansion $U_a = \bigcup\{U_b \mid \mathcal{K}_b \ni U_a\}$ in Lemma 5.9 from their definition in terms of points in Definition 5.2 back into the properties of $\prec\!\!\!\prec$. Indeed it was the $U_k \subset U$ in Lemma 5.4 (which was needed to make $\mathcal{K}_a$ upper) that obliged us to do this.

Maybe we should define the open subspaces directly from the abstract basis without this diversion *via* formal points.

6 Locales

The applications of topology to other disciplines are often called ***spectra***, in which the "points" are structures such as prime ideals that have fairly complicated definitions (*cf.* Definitions 3.13 and 5.1) and can be difficult to find (*cf.* Lemma 5.7). On the other hand, the "open subspaces" typically correspond directly to much simpler features of the mathematical system under study. Peter Johnstone's book [Joh82] explores many examples of this phenomenon. Following him, we make the

Definition 6.1 A ***frame*** Ω is a lattice with arbitrary joins ($\bigvee$) over which meets ($\wedge$) distribute,

$$U \wedge \bigvee V_i = \bigvee (U \wedge V_i),$$

so the lattice ΩX of open subspaces of any topological space X is an example. Accordingly, a ***frame homomorphism*** $f^* : \Omega_2 \to \Omega_1$ is a function that preserves $\bigvee$, $\top$ and $\wedge$, just as the inverse image operator $f^{-1} : \Omega Y \to \Omega X$ does for any continuous function $f : X \to Y$. Frames and homomorphisms form a category, but when we want to use them to discuss topological ideas we use the names ***locale*** and ***continuous map*** instead for the objects and morphisms of the *opposite* category.

For compatibility with Point–Set Topology, we (sometimes) continue to use capital letters for elements of a frame. However, we write $U \leq V$ instead of $U \subset V$ for the order, because it is abstract and not necessarily represented by an inclusion (Warning 6.12). As we have already done, we also use $\wedge$ and $\bigvee$ instead of $\cap$ and $\bigcup$ for the operations.

Definition 6.2 In a locale, following Definitions 3.13 and 3.15,
(a) a ***formal point*** is a completely coprime filter $\mathcal{P} \subset \Omega$;
(b) a ***formal open subspace*** is an element $U \in \Omega$ of the frame,
(c) a ***formal compact subspace*** is a Scott-open filter $\mathcal{K} \subset \Omega$ in the frame.

In order to work with locales, we need a technique for constructing frames, so, since they are algebras, we present them by means of generators and equations. Such a set of generators is essentially just a basis in the same sense as for general (not necessarily locally compact) topological spaces (Definition 1.1), *i.e.* using just open subspaces and not compact ones:

Definition 6.3 A ***concrete basis*** for a frame or locale is a family of elements U_a of the frame Ω

such that
(a) if $a \sqsubseteq b$ then $U_a \leq U_b$;
(b) $U_a \wedge U_b \ = \ \bigvee \{U_c \mid a \sqsupseteq c \sqsubseteq b\}$; and
(c) $U = \bigvee \{U_a \mid U_a \leq U\}$ for any $U \in \Omega$.

The most efficient way of expressing the equations for a frame is this:

Definition 6.4 A ***formal cover*** $(A, \sqsubseteq, \lhd)$ consists of a preorder $\sqsubseteq$ on a set A together with a relation $a \lhd u$ between elements and (possibly infinite) subsets of A such that

$$a \in u \Longrightarrow a \lhd u, \qquad b \sqsubseteq c \lhd u \sqsubseteq v \Longrightarrow b \lhd v,$$

$$a \lhd u \lhd v \Longrightarrow a \lhd v \qquad \text{and} \qquad c \lhd u \wedge c \lhd v \iff c \lhd u \sqcap v,$$

where $\qquad u \lhd v \ \equiv \ \forall b \in u.\, b \lhd v, \qquad u \sqsubseteq v \ \equiv \ \forall b \in u.\, \exists c \in v.\, b \sqsubseteq c$

and $\qquad u \sqcap v \ \equiv \ \{b \mid (\exists c \in u.\, b \sqsubseteq c) \ \wedge \ (\exists d \in v.\, b \sqsubseteq d)\}.$

Therefore $\sqcap$ and not $\cap$ is the meet operation corresponding to the preorder $\sqsubseteq$ and beware that our use of $\sqcap$ is temporarily different from that in Notation 1.12, as far as Lemma 7.1.

Lemma 6.5 Given any A-indexed concrete basis for a topological space or frame, the relation defined by

$$a \lhd u \quad \equiv \quad U_a \leq \bigvee \{U_b \mid b \in u\}$$

is a formal cover. □

Lemma 6.6 Given any formal cover $(A, \sqsubseteq, \lhd)$, the map j on subsets of A that takes

$$u \subset A \qquad \text{to} \qquad ju \ \equiv \ \{a \mid a \lhd u\} \subset A$$

is a closure operation and also satisfies

$$ja \subset ju \iff a \lhd u \qquad \text{and} \qquad ju \cap jv \ = \ j(u \sqcap v).$$

In particular, if $u = ju$ then u is lower, but see Proposition 7.8 for the precise characterisation.

Proof If $u \subset v$ then $\forall a.\, a \lhd u \Longrightarrow a \lhd v$ so $ju \subset jv$.
If $a \in u$ then $a \lhd u$, so $u \subset ju$.
Therefore $a \lhd u \iff a \in ju \iff ja \subset ju$ and $u \lhd v \iff u \subset jv \iff ju \subset jv$.
If $a \in ju$ then $a \lhd u$ so $ju \lhd u$ and $j(ju) = ju$.
For the intersection, $ju \cap jv = j(u \sqcap v)$ because $a \lhd u \wedge a \lhd v \iff a \lhd u \sqcap v$. □

Theorem 6.7 Every formal cover presents a frame, *i.e.* it arises from some concrete basis on some frame.

Proof Let $\Omega \equiv \{u \subset A \mid u = ju\}$ with $(u \leq v) \equiv (u \subset v)$,

$$\top \ \equiv \ A \ \in \ \Omega \qquad u \wedge v \ \equiv j(u \sqcap v) \qquad \text{and} \qquad \bigvee u_i \ \equiv \ j(\bigcup u_i).$$

If $u, v, u_i \in A$ then $\top, u \wedge v, \bigvee u_i \in A$ too, by the Lemma. We recover the formal $\lhd$ relation because

$$ja \leq \bigvee u_i \ \equiv \ ja \subset j(\bigcup u_i) \iff a \lhd \bigcup u_i$$

by the Lemma. Note that we put no countability restriction on this result as we did in the previous section: it holds for *any* formal cover.

We have $\bigvee(u \wedge v_i) \leq u \wedge \bigvee v_i$ trivially. Conversely, writing $v \equiv \bigcup v_i$,

$$
\begin{array}{rcl}
u \sqcap v \;\equiv\; u \sqcap \bigcup v_i & = & \{d \mid \exists a \in u.\, \exists i.\, \exists b \in v_i.\, a \sqsupseteq d \sqsubseteq b\} \\
& = & \bigcup(u \sqcap v_i) \;\lhd\; \bigcup j(u \sqcap v_i) \;\equiv\; \bigcup(u \wedge v_i). \\
c \in u \wedge \bigvee v_i & \Rightarrow & c \lhd u \sqcap \bigvee v_i \implies c \lhd u \;\wedge\; c \lhd \bigvee v_i \\
& \Rightarrow & c \lhd u \;\wedge\; c \lhd v \implies c \lhd u \sqcap v \lhd \bigcup(u \sqcap v_i) \\
& \Rightarrow & c \in \bigvee(u \wedge v_i).
\end{array}
$$
□

Whilst we have introduced Ω here as a *subset* of the powerset $\mathcal{P}(A)$, it is actually a retract and we shall often find it more convenient to regard it as a *quotient*. That is, we use a general subset $u \subset A$ to denote an element $ju \in \Omega$ of the frame. Indeed, it is the surjection and not the inclusion of frames that is the homomorphism, defining a *sublocale*.

Proposition 6.8 Any frame with a concrete basis (Definition 6.3) is recovered up to isomorphism from the formal cover that it defines, where

$$u \mapsto \bigvee \{U_b \mid b \in u\} \qquad \text{and} \qquad U \mapsto \{a \mid U_a \subset U\},$$

the basic open subspaces being U_a and $ja \;=\; \{b \mid b \lhd a\} \;=\; \{b \mid U_b \subset U_a\}$. □

Therefore a formal cover corresponds bijectively to a locale that is equipped with a specified concrete basis. Using arguments similar to those in Section 4, we can go on to express frame homomorphisms or continuous functions between locales in terms of a basis and therefore a formal cover:

Proposition 6.9 There is a bijective correspondence between frame homomorphisms and matrices defined by

$$\big[\, a \,\big|\, f \,\big|\, b \,\big] \;=\; \big(a \in f^*(jb)\big) \qquad \text{and} \qquad f^*v \;=\; \{a \mid \exists b.\, \big[\, a \,\big|\, f \,\big|\, b \,\big] \;\wedge\; b \lhd v\},$$

where the matrices satisfy

$$
\begin{array}{lcll}
a \sqsubseteq a' \;\wedge\; \big[\, a' \,\big|\, f \,\big|\, b' \,\big] \;\wedge\; b' \sqsubseteq b & & & \\
& \Rightarrow & \big[\, a \,\big|\, f \,\big|\, b \,\big] & \text{co- \& contravariance} \\
& & \exists b.\, \big[\, a \,\big|\, f \,\big|\, b \,\big] & \text{boundedness} \\
\big[\, a \,\big|\, f \,\big|\, b_1 \,\big] \;\wedge\; \big[\, a \,\big|\, f \,\big|\, b_2 \,\big] & \Rightarrow & \exists b.\, \big[\, a \,\big|\, f \,\big|\, b \,\big] \;\wedge\; b_1 \sqsupseteq b \sqsubseteq b_2 & \text{filteredness} \\
\big[\, a \,\big|\, f \,\big|\, b \,\big] \;\wedge\; b \lhd v & \Rightarrow & \exists u.\, a \lhd u \;\wedge\; \forall a' \in u.\, \exists b' \in v.\, \big[\, a' \,\big|\, f \,\big|\, b' \,\big] & \text{partition} \\
a \lhd u \;\wedge\; \forall a' \in u.\, \big[\, a' \,\big|\, f \,\big|\, b \,\big] & \Rightarrow & \big[\, a \,\big|\, f \,\big|\, b \,\big]. & \text{saturation}
\end{array}
$$

I don't think the bounded and filtered rules are correct, for the same reason as in Example 4.19.

We can deduce the characterisation of formal points from this as we did in Definition 5.1, but since we intend to use it we prove it in detail.

Proposition 6.10 A ***formal point*** of a formal cover is a subset $p \subset A$ such that

$$
\begin{array}{lcll}
& & \exists a.\, a \in p & \text{bounded} \\
a \sqsupseteq b \in p & \Rightarrow & a \in p & \text{upper} \\
(a \in p) \;\wedge\; (b \in p) & \Rightarrow & \exists c.\, (a \sqsupseteq c \sqsubseteq b) \;\wedge\; c \in p & \text{filtered} \\
(a \in p) \;\wedge\; (a \lhd u) & \Rightarrow & u \between p \;\equiv\; \exists b.\, (b \in u) \;\wedge\; (b \in p). & \text{positive}
\end{array}
$$

The correspondence with Definition 3.13 is

$$p \;\equiv\; \{a \mid ja \in \mathcal{P}\} \subset A \qquad \text{and} \qquad \mathcal{P} \;\equiv\; \{u \mid p \between u = ju\} \subset \Omega.$$

Proof Given a completely coprime filter $\mathcal{P} \subset \Omega$, the set p is upper because $\mathcal{P}$ is and j preserves inclusions. Also p is bounded because $\mathcal{P} \ni A = \bigvee\{ja \mid a \in A\}$ so $\exists a.\, ja \in \mathcal{P}$ since it is completely coprime. For the filter property of p,

$$\begin{aligned} a \in p \ni b \quad &\equiv \quad ja \in \mathcal{P} \ni jb \\ &\Rightarrow \quad \mathcal{P} \ni ja \cap jb = j(a \sqcap p) = \bigvee\{jc \mid c \in a \sqcap b\} \\ &\Rightarrow \quad \exists c.\, \mathcal{P} \ni jc \;\wedge\; (a \sqsupseteq c \sqsubseteq b) \implies \exists c.\, p \ni c \in a \sqcap b. \end{aligned}$$

For positivity,

$$\begin{aligned} p \ni a \lhd u \quad &\Rightarrow \quad \mathcal{P} \ni ja \subset ja \;=\; \bigvee\{jb \mid b \in u\} \\ &\Rightarrow \quad \exists b.\, \mathcal{P} \ni jb \;\wedge\; b \in u \implies \exists b.\, p \ni b \in u. \end{aligned}$$

Conversely, given p, the family $\mathcal{P}$ is upper since $p \between u \subset v \Rightarrow p \between v$ and bounded since p is and so $\exists a.\, a \in p \wedge ja \in \mathcal{P}$. For the filter property of $\mathcal{P}$,

$$\begin{aligned} u \in \mathcal{P} \ni v \quad &\Rightarrow \quad u \between p \between v \implies \exists ab.\, u \ni a \in p \ni b \in v \\ &\Rightarrow \quad \exists c.\, u \sqcap v \ni c \in p \implies u \cap v \in \mathcal{P}. \end{aligned}$$

We recover $\mathcal{P}$ from p because $\mathcal{P}$ is completely coprime and

$$\begin{aligned} \{u \mid p \between u = ju\} \quad &= \quad \{u \mid \exists a.\, \mathcal{P} \ni ja \wedge a \in u = ju\} \\ &= \quad \{u \mid u = \bigvee\{ja \mid \mathcal{P} \ni ja \wedge a \in u\}\} \;=\; \mathcal{P}. \end{aligned}$$

We recover p from $\mathcal{P}$ because it is positive and

$$\{a \mid ja \in \mathcal{P}\} \;=\; \{a \mid p \between ja\} \;=\; \{a \mid \exists b.\, p \ni b \lhd a\} \;=\; p. \qquad \square$$

When we make the connection between $\lhd$ and $\prec$ in Lemma 7.11 we will see that the notions of formal point for these two relations also agree, with positivity playing the same role as locatedness and roundedness together.

Corresponding with Definition 3.13, $ju \in \mathcal{P} \iff \exists a.\, p \ni a \lhd u$. We say that such a point ***lies in*** a formal open set $u \subset A$ if $p \between u$, and then we write $U_u \equiv \{p \mid p \between u\}$ for the ***extent*** of u. This is the same as Definition 5.2 and we have

Proposition 6.11 Extent (the map $u \mapsto U_u$) is a frame homomorphism.

Proof From the first three axioms, $\top \between p$ and $u \between p \between v \implies p \between (u \sqcap v)$, so extent preserves finite meets. By the last, $p \between ju \iff p \between u$, so $p \between \bigvee u_i \iff p \between \bigcup u_i \iff \exists i.\, p \between u_i$ and extent preserves joins. $\square$

Warning 6.12 Although the formal opens $u \in \Omega$ in Theorem 6.7 are sets, they are sets of basis elements and not sets of (formal) *points* as they were in Section 5. Indeed, *the formal opens of a locale need not in general be faithfully representable as sets of points at all*, since the extent need not be an isomorphism [Joh82]. A frame, locale or formal cover for which this is an isomorphism is called ***spatial*** or is said to ***have enough points***. Since we just need $U_a \subset U_u \implies a \lhd u$, the characterisation in terms of $\lhd$ is this:

Proposition 6.13 A formal cover $\lhd$ has enough points iff

$$\big(\forall p.\, a \in p \Rightarrow p \between u\big) \implies a \lhd u. \qquad \square$$

Now we return to locales and local compactness. Definition 2.6 of a basis for a topological space using Scott-open families can easily be transferred because it only mentions opens and not points. We find that, when a locale has *some* basis of this kind then is has a *canonical* one, in which $\mathcal{K}_a$ is determined order-theoretically by U_a:

Lemma 6.14 If a frame Ω has a basis $(U_a, \mathcal{K}_a)$ using Scott-open families then

$$\mathcal{K}_a \ni V \implies U_a \ll V \qquad \text{and so} \qquad a \prec\!\!\!\prec \ell \equiv \mathcal{K}_a \ni U_\ell \implies U_a \ll U_\ell,$$

where we say that U is ***way below*** V in Ω, written

$$U \ll W, \qquad \text{if} \qquad \forall (W_i).\ V \leq \bigvee_{i \in I} W_i \implies \exists \ell.\ U \leq \bigvee_{i \in \ell} W_i.$$

In such a frame, the subset $\Uparrow U \equiv \{V \mid U \ll V\} \subset \Omega$ is itself Scott-open.

Proof If $\mathcal{K}_a \ni V \leq \bigvee W_i$ then, since $\mathcal{K}_a$ is Scott-open, a finite join $j \subset I$ will do, so $\mathcal{K}_a \ni W \equiv \bigvee \{W_i \mid i \in \ell\}$. The latter means that U_a contributes to the expansion of W, so $U_a \leq W$, as required for the definition of $U_a \ll V$. $\square$

Corollary 6.15 A locale is ***locally compact*** in the sense that its frame has *some* basis using Scott-open families, with

$$V = \bigvee \{U_a \mid \mathcal{K}_a \ni V\}, \qquad \text{iff it is } \textbf{\textit{continuous}}, \qquad V = \bigvee \{U \mid U \ll V\}. \qquad \square$$

However, we retain the distinction that a *locally compact* locale comes equipped with the *additional structure* of an arbitrary but specified concrete basis $(U_a, \mathcal{K}_a)$, *cf.* Definition 1.3, whilst a *continuous* one just has the relation $\ll$ that is defined from the order on the frame.

The notion of a continuous lattice arose during the 1970s in theoretical computer science, topological lattice theory and spectral theory, leading to the six-author *Compendium* [GHK+80]: see in particular the historical notes at the end of its Section I 1.

We can proceed in the same way as in Sections 4 and 5:

Proposition 6.16 For any concrete basis on a locale using Scott-open families, in particular the one with $\mathcal{K}_a \equiv \Uparrow U_a$, the relation $(a \prec\!\!\!\prec \ell) \equiv (\mathcal{K}_a \ni U_\ell)$ makes $(A, \sqsubseteq, \prec\!\!\!\prec)$ an abstract basis. $\square$

Also summarise the translations of the results in Sections 2–4 into locale theory.

Next, Inger Sigstam [Sig95] translated the way-below relation $\ll$ from continuous lattices to their generating formal covers. Then Sara Negri [Neg02] considered a possibly sparser relation like our $\prec\!\!\!\prec$, writing $i(a)$ for our $\{b \mid b \prec\!\!\!\prec a\}$ in her Definition 4.10 for a "locally Stone" formal cover, *i.e.* one that we simply call locally compact.

Lemma 6.17 For any formal cover $(A, \sqsubseteq, \lhd)$, the frame that it presents is continuous iff

$$a \lhd \{b \mid b < a\} \qquad \text{where} \qquad (u < v) \equiv (\forall w.\, v \lhd w \Rightarrow \exists \ell.\, u \lhd \ell \subset w).$$

We then say that A is a ***continuous formal cover***.

Here we are temporarily using $<$ for a binary relation on A that we are comparing with $\ll$ on the frame Ω that we constructed in Theorem 6.7. Afterwards, we shall write $\ll$ for both relations.

Proof By Lemma 6.6, the $u < v$ relation on subsets of A is equivalent to

$$\forall w.\, jv \subset jw \implies \exists \ell.\, ju \subset j\ell \wedge \ell \subset w,$$

which is $ju \ll jv$ in Ω. However, we are claiming that it is enough to use *single* elements of the basis to test continuity of the frame. The set-wise continuity condition (Corollary 6.15), for $v \equiv \{a\}$, implies

$$a \vartriangleleft \bigcup \{u \mid u < a\} \;=\; \{b \mid \exists u.\, b \in u < a\} \;=\; \{b \mid b < a\},$$

as follows from the fact that $b \in u < a \Rightarrow b < a$. Conversely, the singleton condition gives

$$\forall a \in u.\; a \vartriangleleft \{b \mid b < a\} \subset \bigcup \{v \mid v < u\}, \qquad \text{so} \qquad u \vartriangleleft \bigcup \{v \mid v < u\},$$

since $b < a \in u \Rightarrow v \equiv \{b\} < u$. □

Traditional topology relied to its detriment on points, where the natural mathematical structure often lies in the open subspaces instead. On the other hand, it may be said of locale theory that it makes excessively heavy use of infinitary lattice theory. Analysts, for example, would tend to refer to open balls and other neighbourhoods, but not to the lattice of *all* of them.

The use of complete lattices makes it very tempting to focus on the *biggest* or *smallest* example of something. In our case, we saw earlier that a binary relation $\prec\!\!\!\prec$ arises *naturally* from a concrete basis using compact subspaces. In the localic presentation, Lemma 6.14 showed that there is a *densest* example of $\prec\!\!\!\prec$, namely the relation $\ll$ that is defined infinitarily from the open-set lattice. However, for example in the last two results, it is not actually necessary to use the densest relation: any $\prec\!\!\!\prec$ satisfying our axioms will do.

We have not yet completed the task of constructing a locally compact locale or topological space with a given abstract basis. We do this in the next section by translating a *general* $\prec\!\!\!\prec$ relation into a formal cover.

7 Formal topology

We now focus more directly on formal covers *per se*, instead of just using them as a tool to construct locales. We take Lemma 6.17, which was inspired by continuous frames, as our starting point, but we shall see that the modification that Sara Negri made to this definition actually agrees with our Scott-open families and so our usual definition of local compactness. The outcome of this is that we succeed in proving completeness of the axioms for abstract bases for sober topological spaces and locales as well as for formal topology.

Lemma 7.1 For any formal cover $(A, \sqsubseteq, \vartriangleleft)$, the relation $\ll$ defined by

$$a \ll u \quad \equiv \quad (\forall v.\, u \vartriangleleft v \Longrightarrow \exists k.\, a \vartriangleleft k \subset v)$$

satisfies

$$\begin{array}{lcl} a \ll u & \Longrightarrow & a \vartriangleleft u \\ a \sqsubseteq b \ll u \sqsubseteq v & \Longrightarrow & a \ll v \\ a \vartriangleleft u \ll v \vartriangleleft w & \Longrightarrow & a \ll w \\ a \ll u \ll v & \Longrightarrow & a \ll v \\ a \ll k \ll u \;\wedge\; k \ll v & \Longrightarrow & a \ll u \sqcap v, \end{array}$$

in which the subset $u \sqcap v \subset A$ is given by Definition 6.4.

Proof If $a \ll u$ then by putting $v \equiv u$ (so $u \vartriangleleft v$) in its definition we deduce $a \vartriangleleft u$.

If $a \sqsubseteq b \ll u \sqsubseteq v \vartriangleleft w$, so $\forall c \in u.\, \exists d \in v.\, c \sqsubseteq d \vartriangleleft w$, then $a \sqsubseteq b \ll u \vartriangleleft w$ and $a \vartriangleleft w$ using the variance properties of $\vartriangleleft$ twice. Hence $a \sqsubseteq b \ll u \sqsubseteq v \Longrightarrow a \ll v$.

Using transitivity of $\lhd$ twice, if $a \lhd u \ll v \lhd w \lhd w'$ then $\exists k.\, a \lhd k \subset w'$. Therefore $a \lhd u \ll v \lhd w \Rightarrow a \ll w$.

If $a \ll u \ll v$ then $u \lhd v$ and $a \ll v$.

If $a \ll k \ll u \wedge a \ll v$ then $k \lhd u \wedge k \lhd v$, so $a \ll k \lhd u \sqcap v$ by the intersection property of $\lhd$ and then $a \ll u \sqcap v$. $\square$

Lemma 7.2 If the formal cover is continuous in the sense of Lemma 6.17 then the relation $\ll$ is finitely represented (or Scott-continuous),

$$a \ll u \iff \exists \ell.\, a \ll \ell \subset u,$$

and satisfies the Wilker rule,

$$a \ll \ell \iff \exists k.\, a \ll k \ll^1 \ell \;\equiv\; \exists k.\, a \ll k \;\wedge\; \forall b \in k.\, \exists c.\, b \ll c \in \ell.$$

Also, the property $a \ll u \sqcap v$ is then equivalent to Notation 1.12.

Proof Given $a \ll u$, we use continuity twice (for the elements of u and then v) to obtain

$$a \;\ll\; u \;\lhd\; v \equiv \{c \mid \exists b.\, c \ll b \in u\} \;\lhd\; w \equiv \{d \mid \exists bc.\, d \ll c \ll b \in u\}.$$

Then transitivity of $\lhd$ and the definition of $\ll$ give some h such that

$$a \lhd h \subset w, \qquad \text{so} \qquad \forall d \in h.\, \exists bc.\, d \ll c \ll b \in u.$$

Now let k and ℓ be choices of cs and bs for $d \in h$ in this, so

$$a \lhd h \ll^1 k \ll^1 \ell \subset u,$$

whence $a \ll k \ll^1 u$ and $a \ll \ell \subset u$ as required. The converses are given by transitivity and covariance. Applying this to the definition of $u \sqcap v$, we recover Notation 1.12:

$$a \ll u \sqcap v \iff \exists \ell.\, a \ll \ell \;\wedge\; \forall b \in \ell.\, \exists c \in u.\, \exists d \in v.\, c \sqsupseteq b \sqsubseteq d. \qquad \square$$

Proposition 7.3 If $(A, \sqsubseteq, \lhd)$ is a continuous formal cover then $(A, \sqsubseteq, \ll)$ is an abstract basis in our sense and

$$b \lhd v \iff (\forall a.\, a \ll b \Rightarrow \exists \ell.\, a \ll \ell \subset v).$$

Proof Only the last part remains. [$\Rightarrow$] If $a \ll b \lhd v$ then $\exists \ell.\, a \ll \ell \subset v$ by the previous results. [$\Leftarrow$] Let $u \equiv \{a \mid a \ll b\}$, so $u \lhd v$ because $a \ll \ell \subset v \Longrightarrow a \lhd v$, and then $b \lhd u \lhd v$. $\square$

Notation 7.4 Conversely, we now show that any abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ defines a continuous formal cover by

$$(b \lhd u) \;\equiv\; (\forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u).$$

Lemma 7.5 This cover relation satisfies

$$b \in u \Longrightarrow b \lhd u, \qquad b \sqsubseteq c \lhd u \sqsubseteq v \Longrightarrow b \lhd v, \qquad c \lhd u \lhd v \Longrightarrow c \lhd v,$$

$$b \lhd \{a \mid a \prec\!\!\!\prec b\} \qquad \text{and} \qquad a \prec\!\!\!\prec \ell \;\Longrightarrow\; a \lhd \ell.$$

Proof

$$\begin{array}{lcll}
b \lhd u & \equiv & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u & \\
 & \Leftarrow & \forall a.\, (a \prec\!\!\!\prec b \Rightarrow a \prec\!\!\!\prec b \in u) \;\Longleftarrow\; b \in u & \\
b \sqsubseteq c \lhd u & \equiv & b \sqsubseteq c \;\wedge\; \forall a.\, a \prec\!\!\!\prec c \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u & \\
 & \Rightarrow & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u \;\equiv\; b \lhd u & \text{contravariance} \\
c \lhd u \sqsubseteq v & \equiv & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u \sqsubseteq v & \\
 & \Rightarrow & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \sqsubseteq v \;\equiv\; c \lhd v & \\
b \lhd \ell & \equiv & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists k.\, a \prec\!\!\!\prec k \subset \ell & \\
 & \Leftarrow & \forall a.\, a \prec\!\!\!\prec b \Rightarrow a \prec\!\!\!\prec \ell \;\Leftarrow\; b \prec\!\!\!\prec \ell & \text{transitivity of } \prec\!\!\!\prec \\
c \lhd \{b \mid b \prec\!\!\!\prec c\} & \equiv & \forall a.\, a \prec\!\!\!\prec c \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset \{b \mid b \prec\!\!\!\prec c\} & \\
 & \Leftarrow & \forall a.\, a \prec\!\!\!\prec c \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \prec\!\!\!\prec c. & \text{interpolation} \quad \square
\end{array}$$

The proof of the other two properties of $\lhd$ of course depends on the Wilker and weak intersection rules for $\prec\!\!\!\prec$.

Lemma 7.6 If $c \lhd u \lhd v$ then $c \lhd v$.

Proof Suppose that $a \prec\!\!\!\prec c \lhd u \lhd v$. Since $c \lhd u$ means

$$\forall a.\, (a \prec\!\!\!\prec c) \Rightarrow \exists \ell.\, (a \prec\!\!\!\prec \ell \subset u),$$

there is some finite set ℓ with $a \prec\!\!\!\prec \ell \subset u$. Then by the Wilker rule there is another finite set k with

$$a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell \subset u \;\equiv\; (a \prec\!\!\!\prec k) \;\wedge\; \forall b \in k.\, \exists c \in \ell.\, (b \prec\!\!\!\prec c \in u).$$

We combine this with $u \lhd v \;\equiv\; \forall bc.\, (b \prec\!\!\!\prec c \in u \Rightarrow \exists h.\, b \prec\!\!\!\prec h \subset v)$ to give

$$a \prec\!\!\!\prec k \;\wedge\; \forall b \in k.\, \exists h_b.\, b \prec\!\!\!\prec h_b \subset v.$$

Taking $h \equiv \bigcup\{h_b \mid b \in k\} \subset v$, we obtain $a \prec\!\!\!\prec k \prec\!\!\!\prec h \subset v$, from which $a \prec\!\!\!\prec h \subset v$ follows by transitivity of $\prec\!\!\!\prec$. Hence $c \lhd v$. $\square$

Lemma 7.7 If $c \lhd u$ and $c \lhd v$ then $c \lhd u \sqcap v$.

Proof Given $a \prec\!\!\!\prec c$, we first interpolate $a \prec\!\!\!\prec \ell \prec\!\!\!\prec c$, so $a \prec\!\!\!\prec \ell \;\wedge\; \forall b \in \ell.\, b \prec\!\!\!\prec c$.

Combining this with $c \lhd u$ and $c \lhd v$ gives

$$a \prec\!\!\!\prec \ell \;\wedge\; (\forall b \in \ell.\, \exists h_b.\, b \prec\!\!\!\prec h_b \subset u) \;\wedge\; (\forall b \in \ell.\, \exists k_b.\, b \prec\!\!\!\prec k_b \subset v).$$

Taking $h \equiv \bigcup\{h_b \mid b \in \ell\} \subset u$ and $k \equiv \bigcup\{k_b \mid b \in \ell\} \subset v$, we obtain

$$a \prec\!\!\!\prec \ell \prec\!\!\!\prec h \subset u \;\wedge\; \ell \prec\!\!\!\prec k \subset v.$$

Then the weak intersection rule gives $a \prec\!\!\!\prec h \sqcap k$, which means

$$\exists \ell'.\, a \prec\!\!\!\prec \ell' \;\wedge\; \forall b \in \ell'.\, (\exists c.\, b \sqsubseteq c \in h \subset u) \;\wedge\; (\exists d.\, b \sqsubseteq d \in k \subset v),$$

but this is $a \prec\!\!\!\prec \ell' \subset u \sqcap v$. Hence $c \lhd u \sqcap v$. $\square$

Having derived a formal cover $\lhd$ from an abstract way-below relation $\prec\!\!\!\prec$, we can use it in the construction in Theorem 6.7:

Proposition 7.8 Any abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ defines a frame Ω whose elements are the subsets $u \subset A$ such that

$$b \prec\!\!\!\prec \ell \subset u \implies b \in u \qquad \text{and} \qquad \downarrow\!\! b \;\equiv\; \{a \mid a \prec\!\!\!\prec b\} \subset u \implies b \in u.$$

Proof If the left hand side of either of the implications holds then $b \lhd u$ and so $b \in u$ since $u = ju$.

Conversely, suppose that u is closed under these two conditions and $b \lhd u$. Then for any $a \prec\!\!\!\prec b$ we have some ℓ with $a \prec\!\!\!\prec \ell \subset u$ by definition of $b \lhd u$, so $a \in u$ by the first condition. Hence $\Downarrow b \subset u$, so $b \in u$ by the second condition. □

The way-below relation $\ll$ that is derived from the formal cover $\lhd$ which is itself defined from $\prec\!\!\!\prec$ satisfies $a \prec\!\!\!\prec \ell \Rightarrow a \ll \ell$ but not necessarily the converse. This is because the given $\prec\!\!\!\prec$ relation may encode *smaller* Scott-open families $\mathcal{K}_a$ than the canonical $\Uparrow U_a$, as we saw in Corollary 6.15:

Lemma 7.9 This frame Ω has a basis $(U_a, \mathcal{K}_a)$ using Scott-open families that is given by

$$U_a \equiv ja \quad \text{and} \quad \mathcal{K}_a \equiv \{u \in \Omega \mid \exists \ell.\, a \prec\!\!\!\prec \ell \subset u\} \quad \text{so} \quad \mathcal{K}_a \ni U_k \iff a \prec\!\!\!\prec k.$$

Proof By the form of the definition, $\mathcal{K}_a$ is Scott-open. and $a \prec\!\!\!\prec k \Longrightarrow \mathcal{K}_a \ni U_k$. For $u \in \Omega$, let

$$v \equiv \{a \mid \mathcal{K}_a \ni u\} \equiv \{a \mid \exists \ell.\, a \prec\!\!\!\prec \ell \subset u\},$$

so $v \subset u$ by the first part of the previous result. Also, if $b \in u$ then $b \lhd \{a \mid a \prec\!\!\!\prec b\} \subset v$, so $u \lhd v$. Hence

$$u = ju = jv = \bigvee \{U_a \mid \mathcal{K}_a \ni u\},$$

which is the basis expansion. For the way-below relation,

$$\begin{array}{rcll} \mathcal{K}_a \ni U_k & \equiv & \exists \ell.\, a \prec\!\!\!\prec \ell \subset jk & \\ & \Rightarrow & \exists b\ell.\, a \prec\!\!\!\prec b \prec\!\!\!\prec \ell \subset jk & \text{single interpolation} \\ & \Rightarrow & \exists b\ell.\, a \prec\!\!\!\prec b \lhd \ell \lhd k & \text{Lemmas 7.5, 6.6} \\ & \Rightarrow & a \prec\!\!\!\prec k. & \text{Lemma 7.6, Notation 7.4} \end{array}$$

□

We have now proved the analogue of Lemma 5.8 for locales in complete generality (not just countably based ones) without using either points or the Axiom of Choice.

Theorem 7.10 Every abstract basis arises from some concrete basis using Scott-open families on a locally compact formal cover, or locale. □

On the other hand, we may take another look at formal points and prove the classical version of the theorem, but now without the countability restriction.

Lemma 7.11 Definitions 5.1 and 6.10 for formal points in terms of $\prec\!\!\!\prec$, $\ll$ and $\lhd$ agree.

Proof They share the properties of being upper, bounded and filtered, so we just have to show that p is rounded and located (with respect to $\prec\!\!\!\prec$) iff it is positive (with respect to $\lhd$). Substituting the definition of $\lhd$ from $\prec\!\!\!\prec$, the subset $p \subset A$ is positive iff, for all $b \in A \supset u$,

$$b \in p \wedge b \lhd u \quad \equiv \quad b \in p \wedge (\forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u) \quad \Longrightarrow \quad p \between u.$$

If p is positive and $b \in p$, put $u \equiv \{a \mid a \prec\!\!\!\prec b\}$. Then $b \lhd u$ (the bracketed clause holds) because if $a \prec\!\!\!\prec b$ then $a \in u$ and we may interpolate $a \prec\!\!\!\prec c \prec\!\!\!\prec b$, so $a \prec\!\!\!\prec \ell \equiv \{c\} \subset u$. Then by positivity there is some $d \in p \cap u$, so $p \ni d \prec\!\!\!\prec b$. Hence p is rounded.

If further $p \ni b \prec\!\!\!\prec \ell$ then $b \lhd u \equiv \ell$ because $a \prec\!\!\!\prec b \Longrightarrow a \prec\!\!\!\prec \ell$. So by positivity $p \between u \equiv \ell$ and p is located.

Conversely, suppose that p is rounded and located and $b \in p$, so we have $p \ni a \prec\!\!\!\prec b$ by roundedness. Then if $b \lhd u$ we have $a \prec\!\!\!\prec \ell \subset u$ by the bracketed clause and so $p \between \ell \subset u$ by locatedness. Hence p is positive.

Given that we have shown equivalence of any $\prec\!\!\!\prec$ with $\lhd$, the choice of $\prec\!\!\!\prec$ or $\ll$ doesn't matter. $\square$

Although locales and formal covers in general need not have enough points (Warning 6.12), locally compact ones do. The underlying idea here is actually the one that we were unable to use in Remark 5.11, which we expressed there using compact subspaces. Here we exploit the way-below relation $\ll$ instead, but we use that on the formal cover, where the same proof in [Joh82, Theorem VII 4.3] used the frame.

Theorem 7.12 Any continuous formal cover or locale has enough points.

Proof Recall from Proposition 6.13 that we need to show that

$$\big(\forall p.\, c \in p \Rightarrow p \between u\big) \implies c \lhd u,$$

so suppose that $c \not\lhd u$. Then $c \in r \equiv A \setminus ju$ and we require $c \in p \subset r$.

By Proposition 7.8, r is rounded, so by Lemma 3.10 there is a $\prec\!\!\!\prec$-filter $s \equiv \{a \mid \exists i.\, c_i \prec\!\!\!\prec a\}$ with $s \subset r$, where $\cdots \prec\!\!\!\prec c_2 \prec\!\!\!\prec c_1 \prec\!\!\!\prec c_0 \equiv c$ and $c_i \in r$. Then by Lemma 3.11, $\mathcal{K} \equiv \{v \mid \exists a \in s.\, \mathcal{K}_a \ni v\} = \{v \mid \exists i\ell.\, c_i \prec\!\!\!\prec \ell \subset v\} \subset \Omega$ is a Scott-open filter. If $ju \in \mathcal{K}$ then $\exists i\ell.\, c_i \prec\!\!\!\prec \ell \subset ju$, so $c_i \in ju$ by Proposition 7.8, but by construction this is not the case, so $ju \notin \mathcal{K}$.

Now, by Lemma 3.12, which relies on the Axiom of Choice and applies to locales as well as traditional topology, there is a completely coprime filter $\mathcal{P}$ with $ju \notin \mathcal{P} \supset \mathcal{K}$.

By Proposition 6.10, $p \equiv \{d \mid jd \in \mathcal{P}\}$ is a formal point in the sense of $\lhd$, which is the same as that of $\prec\!\!\!\prec$ by Lemma 7.11, and $\mathcal{P} = \{v \mid p \between v\}$.

If $d \in p$ then $jd \in \mathcal{P}$ whilst $ju \notin \mathcal{P}$ and $\mathcal{P}$ is upper, so $jd \not\subset ju$ and $d \notin ju$ by Lemma 6.6.

Also, $c_1 \prec\!\!\!\prec c_0 \equiv c \in jc \in \mathcal{K} \subset \mathcal{P}$, so $c \in p \subset r \equiv A \setminus ju$ as required. $\square$

Corollary 7.13 Every abstract basis arises from some concrete basis on a locally compact sober topological space. In particular,

$$(b \prec\!\!\!\prec c) \;\wedge\; (\forall p.\, c \in p \Rightarrow p \between k) \implies (b \prec\!\!\!\prec k).$$

Proof Combine the Theorem with Notation 7.4 and the results of Section 5. $\square$

This completes the proof of the equivalence of categories between locally compact sober spaces or locales and continuous functions on the one hand and bases and matrices on the other. We defer the summary to the Conclusion and return to discussing the definition of local compactness in Formal Topology.

Remark 7.14 Sara Negri introduced $i(a) \equiv \{b \mid b \prec\!\!\!\prec a\}$ in [Neg02, Definition 4.10], where $\prec\!\!\!\prec$ may be sparser than the $\ll$ relation that comes from continuous lattices. This was to allow $i(a)$ to be a legitimate "set" in the sense of Martin-Löf Type Theory, in terms of which Formal Topology is usually presented. In our notation, she required

$$b \lhd \{a \mid a \prec\!\!\!\prec b\}, \quad a \prec\!\!\!\prec k \implies a \lhd k \quad \text{and} \quad a \lhd u \implies (\forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u).$$

The $\ll$ relation in Lemma 6.17 is not permitted in MLTT because the quantification $\forall w$ is ***impredicative***. However, Giovanni Curi [Cur07, Section 7.3] gave a predicative formula that is equivalent to this (maximal) $\ll$, based on an observation of Peter Aczel [Acz06, Section 4.3].

There is an extensive discussion of the proof-theoretic issues regarding formal covers in general in [CSSV03]. Abstract bases provide a very simple example of this:

Proposition 7.15 For any abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$, the families

$$I(a) \;\equiv\; \{k \mid a \prec\!\!\!\prec k\} + \{\downarrow\!\!\!\downarrow\}, \quad C(a,k) \;\equiv\; k \quad \text{and} \quad C(a,\downarrow\!\!\!\downarrow) \;\equiv\; \{b \mid b \prec\!\!\!\prec a\}$$

inductively generate the cover $\triangleleft$ in the sense that $a \triangleleft u$ holds iff it is provable using just the axioms

$$a \in u \implies a \triangleleft u \quad \text{and} \quad C(a,i) \triangleleft u \implies a \triangleleft u,$$

which are called *reflexivity* and *infinity*.

Proof Any such proof is sound by transitivity because $a \triangleleft C(a,i)$. Conversely, these axioms are complete because we have the following deduction, using reflexivity and infinity but not transitivity:

$$\begin{array}{lll}
\cdots & \vdash & \forall a \in \Downarrow b.\, \exists k.\, a \prec\!\!\!\prec k \subset u \\
\cdots,\ a \in \Downarrow b & \vdash & \exists k \in I(a).\, k \subset u \\
\cdots,\ a \in \Downarrow b & \vdash & \exists k \in I(a).\, C(a,k) \equiv k \triangleleft u \qquad \text{reflexivity} \\
\cdots,\ a \in \Downarrow b & \vdash & a \triangleleft u \qquad \text{infinity} \\
\cdots & \vdash & \forall a \in \Downarrow b.\ a \triangleleft u \\
\cdots & \vdash & C(b, \Downarrow) \equiv \Downarrow b\ \triangleleft u \\
\cdots & \vdash & b \triangleleft u. \qquad \text{infinity} \quad \square
\end{array}$$

Similar methods could be used to say how some more manageable sparser system might generate $\prec\!\!\!\prec$ in the way that we wanted in $\mathbb{R}^n$ (Example 1.6). We would need to consider how intersections are managed. For the same issue in what we have just done, and we leave the interested reader to use Lemmas 3.2 and 3.6 to show that if instead

$$C(a,k) \equiv \bigcup \{k' \mid \exists a'.\, a' \prec\!\!\!\prec k' \prec\!\!\!\prec a \ \wedge\ k' \prec\!\!\!\prec^1 k\}$$

then we obtain a *localised* inductive cover in the sense of [CSSV03, Definition 3.4].

Our contribution to this topic is to provide the topological motivation and the abstract axiomatisation for this sparser $\prec\!\!\!\prec$ relation. We also regard this instead of $\triangleleft$ as primary. Indeed, the examples that are usually given, in particular $\mathbb{R}$, are already of this form.

Remark 7.16 Even for those who specifically wish to study $\triangleleft$ using Martin-Löf Type Theory, our account and those of Negri, Aczel and Curi make a compelling case for presenting $\triangleleft$ in terms of $\prec\!\!\!\prec$ whenever the space happens to be locally compact.

If a specific formal cover is inductively generated in some more complicated way and is locally compact in the sense of Curi then by our Propositions 7.3 and 7.15 it has an abstract basis and hence a simple inductive generation. In particular, the proof of the Curi property for the cover also serves to show that it satisfies our definition.

On the other hand, if we wish to work with locally compact formal covers in general, is it more convenient to assume that they are presented in our simpler way.

Therefore the more complicated definition of locally compact inductively generated formal topologies is redundant. $\square$

Such a presentation also makes a much clearer connection between Formal Topology and the theory that we consider next.

Proposition 7.17 The correspondence between matrices with respect to $\triangleleft$ and $\prec\!\!\!\prec$ is:

$$\begin{array}{lcl}
\left[\, a \,\middle|\, f \,\middle|\, b \,\right] & \Leftrightarrow & \left(\forall a'.\, a' \prec\!\!\!\prec a \implies \left\langle\, a' \,\middle|\, f \,\middle|\, b \,\right\rangle\right) \\
\left\langle\, a \,\middle|\, f \,\middle|\, b \,\right\rangle & \Leftrightarrow & \exists k.\, a \prec\!\!\!\prec k \ \wedge\ \forall a' \in k.\, \left[\, a' \,\middle|\, f \,\middle|\, b \,\right].
\end{array}$$

It would take another three or four pages to prove this, maybe in a new section.

8 Intrinsic structure

Our fourth account of general topology exploits the *topological* operations that we may perform within the category of locally compact spaces, instead of relying on the *set theory* of their classical points or the *algebra* of their frames of open subspaces. We set out these properties here using the notation and technology that we have already developed. Then we show in the next section how they can be used to develop an intrinsic language for locally compact spaces.

Proposition 8.1 There is an object, called the ***Sierpiński space*** and written Σ, that has an open point $\top$ and a closed one $\bot$, with the property that, for any space X, there is a bijection amongst
(a) an open subspace $U \subset X$,
(b) a continuous function $\phi : X \to \Sigma$ and
(c) a closed subspace $C \subset X$,
where $U = \phi^{-1}(\top)$ and $C = \phi^{-1}(\bot)$.

Moreover, Σ is a topological distributive lattice, with respect to which

$$(\sigma \in U) \iff (\bot \in U) \vee \sigma \wedge (\top \in U).$$

Proof Classically $\Sigma \equiv \{\top, \bot\}$ with open subspaces $\emptyset$, $\{\top\}$ and Σ. It has a concrete basis indexed by $A \equiv (\odot \sqsubseteq \bullet)$, where

$$\begin{array}{rcl} U_\odot & \equiv & \{\top\} \\ U_\bullet & \equiv & \Sigma \end{array} \qquad \begin{array}{rcl} K_\odot & \equiv & \{\top\} \\ K_\bullet & \equiv & \{\bot\} \quad \text{or} \quad \Sigma \end{array}$$

and the basis expansion is the formula above. Notice that we have a choice between a singleton and a compact saturated subspace for $K_\bullet$, *cf.* the ambiguity in Definitions 1.3 and 3.15. The way-below relation $\prec\!\!\!\prec$ for the abstract basis and the cover relation $\lhd$ for the formal topology are the same as $\sqsubseteq$.

The topology on Σ is the free frame on one generator, so frame homomorphisms from it correspond to elements of the target frame. Hence in locale theory continuous maps $X \to \Sigma$ are given by elements $U \in \Omega$ of the frame corresponding to X. Sublocales are defined by nuclei, written j and satisfying $\mathsf{id} \leqslant j = j^2$ and $j(U \wedge V) = jU \wedge jV$, *cf.* Lemma 6.6. In particular, the open and closed sublocales named by $U \in \Omega$ are given by the nuclei $U \Rightarrow (-)$ and $U \vee (-)$ respectively. If $V \in \Omega$ gives rise to an isomorphic sublocale of *either* kind then the corresponding nuclei are equal as endofunctions, but by applying them both to $\emptyset$, U and V, we deduce that $U = V$. Hence the Sierpiński locale enjoys the same universal property as its classical analogue for both open and closed sublocales, *cf.* [Joh82, Lemma II 2.6]. □

Remark 8.2 The familiar Tychonov construction provides finite products in the category of locally compact sober spaces, but the description of the abstract basis for a product is beyond the scope this paper: see [work in progress].

Proposition 8.3 For any locally compact space X, the exponential Σ^X exists in the category.

Proof As Proposition 8.1 suggests, the points of Σ^X are the open subspaces of X itself, or the elements of the frame ΩX in locale theory. A typical open subspace $\mathcal{V} \subset \Sigma^X$ is a Scott-open family of open subspaces of X, *i.e.* Σ^X carries the Scott topology (Proposition 2.11).

There is then a bijection that is natural in Γ between continuous maps

$$\sigma : \Gamma \times X \longrightarrow \Sigma \qquad \text{and} \qquad \phi : \Gamma \longrightarrow \Sigma^X,$$

given classically by $\sigma(\gamma, x) = \top \iff x \in \phi(\gamma)$. This is valid exactly when X is a locally compact space; for the long history of the ideas behind this fact see [Isb86]; the same holds for locales [Hyl81] [Joh82, Theorem VII 4.11] and formal topology [Sig95] [Mai05] [Cur07, appendix]. However, we are unable to give the proof of the adjunction here since we do not want to discuss the Tychonov topology on the product $\Gamma \times X$.

Now let $(U_\ell, \mathcal{K}_\ell)$ be a *directed* basis using Scott-open families for X (Lemma 3.4), so that

$$U \;=\; \bigcup\nolimits^{\uparrow}\{U_\ell \mid \mathcal{K}_\ell \ni U\}.$$

Then for any Scott-open family $\mathcal{V} \subset \Sigma^X$ we have

$$\mathcal{V} \ni U \iff \exists \ell.\, \mathcal{V} \ni U_\ell \;\wedge\; \mathcal{K}_\ell \ni U.$$

Hence Σ^X has a concrete basis $(\mathcal{V}_{(\ell)}, \mathcal{L}_{(\ell)})$ indexed by $\mathsf{Fin}\,(A)$ and given by

$$\mathcal{V}_{(\ell)} \;\equiv\; \mathcal{K}_\ell \qquad \text{and} \qquad \mathcal{L}_{(\ell)} \;\equiv\; \{U_\ell\} \quad \text{or} \quad \{V \mid U_\ell \subset V\},$$

where, as in Remark 2.7, the parentheses on the subscripts denote formal intersections instead of unions as in a directed basis. Also, even more than with Σ, we see the utility of using a singleton $\{U_\ell\} \subset \Sigma^X$ instead of a compact saturated subspace.

Since U_ℓ and $\mathcal{K}_\ell$ essentially swap their roles, the pre-order $\sqsubseteq_{\Sigma^X}$ on $\mathsf{Fin}\,(A)$ that we use for the basis on Σ^X is the opposite of $\sqsubseteq_X$ for X. A similar thing happens with the way-below relation,

$$(k \prec\!\!\!\prec_{\Sigma^X} L) \;\equiv\; (\mathcal{L}_{(k)} \subset \mathcal{V}_{(L)}) \;\equiv\; \exists \ell \in L.\, \forall a \in \ell.\, (K_a \subset U_k) \;\equiv\; \exists \ell \in L.\, (\ell \prec\!\!\!\prec_X k),$$

where $L \in \mathsf{Fin}\,(\mathsf{Fin}\,(A))$ denotes a finite set of finite sets or list of lists.

This is a *stable* basis (Definition 2.2) in which the empty list $\emptyset \equiv \circ_X$ provides the top element $\bullet_{\Sigma^X}$ for $\sqsubseteq_{\Sigma^X}$, whilst union or concatenation $(\sqcup_X)$ in the directed basis for X is now conjunction $\sqcap_{\Sigma^X}$ for $\sqsubseteq_{\Sigma^X}$. The strong intersection rule holds because

$$\begin{aligned}(k \prec\!\!\!\prec_{\Sigma^X} L_1) \wedge (k \prec\!\!\!\prec_{\Sigma^X} L_2) \quad &\equiv \quad \exists \ell_1 \in L_1.\, \exists \ell_2 \in L_2.\, (\ell_1 \prec\!\!\!\prec_X k) \wedge (\ell_2 \prec\!\!\!\prec_X k) \\ &\Leftrightarrow \quad k \prec\!\!\!\prec_{\Sigma^X} (L_1 \sqcap_{\Sigma^X} L_2),\end{aligned}$$

where $(L_1 \sqcap_{\Sigma^X} L_2) \;\equiv\; \{\ell_1 \sqcup_X \ell_2 \mid \ell_1 \in L_1,\ \ell_2 \in L_2\}$. Single interpolation is

$$\begin{aligned}(k \prec\!\!\!\prec_{\Sigma^X} L) \quad &\equiv \quad \exists \ell \in L.\, (\ell \prec\!\!\!\prec_X k) \\ &\Rightarrow \quad \exists h.\, \exists \ell \in L.\, (\ell \prec\!\!\!\prec_X h \prec\!\!\!\prec_X k) \;\equiv\; \exists h.\, (k \prec\!\!\!\prec_{\Sigma^X} h \prec\!\!\!\prec_{\Sigma^X} L)\end{aligned}$$

and this also gives Wilker with $H \equiv \{h\} \prec\!\!\!\prec^1_{\Sigma^X} \{\ell\} \subset L$.

Using the translations in Section 7, the formal cover for the exponential Σ^X is

$$(\ell \lhd_{\Sigma^X} V) \quad \equiv \quad \forall k.\, (\ell \prec\!\!\!\prec_X k \implies \exists h \in V.\, h \prec\!\!\!\prec_X k),$$

where V is a possibly infinite set of finite subsets of A and $\prec\!\!\!\prec_X$ is an abstract basis that generates $\lhd_X$. ☐

Corollary 8.4 Every Scott-open filter is expressible in the manner of Lemma 3.11, as

$$\mathcal{K} \;=\; \bigcup\{\mathcal{K}_\ell \mid \ell \in s\} \subset \Omega, \qquad \text{where} \qquad s \equiv \{\ell \mid \mathcal{K} \ni U_\ell\} \subset \mathsf{Fin}\,(A)$$

is a $\prec\!\!\!\prec$-filter in the directed basis (Lemma 3.4).

Proof This is the basis expansion of $\mathcal{K} \ni U$ in Σ^X. ☐

In Section 3 we relied on the fact that the directed basis automatically satisfies the bounded-below and rounded unions rules to say that *without loss of generality* we may choose the bases for spaces to have these properties, without necessarily requiring them to be closed under unions.

That is a reasonable attitude for *base* types, in the sense of the next section, *i.e.* spaces such as $\mathbb{R}$ that we introduce on their own merits. However, for type *constructors* the situation is different: we want to assign bases to product and exponential types in a canonical way that makes use of the given bases for the base types. The simple way in which the basis for the exponential is constructed above gives us a strong preference for this over modified forms, for example the directed basis would be indexed by lists of lists (or by lattices).

We would therefore like to know when the foregoing construction yields a basis with the rounded union property:

Lemma 8.5 If X has a basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ that satisfies the strong intersection rule then the basis for Σ^X satisfies the rounded union rule.

Proof By the strong intersection and Wilker rules,

$$\begin{aligned}(L \prec\!\!\!\prec_{\Sigma^X} \ell) &\Leftrightarrow \forall b \in \ell.\, \forall k \in L.\, (b \prec\!\!\!\prec_X k)\\ &\Rightarrow \forall b \in \ell.\, \exists \ell' \ell'.\, (b \prec\!\!\!\prec_X \ell'' \prec\!\!\!\prec^1_X \ell') \ \wedge\ \forall k \in L.\, \ell' \sqsubseteq k\\ &\Rightarrow \exists \ell'''.\, (\ell \prec\!\!\!\prec_X \ell''') \ \wedge\ \forall k \in L.\, \ell''' \prec\!\!\!\prec^1_X k\\ &\equiv \exists \ell'''.\, (\ell''' \prec\!\!\!\prec_{\Sigma^X} \ell) \ \wedge\ \forall k \in L.\, (k \prec\!\!\!\prec_{\Sigma^X} \ell''')\\ &\equiv \exists \ell'''.\, (L \prec\!\!\!\prec_{\Sigma^X} \ell''' \prec\!\!\!\prec_{\Sigma^X} \ell),\end{aligned}$$

where ℓ''' is the union of the lists ℓ'' for $b \in \ell$.

Conversely, we can deduce $\exists \ell.\, a \prec\!\!\!\prec_X \ell \prec\!\!\!\prec_X c, d$ from the rounded union rule for Σ^X with $L \equiv \{\{c\}, \{d\}\}$ and $\ell \equiv \{a\}$, along with single interpolation, whence Lemma 3.3 gives the strong intersection rule. $\square$

Example 8.6 The space $\Sigma^{\mathbb{N}}$ is classically the powerset $\mathcal{P}(\mathbb{N})$ with the Scott topology. This topology is the free frame on $\mathbb{N}$. It has a basis indexed by finite sets ℓ of natural numbers, where

$$U_{(\ell)} \equiv \{u \mid \ell \subset u \subset \mathbb{N}\} \subset \mathcal{P}(\mathbb{N}) \qquad \text{and} \qquad K_{(\ell)} \equiv \{\ell\} \quad \text{or} \quad U_{(\ell)}.$$

The basis expansion is $u \in U \iff \exists \ell.\, (\ell \subset u) \wedge (\ell \in U)$. $\square$

The space Σ^N is also an example, with $A \equiv \mathsf{Fin}\,(N)$ and $(k \sqsubseteq \ell) \equiv (k \supset \ell)$, of the following class of spaces. These are the starting point for the construction of a space from an abstract basis using the ideas of this section:

Proposition 8.7 For any preorder $(A, \sqsubseteq)$, the relations

$$a \prec\!\!\!\prec^0 \ell \equiv \exists b.\, a \sqsubseteq b \in \ell \qquad \text{and} \qquad a \lhd^0 u \equiv \exists b.\, a \sqsubseteq b \in u$$

define an abstract basis satisfying the strong intersection rule and a formal cover, for which any upper subset $p \subset A$ is trivially rounded, located and positive (Definitions 5.1 and 6.10). In the corresponding locally compact sober space,

(a) the formal points are (upper, bounded) filters $p \subset A$, so

$$b \sqsupseteq a \in p \Longrightarrow b \in p, \quad \exists a.\, a \in p \quad \text{and} \quad a \in p \wedge b \in p \Longrightarrow \exists c.\, c \in p \wedge a \sqsupseteq c \sqsubseteq b;$$

(b) in particular each $a \in A$ defines a so-called ***compact point*** $p \equiv \uparrow a \equiv \{b \mid a \sqsubseteq b\}$;

(c) the specialisation order on compact points is the reverse of the usual one in domain theory: if $a \sqsubseteq b$ then $\uparrow a \supset \uparrow b$;

(d) the open subspaces are Scott-open sets of points;

(e) the basic open and compact subspaces are

$$V_a \equiv \{p \mid a \in p\} \qquad \text{and} \qquad L_a \equiv \{\uparrow a\} \quad \text{or} \quad L_a \equiv \uparrow\uparrow a \equiv \{p \mid a \in p\};$$

(f) and the basis expansion is

$$p \in V \iff \exists a.\, p \in V_a \wedge L_a \subset V \iff \exists a.\; a \in p \wedge \uparrow a \in V.$$

This space is called $\mathbf{Filt}(A, \sqsubseteq)$ or $\mathbf{Idl}\,(A, \sqsubseteq^{\mathsf{op}})$ and is (the typical example of) an ***algebraic dcpo*** (***directed-complete partial order***). □

This space allows us to give a simple categorical meaning to concrete bases:

Proposition 8.8 For any locally compact space X with a concrete basis $(U_a, \mathcal{K}_a)$ indexed by $(A, \sqsubseteq)$ there are maps

$$i : X \longrightarrow Y \equiv \mathbf{Filt}(A, \sqsubseteq) \quad \text{and} \quad I : \Sigma^X \longrightarrow \Sigma^Y \quad \text{such that} \quad ix \in IU \iff x \in U$$

that are defined by

$$ix \equiv \{a \mid x \in U_a\} \quad \text{and} \quad IU \equiv \bigcup\{V_a \mid \mathcal{K}_a \ni U\} \equiv \{p \mid \exists a.\, (\mathcal{K}_a \ni U) \wedge (a \in p)\}.$$

Conversely, any such pair (i, I) defines a concrete basis on X by

$$U_a \equiv i^{-1}V_a \equiv \{x : X \mid a \in ix\} \qquad \text{and} \qquad \mathcal{K}_a \equiv \{U \mid \{b \mid a \sqsubseteq b\} \in IU\}.$$

Moreover, these translations are inverse.

Proof The filteredness conditions for a concrete basis (Definition 2.6(b,c)) give those for ix in the Proposition, so this is a point of $\mathbf{Filt}(A, \sqsubseteq)$. The subspace IU is a union of basic open subspaces. Then

$$ix \in IU \equiv \exists a.\, x \in U_a \wedge \mathcal{K}_a \ni U \iff x \in U$$

by the basis expansion for X.

Conversely, U_a is an inverse image of an open subspace, whilst $\mathcal{K}_a$ is Scott-open because I is Scott-continuous. The basis expansion for X follows from that for Y and the equation for (i, I) because

$$\begin{aligned} x \in U \;&\Leftrightarrow\; ix \in IU \iff \exists a.\, ix \in V_a \wedge L_a \subset IU \\ &\Leftrightarrow\; \exists a.\; a \in ix \wedge \{b \mid a \sqsubseteq b\} \in IU \\ &\Leftrightarrow\; \exists a.\; x \in U_a \wedge \exists c.\, \mathcal{K}_c \ni U \wedge c \in \{b \mid a \sqsubseteq b\} \\ &\Leftrightarrow\; \exists a.\, x \in U_a \wedge \mathcal{K}_a \ni U. \end{aligned}$$

Finally, the definitions are inverse because

$$ix \ni a \iff x \in U_a \qquad \text{and} \qquad \uparrow a \in IU \iff \mathcal{K}_a \ni U.$$ □

We now describe this categorical structure more formally, because it offers a way of constructing a general locally compact space from data on $\mathbf{Filt}(A, \sqsubseteq)$.

Definition 8.9 For locally compact sober spaces X and Y, a ***Σ-split inclusion*** is a continuous map $i : X \rightarrowtail Y$ together with a Scott-continuous map $I : \Sigma^X \to \Sigma^Y$ such that

$$ix \in IU \iff x \in U \qquad \text{or} \qquad \Sigma^i \cdot I = \mathsf{id}_{\Sigma^X}.$$

The other composite, $\mathcal{E} \equiv I \cdot \Sigma^i : \Sigma^Y \to \Sigma^Y$, is called a ***nucleus*** and satisfies

$$\mathcal{E}(U \wedge V) \;=\; \mathcal{E}(\mathcal{E}U \wedge \mathcal{E}V) \qquad \text{and} \qquad \mathcal{E}(U \vee V) \;=\; \mathcal{E}(\mathcal{E}U \vee \mathcal{E}V).$$

If nuclei $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy $\mathcal{E}_1 \cdot \mathcal{E}_2 = \mathcal{E}_2 = \mathcal{E}_2 \cdot \mathcal{E}_1$ then the subspace defined by $\mathcal{E}_2$ is a Σ-split subspace of that defined by $\mathcal{E}_1$ and then we write $\mathcal{E}_2 \subset \mathcal{E}_1$.

Lemma 8.10 The following diagram is then an equaliser in the category of locally compact sober spaces:

$$X \overset{i}{\rightarrowtail} Y \underset{y \mapsto \{V \mid y \in V\}}{\overset{y \mapsto \{V \mid y \in \mathcal{E}V\}}{\rightrightarrows}} \Sigma^{\Sigma^Y}$$

so the points of X are those $y : Y$ that are ***admissible***, $\quad \forall V \in \Sigma^Y.\; y \in \mathcal{E}V \iff y \in V$.

Proof For any $y \in Y$ that satisfies $y \in V \iff y \in \mathcal{E}V \equiv I\Sigma^i V$ for all open $V \subset Y$, let $\mathcal{P} \equiv \{U \subset X \mid y \in IU\}$. Then

(a) $y \in IX \equiv IX \equiv I(\Sigma^i Y) \equiv \mathcal{E}Y \iff y \in Y$, which is true;

(b) dually $y \notin I\emptyset$ since $y \notin \emptyset \subset Y$;

(c) $y \in IU \wedge y \in IV \iff y \in IU \cap IV \iff y \in I\Sigma^i(IU \cap IV) \equiv I(\Sigma^i IU \cap \Sigma^i IV) \equiv I(U \cap V)$;

(d) dually $y \in IU \vee y \in IV \iff y \in IU \cup IV \iff y \in I(U \cup V)$; and

(e) the family $\mathcal{P} \equiv \{U \mid y \in IU\} \subset \Sigma^X$ is Scott-open because I is Scott-continuous.

Hence $\mathcal{P}$ is a formal point (Definition 3.13) of X, so by sobriety of X there is a unique point $x \in X$ with $x \in U \iff y \in IU$, but $x \in U \iff ix \in IU$ so $y = ix$. Then $ix \in V \iff x \in U \equiv i^*V \iff u \in IU \equiv I(i^*V) \equiv \mathcal{E}V \iff y \in V$, so $y = ix$ by sobriety of Y. □

We can describe the same structure using locales instead of Point–Set Topology:

Lemma 8.11 The trivial formal cover in Proposition 8.7 generates the ***Alexandrov topology*** $\mathcal{D}(A, \sqsubseteq)$ on $(A, \sqsubseteq)$, consisting of its lower subsets.

Proof In Proposition 7.8, $u = ju$ for this cover iff u is lower. □

To make a closer connection with the previous version, we may regard $\mathcal{D}(A, \sqsubseteq)$ as the lattice of open sets of *compact* points (Proposition 8.7(b)) and $\mathbf{Filt}(A, \sqsubseteq)$ as its sobrification.

Proposition 8.12 Let Ω be a frame with concrete basis $(U_a, \mathcal{K}_a)$. Then there are maps $i^* : \mathcal{D}(A) \twoheadrightarrow \Omega$ and $i_*, I : \Omega \rightarrowtail \mathcal{D}(A)$, where i^* is a frame homomorphism, i_* preserves arbitrary meets and I is Scott continuous. These satisfy the equations

$$i^* \cdot i_* \;=\; i^* \cdot I \;=\; \mathsf{id}_\Omega \qquad i_* \cdot i^* \;=\; j \qquad \text{and} \qquad I \cdot i^* \;=\; \mathcal{E},$$

where

$$ju \;\equiv\; \{a \mid a \lhd u\} \qquad \text{and} \qquad \mathcal{E}u \;\equiv\; \{a \mid \exists \ell.\, a \prec\!\!\!\prec \ell \subset u\}$$

are localic- and ASD-nuclei respectively.

Proof We already know all of this structure apart from I and $\mathcal{E}$. In particular, i^* is the inverse image operation for the inclusion $i : X \to \mathbf{Filt}(A, \sqsubseteq)$ in Proposition 8.8, which also defined, for $u \in \Omega$ (so $u = ju$),

$$Iu \;\equiv\; \{a \mid \mathcal{K}_a \ni u\} \;\equiv\; \{a \mid \exists k.\, a \prec\!\!\!\prec k \subset u\},$$

by Lemma 7.9. By the first part of Proposition 7.8, if $u = ju$ then $Iu \subset u$, so $i^* Iu \subset i^* u = ju = u$.

Conversely, if $a \in u = ju$ then, using Lemma 6.17,

$$a \;\lhd\; \{b \mid b \prec\!\!\!\prec a\} \;\equiv\; Ij\{a\} \;\subset\; Iju \;\equiv\; Iu,$$

so $u \lhd Iu$ and then $i^* \cdot I \;=\; \mathsf{id}_\Omega$.

The maps I and $\mathcal{E} \equiv I \cdot i^*$ are Scott-continuous because of their definition using the Scott-open families $\mathcal{K}_a$. Also, j and $\mathcal{E}$ inherit the equations for the two kinds of nuclei from the frame homomorphism i^*. □

Corollary 8.13 Any continuous frame Ω is related in the same way to $\mathcal{D}(\Omega)$ by

$$i^* u \;\equiv\; \bigvee u, \qquad i_* a \;\equiv\; \downarrow a \;\equiv\; \{b \mid b \leq a\} \qquad \text{and} \qquad Ia \;\equiv\; \twoheaddownarrow a \;\equiv\; \{b \mid b \ll a\}. \qquad \square$$

Whilst the two kinds of nuclei satisfy different equations, they play the same role in their respective subjects, namely to define subspaces.

Then every ASD nucleus splits in this way:

Theorem 8.14 Let $\mathcal{E}$ be a Scott-continuous endofunction of continuous frame Ω (for a locale Y) such that

$$\mathcal{E}(U \wedge V) \;=\; \mathcal{E}(\mathcal{E}U \wedge \mathcal{E}V) \qquad \text{and} \qquad \mathcal{E}(U \vee V) \;=\; \mathcal{E}(\mathcal{E}U \vee \mathcal{E}V).$$

Then there is a Σ-split sublocale $i : X \rightarrowtail Y$ with X locally compact and $\mathcal{E} = I \cdot \Sigma^i$. If $Y \equiv \mathbf{Filt}(A, \sqsubseteq)$ then X is given by the formal cover defined by

$$a \lhd u \quad\equiv\quad \mathcal{E}B_a \leqslant \mathcal{E}B_u$$

and has a concrete basis. These are unique up to unique isomorphism.

Proof From either equation, $\mathcal{E}$ is an idempotent on Ω. Splitting it, we write i^* for the epi part because the equations make this a frame homomorphism, with a right adjoint $i^* \dashv i_*$. Then $i^* \cdot i_*$ is also the identity on the smaller lattice, whilst the composite $j \equiv i_* \cdot i^*$ is a localic nucleus, so the splitting is a frame that defines a sublocale $i : X \rightarrowtail Y$. Neither i_* nor j need be Scott continuous, but $\mathcal{E}$ and hence I are, so the smaller frame is a continuous lattice and X is locally compact. The cover relation is

$$\begin{aligned} a \lhd u \;\; &\equiv\;\; a \in ju \;\equiv\; \{a\} \subset i_*(i^* u) \\ &\Leftrightarrow\;\; i^*\{a\} \subset i^* u \iff I(i^*\{a\}) \subset I(i^* u) \\ &\equiv\;\; \mathcal{E}B_a \leqslant \mathcal{E}B_u. \end{aligned} \qquad \square$$

Notice in this proof that we pass irreversibly from using I to i_*. This is where we lose the track of the chosen Scott-open family $\mathcal{K}_a$ and are just left with $\twoheaduparrow U_a$ defined by the order on the frame, *cf.* Corollary 6.15.

Remark 8.15 We have essentially given the proof that the contravariant self-adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$ on the category of locally compact locales is monadic.

In particular, re-interpreting the diagram in Lemma 8.10 using locales, it is an equaliser because it is a "U-split coequaliser" of frames, where U is the forgetful functor to **Set** or **Dcpo**. We leave the reader to prove this, with the help of any account of the Eilenberg–Moore category for a monad, such as [Tay99, Section 7.5].

The idea of Abstract Stone Duality that gave it its name was to take this as the defining property of an abstract category and develop a symbolic calculus for general topology from that [B]. The equation for a nucleus was expressed there using the λ-calculus, but [G] showed that this is equivalent to our lattice-theoretic form. In this calculus the (somewhat unwieldy) notion of nucleus therefore provided the definition (formation rule) for types.

In the next section we shall develop the ideas that we have discussed into a intrinsic language for locally compact spaces, taking types to be specified by abstract bases. Then in Section 10 we will show that any abstract basis defines a nucleus.

9 Abstract Stone Duality

We now introduce a notation based on the ***simply typed λ-calculus*** that is designed to take advantage of the topological facts about the categories of locally compact spaces that we stated in the previous section. Describing such calculi and their equivalence with other mathematical structures takes a lot of space, so we do this rather tersely. For more extensive introductions that show the relationship with real analysis please see [J], with domain theory see [I] and with mathematical foundations see [O]. In the following account we rely on our earlier treatment of locally compact spaces in Point–Set Topology, Locale Theory and Formal Topology, but in [work in progress] we shall construct a model of ASD directly from abstract bases and matrices over an arithmetic universe.

Remark 9.1 We shall take a ***type*** in ASD to be specified by an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ and its ***interpretation*** or ***denotation*** is a locally compact space together with a concrete basis. If this subject is new to you then you should just regard the type *as* the space. On the other hand, using abstract bases as the definition is an innovation of this paper in the ASD programme: previous work specified types as nuclei (Definition 8.9), so in the next section we will prove that these are equivalent.

Remark 9.2 As usual in modern symbolic logic, we write

$$x_1 : X_1,\ \ldots,\ x_n : X_k \quad \vdash \quad t : Y$$

for a ***term*** t of type Y, possibly containing (at most) free variables $x_1, \ldots, x_n$ respectively of types $X_1, \ldots, X_n$. How terms are formed and manipulated remains to be defined, but the ***interpretation*** or ***denotation*** of t is a continuous function

$$[\![\, t \,]\!] : [\![X_1]\!] \times \cdots \times [\![X_k]\!] \longrightarrow [\![Y]\!],$$

where $[\![X_1]\!]$, ..., $[\![X_k]\!]$, $[\![Y]\!]$ are locally compact spaces that have been chosen as the denotations of the types $X_1, \ldots, X_n, Y$. The validity of processes such as substitution depends on having finite products in the category. We shall omit the brackets because we do not really need to distinguish between (syntactic) terms and their (topological) denotations in this paper.

Remark 9.3 Sets and lists or (Kuratowski-) finite subsets, with induction over them.

Remark 9.4 Very brief introduction to the restricted λ-calculus to exploit the exponential Σ^X

from Proposition 8.3. Give the β- and η-rules; also λ-abstraction preserves $=$, $\leqslant$ and $\Rightarrow$.

Remark 9.5 The other topological facts in the previous section may be taken as symbolic operations and equations:

(a) the types Σ and Σ^X are distributive lattices and we may use $\top$, $\bot$, $\wedge$ and $\vee$ on their terms, because of Proposition 8.1;

(b) equality $n = m$ between members of a *set* such as $\mathbb{N}$ is a term of type Σ because the diagonal subspace $\mathbb{N} \hookrightarrow \mathbb{N} \times \mathbb{N}$ is open;

(c) combining these operations with induction over a list, we have membership $(a \in \ell)$ and both forms of quantification over a list or (Kuratowski-) finite subset ($\forall a \in \ell.\, \phi a$ and $\exists a \in \ell.\, \phi a$) as terms of type Σ;

(d) the ***Phoa***[1] ***principle***

$$F\sigma \iff F\bot \ \vee\ \sigma\ \wedge\ F\top$$

holds for any $\sigma : \Sigma$ and $F : \Sigma^\Sigma$, also because of Proposition 8.1;

(e) existential quantification $\exists n.\, \phi n$ of any system of Σ-terms indexed by a set gives another Σ-term, since the union of a family of open subspaces is open;

(f) the ***Scott principle***

$$F\xi \iff \exists \ell.\, (\forall n \in \ell.\, \xi n)\ \wedge\ F(\lambda n.n \in \ell)$$

holds for any $\xi : \Sigma^{\mathbb{N}}$ and $F : \Sigma^{\Sigma^{\mathbb{N}}}$, because of Example 8.6; and

(g) in fact the Scott principle holds for any set N (*quâ* discrete space) in place of $\mathbb{N}$ and the case $N \equiv \mathbf{1}$ is the Phoa principle.

Remark 9.6 From the Phoa principle we deduce that

(a) if $\sigma \Rightarrow \tau$ then $F\sigma \Rightarrow F\tau$;

(b) more generally, any $F : \Sigma^Y \to \Sigma^X$ preserves the lattice order, which we therefore call ***intrinsic*** and write as $\leqslant$;

(c) the symbols $\neg$, $\Rightarrow$ and $\Leftrightarrow$ are therefore not allowed *within* terms of type Σ, but we use $\Rightarrow$ and $\Leftrightarrow$ instead of $\leqslant$ and $=$ for the order and equality *between* such terms and $\neg\sigma$ for $\sigma \Leftrightarrow \bot$;

(d) if $F\top \Rightarrow G\top$ then $\sigma \wedge F\sigma \Leftrightarrow \sigma \wedge (F\bot \vee F\top) \Rightarrow \sigma \wedge F\top \Rightarrow \sigma \wedge G\top \Rightarrow G\sigma$; and

(e) similarly if $F\bot \Rightarrow G\bot$ then $F\sigma \Rightarrow G\sigma \vee \sigma$.

The last two observations may be formulated as the following two fundamental rules for topological reasoning, where the terms $\alpha, \beta : \Sigma$ may depend on $\sigma : \Sigma$ (so $\alpha \equiv F\sigma$ and $\beta \equiv G\sigma$) and the variables in Γ:

$$\frac{\Gamma,\ \sigma \Leftrightarrow \top \quad \vdash \quad \alpha \Longrightarrow \beta}{\Gamma \quad \vdash \quad \sigma \wedge \alpha \Longrightarrow \beta} \qquad \text{and} \qquad \frac{\Gamma,\ \sigma \Leftrightarrow \bot \quad \vdash \quad \alpha \Longrightarrow \beta}{\Gamma \quad \vdash \quad \alpha \Longrightarrow \beta \vee \sigma}$$

The top lines say that $\alpha \Rightarrow \beta$ holds in the subspace U or C of Γ on which $\sigma \Leftrightarrow \top$ or $\bot$. Then the rules allow us to deduce the more complex implications in the *whole* space. We call these principles after Gerhard ***Gentzen*** because of the loose resemblance to his rules for implication and negation in the sequent calculus [Gen35, Section III].

The rule on the left is used very commonly and is easily overlooked, so for illustration we spell out its use in the proof of Lemma 10.1. The one on the right, on the other hand, may be surprising to an intuitionistic *set* theorist, but is a theorem of intuitionistic *locale* theory.

Lemma 9.7 Let $G : \Sigma^{\Sigma^X}$, $\phi_\ell : \Sigma^X$ and $\alpha_\ell : \Sigma$ for $\ell \in \mathsf{Fin}\,(N)$ be such that

$$\phi_k \Longrightarrow \phi_{k \sqcup \ell} \Longleftarrow \phi_\ell, \quad \alpha_\circ \iff \top \quad \text{and} \quad \alpha_{k \sqcup \ell} \iff \alpha_k \wedge \alpha_\ell.$$

[1]After Wesley Phoa, whose name is of southeast Asian origin and is pronounced a little like French *poire*.

Then $$G\left(\exists\ell.\,\phi_\ell \;\wedge\; \alpha_\ell\right) \iff \exists\ell.\,(G\phi_\ell) \;\wedge\; \alpha_\ell.$$

Proof Let $\xi \equiv \lambda n.\,\alpha_{\{n\}}$, so $\alpha_\ell \iff \forall n\in\ell.\,\xi n$, and

$$F \;\equiv\; \lambda\zeta.\,G\left(\exists\ell.\,\phi_\ell \;\wedge\; \forall n\in\ell.\,\zeta n\right).$$

Then $$F(\lambda n.\,n\in k) \iff G\left(\exists\ell.\,\phi_\ell \;\wedge\; (\ell\subset k)\right) \iff G\phi_k,$$

so $$\exists\ell.\,(\forall n\in\ell.\,\xi n) \;\wedge\; F(\lambda n.\,n\in\ell) \iff \exists\ell.\,\alpha_\ell \;\wedge\; G\phi_\ell,$$

which is equal by Remark 9.5(f) to $F\xi \iff G\left(\exists\ell.\,\phi_\ell \;\wedge\; \alpha_\ell\right)$, as required. □

Definition 9.8 Turning to the topological interpretation of this calculus,
(a) terms of type X are formal points;
(b) terms of type Σ^X are formal open subspaces;
(c) such terms are also used to describe the complementary closed subspaces, *cf.* Proposition 8.1;
(d) terms of type Σ^{Σ^X} are interpreted as Scott-open families of open subspaces, of which we have made substantial use in this paper;
(e) in particular, a ***formal compact subspace*** of X (*cf.* Proposition 3.14 and Definition 6.2) is a term

$$K:\Sigma^{\Sigma^X} \qquad \text{such that} \qquad K\top \iff \top \qquad \text{and} \qquad K(\phi\wedge\psi) \iff K\phi\wedge K\psi,$$

where ϕ and ψ are terms of type Σ^X that denote open subspaces $U,V\subset X$;
(f) because of the Phoa principle (or the negation-like Gentzen rule, Remark 9.6(e)), any formal compact subspace also satisfies the so-called ***dual Frobenius law***,

$$K\left(\lambda x.\,\sigma\vee\phi x\right) \iff \sigma \;\vee\; K\phi,$$

so long as σ does not depend on x. This rule, which was essentially first observed by Japie Vermeulen for proper maps [Ver94], is valid in *intuitionistic* locale theory and formal topology, because, by Proposition 8.1, the logic is one of *closed* subspaces as well as of *open* ones.

See [J] for the treatment of discrete and Hausdorff spaces in this formalism, along with further discussion of compact subspaces and the way in which we may use K as a *universal quantifier*. Beware, however, that formal compact subspaces are not necessarily representable as spaces or types in our calculus, because not all compact subspaces of a locally compact space are locally compact.

The logic also admits *existential quantifiers*, in particular over $\mathbb{N}$ and more generally over any set *quâ* discrete space. More generally, the class of spaces over which this quantifier may range is a constructively subtle issue, to which we return in Section 12.

We can now use this λ-notation to rewrite the fundamental definition of this paper. We now use terms $\phi,\beta_a:\Sigma$ in place of open subspaces and $K:\Sigma^{\Sigma^X}$ for compact ones or Scott-open families, the distinction between these being whether or not K preserves intersections. Then we have:

Definition 9.9 A ***concrete basis using λ-terms*** consists of
(a) for each $a\in A$, terms $\beta_a:\Sigma^X$ and $K_a:\Sigma^{\Sigma^X}$;
(b) if $a\sqsubseteq b$ then $\beta_a x\Longrightarrow\beta_b x$ and $K_b\phi\Longrightarrow K_a\phi$;
(c) $\beta_a x \;\wedge\; \beta_b x \iff \exists c.\,\beta_c x \;\wedge\; (a\sqsubseteq c\sqsupseteq b)$; and
(d) $\phi x \iff \exists a.\,\beta_a x \;\wedge\; K_a\phi$.

Proposition 9.10 Any concrete basis (β_a, K_a) using λ-terms gives rise to an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$, where

$$(a \prec\!\!\!\prec \ell) \;\equiv\; K_a\beta_\ell \;\equiv\; K_a\,(\exists b \in \ell.\, \beta_b).$$

If the K_a preserve meets, so they are formal compact subspaces, then $\prec\!\!\!\prec$ obeys the strong intersection rule.

Proof Co- and contravariance of $\prec\!\!\!\prec$ follow from that of β_ℓ and A_a respectively. For the Wilker rule we use the basis expansion of β_c, switch to a directed basis and then apply K_a:

$$\begin{array}{lrcl} & \beta_k \;\equiv\; \exists c \in k.\, \beta_c & = & \exists c \in k.\, \exists b.\, \beta_b \;\wedge\; A_b\beta_c \\ & & = & \exists b.\, \beta_b \;\wedge\; \exists c \in k.\, A_b\beta_c \\ & & = & \exists \ell.\, \beta_\ell \;\wedge\; \forall b \in \ell.\, \exists c \in k.\, A_b\beta_c. \\ \text{Hence} & a \prec\!\!\!\prec k \;\equiv\; A_a\beta_k & \Leftrightarrow & \exists \ell.\, A_a\beta_\ell \;\wedge\; \forall b \in \ell.\, \exists c \in k.\, A_b\beta_c \\ & & \equiv & \exists \ell.\, a \prec\!\!\!\prec \ell \prec\!\!\!\prec^1 k \end{array}$$

by Lemma 9.7. For the weak intersection rule, the directed basis expansion of β_ℓ gives

$$\beta_\ell \quad = \quad \exists b.\, \beta_b \;\wedge\; A_b\beta_\ell \quad = \quad \exists k.\, \beta_k \;\wedge\; \forall b \in k.\, A_b\beta_\ell \quad \geqslant \quad \beta_k \;\wedge\; (k \prec\!\!\!\prec \ell).$$

Hence, using $\beta_c \wedge \beta_d = \exists e.\, \beta_e \wedge (c \sqsupseteq e \sqsubseteq d)$ in the equality,

$$\beta_k \;\wedge\; (k \prec\!\!\!\prec \ell_1) \;\wedge\; (k \prec\!\!\!\prec \ell_2) \quad \leqslant \quad \beta_{\ell_1} \;\wedge\; \beta_{\ell_2} \quad = \quad \exists h.\, \beta_h \;\wedge\; h \sqsubseteq \ell_1 \sqcap \ell_2$$

and therefore, by Lemma 9.7 again,

$$K_a\beta_k \;\wedge\; (k \prec\!\!\!\prec \ell_1) \;\wedge\; (k \prec\!\!\!\prec \ell_2) \implies \exists h.\, K_a\beta_h \;\wedge\; h \sqsubseteq \ell_1 \sqcap \ell_2.$$

The strong case is similar but simpler. □

In future work, we intend the ASD calculus to use the following rules to define general types and their terms.

Definition 9.11 The ***formation rule*** for a type X in ASD takes an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$. This yields not only the type itself but also a Σ-split inclusion $i : X \to \Sigma^A$ with $I : \Sigma^X \to \Sigma^{\Sigma^A}$ and a concrete basis $\beta_a : \Sigma^X$ and $K_a : \Sigma^{\Sigma^A}$.

Definition 9.12 The ***introduction rule*** for terms of the type X specified by $(A, \sqsubseteq, \prec\!\!\!\prec)$ takes a covariant, rounded, bounded filter $\xi : \Sigma^A$ and defines a term $x : X$ with $ix = \xi$ or $\beta_a x \equiv \xi a$.

Two such terms are ***equal*** in X iff they are equal in Σ^A.

Conversely, the ***elimination rules*** recover $\xi \equiv ix : \Sigma^A$ and its properties from a term $x : X$.

Definition 9.13 The elimination rule for $x : X$ also gives an ***introduction rule*** that turns a term $\Phi : \Sigma^{\Sigma^A}$ into $\phi \equiv \Sigma^i\Phi : \Sigma^X$ by $\phi x \equiv \Phi\xi \equiv \Phi(ix)$.

Conversely, the ***elimination rule*** turns $\phi : \Sigma^X$ into $\Phi \equiv I\phi : \Sigma^{\Sigma^A}$ by $\Phi\xi \equiv \exists a.\, \xi a \wedge K_a\phi$, using the dual basis (K_a).

Two such terms Φ and Ψ are ***equal*** in Σ^X if $\mathcal{E}\Phi = \mathcal{E}\Psi$.

We need to justify these rules with respect to the previous ASD literature, which we do in the next section.

We could also show that morphisms in ASD, i.e. *terms $x : X \vdash t : Y$, correspond bijectively to matrices.*

10 Abstract bases and nuclei

In Remark 7.16 we argued that abstract bases are equivalent to but easier to use than the definition of a locally compact Formal Topology that is found in the existing literature on that subject. Here we do the same thing for Abstract Stone Duality, where nuclei (Definition 8.9) provided the earlier definition.

Following Proposition 8.7, we must therefore first introduce $\mathbf{Filt}(A, \sqsubseteq)$ as an object in ASD. We do this by defining a nucleus $\mathcal{E}^0$ on Σ^A and identifying the admissible terms. The following account is a much simplified version of the one in [G].

Lemma 10.1 The term $\mathcal{E}^0$ defined by $\mathcal{E}^0\Phi\xi \;\equiv\; \exists a.\, \xi a \;\wedge\; \Phi(\lambda b.\, a \sqsubseteq b)$ is a nucleus.

Proof We spell out this simple argument in detail because it illustrates the Gentzen rule (Remark 9.6(d)), whilst any $\Phi : \Sigma^A \to \Sigma$ preserves the intrinsic order (Remark 9.6(b)).

$$
\begin{array}{lcll}
a \sqsubseteq b,\ b \sqsubseteq c & \vdash & a \sqsubseteq c & \text{transitivity} \\
a \sqsubseteq b & \vdash & b \sqsubseteq c \implies a \sqsubseteq c & \text{Gentzen} \\
a \sqsubseteq b & \vdash & \lambda c.\, b \sqsubseteq c \;\leqslant\; \lambda c.\, a \sqsubseteq c & \lambda\text{-abstraction} \\
a \sqsubseteq b & \vdash & \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi(\lambda c.\, a \sqsubseteq c) & \text{intrinsic monotonicity} \\
\ldots & \vdash & (a \sqsubseteq b) \;\wedge\; \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi(\lambda c.\, a \sqsubseteq c) & \text{Gentzen} \\
\ldots & \vdash & \exists b.\, (\lambda b.\, a \sqsubseteq b) b \;\wedge\; \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi(\lambda c.\, a \sqsubseteq c), & \exists
\end{array}
$$

where the last line is in fact $\Leftrightarrow$ because we may put $b \equiv a$. Hence $\mathcal{E}^0\Phi(\lambda b.\, a \sqsubseteq b) \Leftrightarrow \Phi(\lambda b.\, a \sqsubseteq b)$. Then, with either $\wedge$ or $\vee$,

$$
\begin{aligned}
\mathcal{E}^0(\mathcal{E}^0\Phi \overset{\vee}{\underset{\wedge}{}} \mathcal{E}^0\Psi)\xi \;&\equiv\; \exists a.\, \xi a \;\wedge\; (\mathcal{E}^0\Phi \overset{\vee}{\underset{\wedge}{}} \mathcal{E}^0\Psi)(\lambda b.\, a \sqsubseteq b) \\
&\Leftrightarrow\; \exists a.\, \xi a \;\wedge\; (\Phi \overset{\vee}{\underset{\wedge}{}} \Psi)(\lambda b.\, a \sqsubseteq b) \;\equiv\; \mathcal{E}^0(\Phi \overset{\vee}{\underset{\wedge}{}} \Psi)\xi.
\end{aligned}
$$

□

Next we verify that $\mathcal{E}^0$ defines the object that we want by proving that a term $\xi : \Sigma^A$ is a filter iff it is *admissible* for $\mathcal{E}^0$ in the sense of Lemma 8.10, satisfying $\mathcal{E}^0\Phi\xi = \Phi\xi$ for all Φ. Note that such a term ξ may have parameters, so these points are "generalised" ones in the sense of sheaf theory; they are test maps to an equaliser from a general object.

Lemma 10.2 If $\xi : \Sigma^A$ is admissible for $\mathcal{E}^0$ then it is covariant, bounded and filtered.

Proof We use admissibility with respect to various Φ. For covariance, let $\Phi \equiv \lambda\zeta.\, \zeta a$, so

$$\xi a \;\equiv\; \Phi\xi \iff \mathcal{E}^0\Phi\xi \;\equiv\; \exists b.\, \xi b \;\wedge\; (b \sqsubseteq a).$$

Then, for filteredness, let $\Phi \equiv \lambda\zeta.\, \zeta b \wedge \zeta c$, so

$$\xi b \wedge \xi c \;\equiv\; \Phi\xi \iff \mathcal{E}^0\Phi\xi \iff \exists a.\, \xi a \;\wedge\; (b \sqsupseteq a \sqsubseteq c).$$

Finally, $\Phi \equiv \lambda\zeta.\, \top$ gives boundedness: $\top \equiv \Phi\xi \Leftrightarrow \mathcal{E}^0\Phi\xi \Leftrightarrow \exists a.\, \xi a$. □

Lemma 10.3 If ξ is covariant then $\mathcal{E}^0\Phi\xi \implies \Phi\xi$.

Proof As in Lemma 10.1, we may write covariance as $\xi b \;\vdash\; (\lambda c.\, b \sqsubseteq c) \;\leqslant\; \xi$.
Since any Φ preserves the intrinsic order $\leqslant$, we have $\xi b \;\vdash\; \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi\xi$.
Using the Gentzen rule we deduce that $\xi b \;\wedge\; \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi\xi$. □

The converse uses the Scott principle, Remark 9.5(f):

Lemma 10.4 If ξ is bounded and filtered then $\Phi\xi \Longrightarrow \mathcal{E}^0\Phi\xi$.

Proof By Scott continuity, $\Phi\xi \iff \exists\ell.\,(\forall b \in \ell.\,\xi b) \wedge \Phi(\lambda b.\,b \in \ell)$.

By induction on ℓ, we claim that

$$\exists c.\,\xi c \wedge \forall b \in \ell.\,(c \sqsubseteq b).$$

In the base case $\ell \equiv \circ$, this is boundedness of ξ, whilst filteredness of ξ gives the induction step. Then $(\lambda b.\,b \in \ell) \leqslant (\lambda b.\,c \sqsubseteq b)$, so $\Phi(\lambda b.\,b \in \ell) \Longrightarrow \Phi(\lambda b.\,c \sqsubseteq b)$ since Φ preserves the intrinsic order. Hence $\exists c.\,\xi c \wedge \Phi(\lambda b.\,c \sqsubseteq b)$, which is $\mathcal{E}^0\Phi\xi$. $\square$

Lemma 10.5 The object $\mathbf{Filt}(A, \sqsubseteq)$ that is defined by the nucleus $\mathcal{E}^0$ on Σ^A has a basis using λ-terms with

$$B_a \equiv \lambda\xi.\,\xi a \qquad \text{and} \qquad \mathcal{L}_a \equiv \lambda\Phi.\,\Phi(\lambda b.\,a \sqsubseteq b),$$

where the general open subspaces are those $\Phi : \Sigma^{\Sigma^A}$ such that $\Phi = \mathcal{E}^0\Phi$ and the basis expansion is

$$\Phi\xi \iff \mathcal{E}^0\Phi\xi \equiv \exists a.\,B_a\xi \wedge \mathcal{L}_a\Phi \equiv \exists a.\,\xi a \wedge \Phi(\lambda b.\,a \sqsubseteq b). \qquad \square$$

Next we prove the analogue of Theorem 8.8 in ASD.

Proposition 10.6 Any concrete basis (β_a, K_a) for X using λ-terms defines a Σ-split inclusion $i : X \to Y \equiv \mathbf{Filt}(A, \sqsubseteq)$ by

$$ix \equiv \lambda a.\,\beta_a x \qquad \text{and} \qquad I\phi \equiv \lambda\xi.\,\exists a.\,K_a\phi \wedge \xi a.$$

Conversely, given such an inclusion, the basis is

$$\beta_a \equiv \lambda x.\,ixa \qquad \text{and} \qquad K_a \equiv \lambda\phi.\,I\phi(\lambda b.\,a \sqsubseteq b)$$

and these translations are inverse.

Proof The basis gives a Σ-splitting because

$$(I\phi)(ix) \equiv \exists a.\,K_a\phi \wedge \beta_a x \iff \phi x.$$

Conversely, the Σ-splitting yields a basis because

$$\begin{aligned} \phi x &\Leftrightarrow I\phi(ix) \equiv \exists a.\,B_a(ix) \wedge \mathcal{L}_a(I\phi) \\ &\equiv \exists a.\,ixa \wedge I\phi(\lambda b.\,a \sqsubseteq b) \equiv \exists a.\,\beta_a x \wedge \mathcal{K}_a\phi. \end{aligned}$$

These translations are inverse because $ixa \iff \beta_a x$ and $K_a\phi \iff I\phi(\lambda b.\,a \sqsubseteq b)$ and we can recover $I\phi\xi$ from the latter. $\square$

Corollary 10.7 Any concrete basis gives rise to a nucleus on $\mathbf{Filt}(A, \sqsubseteq)$ with

$$\mathcal{E}\Phi\xi \equiv \exists a\ell.\,\xi a \wedge (a \prec\!\!\!\prec \ell) \wedge \forall b \in \ell.\,\Phi(\lambda c.\,b \sqsubseteq c).$$

Proof Let Φ be an open subspace of $\mathbf{Filt}(A, \sqsubseteq)$, so $\Phi = \mathcal{E}^0\Phi$, then

$$\begin{array}{llll}
\Sigma^i\Phi & \equiv & \lambda x.\,\Phi(ix) \;\equiv\; \lambda x.\,\Phi(\lambda b.\,\beta_b x) & \\
 & = & \lambda x.\,\mathcal{E}^0\Phi(\lambda b.\,\beta_b x) & \\
 & \equiv & \lambda x.\,\exists b.\,(\lambda b'.\,\beta_{b'} x)b \;\wedge\; \Phi(\lambda c.\,b \sqsubseteq c) & \text{Lemma 10.1} \\
 & = & \exists b.\,\beta_b \;\wedge\; \Phi(\lambda c.\,b \sqsubseteq c) & \\
 & = & \exists \ell.\,\beta_\ell \;\wedge\; \forall b \in \ell.\,\Phi(\lambda c.\,b \sqsubseteq c) & \\
K_a(\Sigma^i\Phi) & \Leftrightarrow & \exists \ell.\,K_a\beta_\ell \;\wedge\; \forall b \in \ell.\,\Phi(\lambda c.\,b \sqsubseteq c) & \text{Lemma 9.7} \\
\mathcal{E}\Phi\xi & \equiv & I(\Sigma^i\Phi)\xi \;\equiv\; \exists a.\,\xi a \;\wedge\; K_a(\Sigma^i\Phi) & \text{Prop 10.6} \\
 & \Leftrightarrow & \exists a\ell.\,\xi a \;\wedge\; a \prec\!\!\!\prec \ell \;\wedge\; \forall b \in \ell.\,\Phi(\lambda c.\,b \sqsubseteq c). &
\end{array}$$

The equations for a nucleus follow from the fact that $\mathcal{E} = I \cdot \Sigma^i$. $\square$

Now we show that any *abstract* basis defines a nucleus too.

Lemma 10.8 Given any co- and contravariant relation $\prec\!\!\!\prec$, let

$$\mathcal{K}_a\Phi \;\equiv\; \exists \ell.\,(a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\,\Phi(\lambda c.\,b \sqsubseteq c)$$

and
$$\mathcal{E}\Phi\xi \;\equiv\; \exists a.\,B_a\xi \;\wedge\; \mathcal{K}_a\Phi \;\equiv\; \exists a\ell.\,\xi a \;\wedge\; (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\,\Phi(\lambda c.\,b \sqsubseteq c).$$

Then we recover

$$\mathcal{K}_a\Phi \iff \mathcal{E}\Phi(\lambda b.\,a \sqsubseteq b) \qquad \text{and} \qquad (a \prec\!\!\!\prec \ell) \iff \mathcal{K}_a B_\ell \iff \mathcal{E}B_\ell(\lambda b.\,a \sqsubseteq b).$$

Also, $\mathcal{E}$ satisfies $\mathcal{E} = \mathcal{E}^0 \cdot \mathcal{E} = \mathcal{E} \cdot \mathcal{E}^0$ and is recovered from $\prec\!\!\!\prec$.

Proof By covariance of $a \prec\!\!\!\prec \ell$ in ℓ,

$$\mathcal{K}_a B_\ell \;\equiv\; \exists k.\,(a \prec\!\!\!\prec k) \;\wedge\; \forall b \in k.\,\exists c \in \ell.\,b \sqsubseteq c \iff (a \prec\!\!\!\prec \ell).$$

Contravariance of $a \prec\!\!\!\prec \ell$ in a transfers to $\mathcal{K}_a$; using this, $\mathcal{K}_a$ is recovered from $\mathcal{E}$. We leave the last part to the reader since we will not use it. $\square$

Now we must use the properties of an abstract basis to prove that $\mathcal{E}$ satisfies the two equations in Definition 8.9. However, since any term preserves the intrinsic order, we already have

$$\mathcal{E}(\Phi \wedge \Psi) \;\leqslant\; (\mathcal{E}\Phi) \wedge (\mathcal{E}\Psi) \qquad \text{and} \qquad (\mathcal{E}\Phi) \vee (\mathcal{E}\Psi) \;\leqslant\; \mathcal{E}(\Phi \vee \Psi),$$

so we need to prove the reverse inequalities. The weak intersection rule gives the first and the Wilker rule the second.

Lemma 10.9 If $\prec\!\!\!\prec$ satisfies the weak intersection rule

$$(a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\,(b \prec\!\!\!\prec k_1 \wedge b \prec\!\!\!\prec k_2) \implies a \prec\!\!\!\prec k_1 \sqcap k_2,$$

then $\mathcal{K}_a(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \implies \mathcal{K}_a(\Phi \wedge \Psi)$ and so $\mathcal{E}(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \;\leqslant\; \mathcal{E}(\Phi \wedge \Psi)$.

Proof Using the formulae for $\mathcal{K}_a$ in Lemma 10.8 three times,

$$\begin{array}{ll}
 & \mathcal{K}_a(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \\
\Leftrightarrow & \exists \ell.\,(a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\,\mathcal{E}\Phi(\lambda c.\,b \sqsubseteq c) \;\wedge\; \mathcal{E}\Psi(\lambda c.\,b \sqsubseteq c) \\
\Leftrightarrow & \exists \ell.\,(a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\,\left\{\begin{array}{ll} & \exists k_1.\,(b \prec\!\!\!\prec k_1) \;\wedge\; \forall c_1 \in k_1.\,\Phi(\lambda d.\,c_1 \sqsubseteq d) \\ \wedge & \exists k_2.\,(b \prec\!\!\!\prec k_2) \;\wedge\; \forall c_2 \in k_2.\,\Psi(\lambda d.\,c_2 \sqsubseteq d). \end{array}\right.
\end{array}$$

Taking the unions of the k-lists for all $b \in \ell$ and using covariance of $\prec\!\!\!\prec$ with respect to k, this implies

$$\exists k_1 k_2. \begin{cases} \qquad \exists \ell. \, (a \prec\!\!\!\prec \ell) \;\wedge\; (\forall b \in \ell. \, b \prec\!\!\!\prec k_1 \wedge b \prec\!\!\!\prec k_2) \\ \wedge \quad \forall c_1 \in k_1. \, \Phi(\lambda d. \, c_1 \sqsubseteq d) \\ \wedge \quad \forall c_2 \in k_2. \, \Psi(\lambda d. \, c_2 \sqsubseteq d). \end{cases}$$

By the weak intersection rule, the top line implies $a \prec\!\!\!\prec k_1 \sqcap k_2$, which is

$$\exists \ell'. \, (a \prec\!\!\!\prec \ell') \;\wedge\; \forall b \in \ell'. \, (\exists c_1 \in k_1. \, b \sqsubseteq c_1) \;\wedge\; (\exists c_2 \in k_2. \, b \sqsubseteq c_2),$$

possibly with a different list ℓ'. Then we match $\exists c$ with $\forall c$ and use

$$(b \sqsubseteq c) \;\wedge\; \Phi(\lambda d. \, c \sqsubseteq d) \implies \Phi(\lambda d. \, b \sqsubseteq d)$$

from Lemma 10.1, the fact that Φ preserves the intrinsic order, the Gentzen rule and Lemma 10.8 to obtain

$$\exists \ell'. \, (a \prec\!\!\!\prec \ell') \;\wedge\; \forall b \in \ell'. \, \Phi(\lambda d. \, b \sqsubseteq d) \;\wedge\; \Psi(\lambda d. \, b \sqsubseteq d) \;\equiv\; \mathcal{K}_a(\Phi \wedge \Psi).$$

Hence we have shown that $\mathcal{K}_a(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \implies \mathcal{K}_a(\Phi \wedge \Psi)$. Then by Lemma 10.8,

$$\begin{aligned} \mathcal{E}(\Phi \wedge \Psi)\xi \quad &\equiv \quad \exists a. \, \xi a \;\wedge\; \mathcal{K}_a(\Phi \wedge \Psi) \\ &\Rightarrow \quad \exists a. \, \xi a \;\wedge\; \mathcal{K}_a(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \;\equiv\; \mathcal{E}(\mathcal{E}\Phi \wedge \mathcal{E}\Psi). \end{aligned}$$ □

We leave the following similar but simpler results to the reader:

Lemma 10.10
(a) If $\mathcal{E}$ satisfies $\mathcal{E}(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \leqslant \mathcal{E}(\Phi \wedge \Psi)$ then $\prec\!\!\!\prec$ obeys the weak intersection rule;
(b) $\mathcal{E} \cdot \mathcal{E} \leqslant \mathcal{E}$ iff $\prec\!\!\!\prec$ is transitive; and
(c) $\mathcal{E}$ preserves meets iff $\prec\!\!\!\prec$ satisfies the strong intersection rule. □

In the Wilker rule it is convenient to consider existential quantification instead of binary disjunction:

Lemma 10.11 If $\prec\!\!\!\prec$ satisfies the Wilker rule

$$a \prec\!\!\!\prec k \implies \exists \ell. \, (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell. \, \exists c \in k. \, (b \prec\!\!\!\prec c),$$

then $\qquad \mathcal{K}_a(\exists i. \, \Phi_i) \implies \mathcal{K}_a(\exists i. \, \mathcal{E}\Phi_i) \qquad$ and so $\qquad \mathcal{E}(\exists i. \, \Phi_i) \leqslant \mathcal{E}(\exists i. \, \mathcal{E}\Phi_i)$

and in particular $\mathcal{E} \leqslant \mathcal{E} \cdot \mathcal{E}$.

Proof Using Lemma 10.8 several times, the Wilker rule in the second line and $h \equiv \{c\}$ half-way down,

$$\begin{aligned} \mathcal{K}_a(\exists i. \, \Phi_i) \quad &\equiv \quad \exists k. \, (a \prec\!\!\!\prec k) \;\wedge\; \forall c \in k. \, \exists i. \, \Phi_i(\lambda d. \, c \sqsubseteq d) \\ &\Rightarrow \quad \exists k \ell. \, (a \prec\!\!\!\prec \ell) \;\wedge\; \big(\forall b \in \ell. \, \exists c \in k. \, b \prec\!\!\!\prec c\big) \\ &\qquad\qquad \wedge \big(\forall c \in k. \, \exists i. \, \Phi_i(\lambda d. \, c \sqsubseteq d)\big) \\ &\Rightarrow \quad \exists \ell. \, (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell. \, \exists c i. \, (b \prec\!\!\!\prec c) \;\wedge\; \Phi_i(\lambda d. \, c \sqsubseteq d) \\ &\Rightarrow \quad \exists \ell. \, (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell. \, \exists i. \\ &\qquad\qquad \exists h. \, (b \prec\!\!\!\prec h) \;\wedge\; \forall c \in h. \, \Phi_i(\lambda d. \, c \sqsubseteq d) \\ &\Rightarrow \quad \exists \ell. \, (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell. \, \exists i. \, \mathcal{E}\Phi_i(\lambda c. \, b \sqsubseteq c) \\ &\equiv \quad \mathcal{K}_a(\exists i. \, \mathcal{E}\Phi_i). \end{aligned}$$

Then $\mathcal{E}(\exists i.\,\Phi_i)\xi \;\equiv\; \exists a.\,\xi a \land \mathcal{K}_a(\exists i.\,\Phi_i) \;\Longrightarrow\; \exists a.\,\xi a \land \mathcal{K}_a(\exists i.\,\mathcal{E}\Phi_i) \;\equiv\; \mathcal{E}(\exists i.\,\mathcal{E}\Phi_i)\xi.$ □

This completes the proof that $\mathcal{E}$ is a nucleus, so we can split it to define a locally compact locale by Theorem 8.14. We would also like to compare this construction with the one *via* Formal Topology, the logic of which we use for the following result:

Lemma 10.12 For the nucleus $\mathcal{E}$, with $B_u\xi \equiv \exists c \in u.\,\xi b$, we have

$$\mathcal{E}B_u\xi \iff \exists a\ell.\,\xi a \;\land\; a \prec\!\!\!\prec \ell \subset u,$$

and so $$(\mathcal{E}B_b \leqslant \mathcal{E}B_u) \iff (\forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u).$$

Proof

$$\begin{aligned}
\mathcal{E}B_u\xi \quad &\Leftrightarrow\quad \exists a\ell.\,\xi a \;\land\; a \prec\!\!\!\prec \ell \;\land\; \forall b \in \ell.\, B_u(\lambda c.\, b \sqsubseteq c)\\
&\Leftrightarrow\quad \exists a\ell.\,\xi a \;\land\; a \prec\!\!\!\prec \ell \;\land\; \forall b \in \ell.\,\exists c \in u.\, b \sqsubseteq c\\
&\Leftrightarrow\quad \exists a\ell\ell'.\,\xi a \;\land\; a \prec\!\!\!\prec \ell \sqsubseteq \ell' \subset u\\
&\Leftrightarrow\quad \exists a\ell'.\,\xi a \;\land\; a \prec\!\!\!\prec \ell' \subset u.\\
\text{Hence}\quad \mathcal{E}B_b \leqslant \mathcal{E}B_u \quad &\Leftrightarrow\quad \forall \xi a.\,\big(\xi a \;\land\; a \prec\!\!\!\prec b \;\Longrightarrow\; \exists a'\ell.\,\xi a' \;\land\; a' \prec\!\!\!\prec \ell \subset u\big)\\
&\Leftrightarrow\quad \forall a.\,(a \prec\!\!\!\prec b \;\Longrightarrow\; \exists \ell.\, a \prec\!\!\!\prec \ell \subset u),
\end{aligned}$$

where $\Leftarrow$ in the last line is easy and we obtain $\Rightarrow$ by putting $\xi \equiv \lambda a'.\, a \sqsubseteq a'$. □

Theorem 10.13 Every abstract basis defines a locally compact formal cover, locale, sober space and object in Abstract Stone Duality. Hence the formation rule for an ASD type X in Definition 9.11 and the elimination rule for Σ^X in Definition 9.13 are justified.

Proof Having verified the equations for a nucleus we may invoke Theorem 8.14, in which the formal cover agrees with that in Section 7. On the other hand, a nucleus directly justifies the introduction of an object in ASD according to its previous literature [B] [G]. This object is a Σ-split subspace of $\mathbf{Filt}(A, \sqsubseteq)$ and carries a concrete basis using the λ-terms

$$B_\ell \;\equiv\; \lambda\xi.\,\exists b \in \ell.\,\xi b \qquad\text{and}\qquad \mathcal{K}_a \;\equiv\; \lambda\Phi.\,\mathcal{E}\Phi(\lambda b.\, a \sqsubseteq b),$$

and the way-below relation is $\mathcal{K}_a B_\ell \iff (a \prec\!\!\!\prec \ell)$ by Lemma 10.8. □

Warning 10.14 If you want to define a new space from $(A, \sqsubseteq, \prec\!\!\!\prec)$, where $a \prec\!\!\!\prec \ell \equiv A_a\beta_\ell$ for some terms, it is advisable to check the Wilker and intersection rules explicitly and not assume that they will follow automatically, *cf.* Theorem 12.10.

Now that we have established the equivalence between abstract bases and nuclei we turn to that between their respective notions of formal point, in Definition 5.1 and Lemma 8.10.

Lemma 10.15 If ξ is admissible then it is ***rounded***, $\xi c \iff \exists a.\,\xi a \land (a \prec\!\!\!\prec c)$.
Conversely, if ξ is rounded then $\mathcal{E}^0\Phi\xi \Longrightarrow \mathcal{E}\Phi\xi$ for any Φ.

Proof Consider $\Phi \equiv \lambda\zeta.\,\zeta c$ and use covariance for $\ell \sqsubseteq \{c\}$.

$$\begin{aligned}
\xi c \quad &\equiv\quad \Phi\xi \iff \mathcal{E}\Phi\xi && \text{def } \Phi\\
&\equiv\quad \exists a\ell.\,\xi a \;\land\; (a \prec\!\!\!\prec \ell) \;\land\; \forall b \in \ell.\,(\lambda c'.\, b \sqsubseteq c')c && \text{Lemma 10.8}\\
&\Leftrightarrow\quad \exists a.\,\xi a \;\land\; \exists \ell.\, a \prec\!\!\!\prec \ell \sqsubseteq c && \text{Notation 1.12}\\
&\Leftrightarrow\quad \exists a.\,\xi a \land (a \prec\!\!\!\prec c). && \text{covariance for } \ell \sqsubseteq \{c\}\\
\mathcal{E}^0\Phi\xi \quad &\equiv\quad \exists b.\,\xi b \;\land\; \Phi(\lambda c.\, b \sqsubseteq c) && \text{Lemma 10.1}\\
&\Leftrightarrow\quad \exists ab.\,\xi a \;\land\; (a \prec\!\!\!\prec b) \;\land\; \Phi(\lambda c.\, b \sqsubseteq c) && \text{rounded}\\
&\equiv\quad \mathcal{E}^1\Phi\xi \;\Longrightarrow\; \mathcal{E}\Phi\xi,
\end{aligned}$$

where the intermediate term $\mathcal{E}^1\Phi\xi \equiv \exists ab.\, \xi a \wedge (a \prec\!\!\prec b) \wedge \Phi(\lambda c.\, b \sqsubseteq c)$ will become relevant in Example 13.3 $\square$

Lemma 10.16 If ξ is admissible then it is ***located***,

$$\xi a \ \wedge\ (a \prec\!\!\prec \ell) \implies \exists b.\, \xi b \ \wedge\ (b \in \ell).$$

Conversely, if ξ is located then $\mathcal{E}\Phi\xi \implies \mathcal{E}^0\Phi\xi$ for any Φ.

In particular, if a is empty $(a \prec\!\!\prec \circ)$ then $\xi a \Leftrightarrow \bot$.

Proof Consider $\Phi \equiv \lambda\zeta.\, \exists b \in \ell.\, \zeta b$. Then with $k \equiv \ell$ and $b \equiv d$,

$$\begin{array}{rcll}
\xi a \ \wedge\ (a \prec\!\!\prec \ell) & \Rightarrow & \exists ak.\, \xi a \ \wedge\ (a \prec\!\!\prec k) \ \wedge\ \forall d \in k.\, \exists b \in \ell.\, d \sqsubseteq b & \\
& \equiv & \mathcal{E}\Phi\xi \iff \Phi\xi \ \equiv\ \exists b \in \ell.\, \xi b. & \text{def } \mathcal{E},\ \Phi \\
\mathcal{E}\Phi\xi & \equiv & \exists a\ell.\, \xi a \wedge (a \prec\!\!\prec \ell) \wedge \forall b \in \ell.\, \Phi(\lambda c.\, b \sqsubseteq c) & \text{def } \mathcal{E} \\
& \Rightarrow & \exists b\ell.\, \xi b \wedge (b \in \ell) \wedge \forall b' \in \ell.\, \Phi(\lambda c.\, b' \sqsubseteq c) & \text{located} \\
& \Rightarrow & \exists b.\, \xi b \ \wedge\ \Phi(\lambda c.\, b \sqsubseteq c) \ \equiv\ \mathcal{E}^0\Phi\xi. & \text{Lemma 10.1} \quad \square
\end{array}$$

Lemma 10.17 If ξ is admissible then it is ***covariant***, $\xi a \ \wedge\ (a \sqsubseteq b) \implies \xi b$.

Proof We need to be careful because the verbatim proof of Lemma 10.2 gave roundedness instead. For $a \in A$, let $\Phi_a \equiv \lambda\zeta.\, \zeta a$, so by Lemma 10.8,

$$\mathcal{E}\Phi_a\xi \iff \exists c\ell.\, \xi c \ \wedge\ (c \prec\!\!\prec \ell) \ \wedge\ \forall d \in \ell.\, d \sqsubseteq a.$$

Then, for admissible ξ, since $\sqsubseteq$ is transitive we have

$$a \sqsubseteq b \quad \vdash \quad \xi a \iff \Phi_a\xi \iff \mathcal{E}\Phi_a\xi \implies \mathcal{E}\Phi_b\xi \iff \Phi_b\xi \iff \xi b$$

and the stated result follows from the Gentzen rule. $\square$

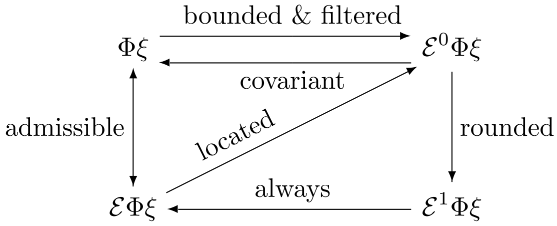
Lemma 10.18 If ξ is admissible then it is ***bounded*** and ***filtered***,

$$\exists a.\, \xi a \qquad \text{and} \qquad \xi b \ \wedge\ \xi c \implies \exists a.\, \xi a \ \wedge\ (b \sqsupseteq a \sqsubseteq c).$$

Proof The proof of boundedness is the same as in Lemma 10.2, but that for filteredness uses the roundedness property above. With $\Phi \equiv \lambda\zeta.\, \zeta b \wedge \zeta c$ as before,

$$\begin{array}{rcl}
\xi b \ \wedge\ \xi c & \equiv & \Phi\xi \iff \mathcal{E}\Phi\xi \\
& \Rightarrow & \mathcal{E}^0\Phi\xi \ \equiv\ \exists a.\, \xi a \ \wedge\ \Phi(\lambda d.\, a \sqsubseteq d) \ \equiv\ \exists a.\, \xi a \ \wedge\ (b \sqsupseteq a \sqsubseteq c).
\end{array}$$ $\square$

Proposition 10.19 A term $\xi : \Sigma^A$ is admissible for $\mathcal{E}$ iff it is rounded, bounded, covariant, filtered and located for $\prec\!\!\prec$. Hence the introduction and elimination rules for terms of type X and the introduction rule for Σ^X in Definitions 9.12 and 9.13 are justified.



Proof The preceding lemmas deduce the other properties from admissibility. Conversely, if ξ is rounded and located then $\mathcal{E}^0\Phi\xi \Leftrightarrow \mathcal{E}\Phi\xi$ by Lemmas 10.15 and 10.16, whilst if it is bounded, covariant and filtered then $\mathcal{E}^0\Phi\xi \Leftrightarrow \Phi\xi$ by Lemmas 10.3 and 10.4. □

This completes the proof of the soundness and completeness of the axioms for a abstract basis as an account of concrete bases for locally compact sober spaces, locales, formal topologies and objects of ASD.

11 Bases using compact subspaces

We began this paper with a natural definition of basis that uses compact subspaces, but in most of our discussion we have used Scott-open families instead, at the cost of a weaker rule for intersections. In this section we use Lawson's Lemma 3.10 and the axiom of Dependent Choice to convert the weaker forms to the stronger ones. We consider the result for abstract bases in detail first and the concrete one afterwards.

Notation 11.1 Let $(A, \sqsubseteq, \prec\!\!\!\prec)$ be an abstract basis that satisfies the single interpolation rule. A ***Lawson sequence*** $\vec{a}$ is one of the form

$$a_\infty \prec\!\!\!\prec \cdots \prec\!\!\!\prec a_2 \prec\!\!\!\prec a_1 \prec\!\!\!\prec a_0, \qquad \text{that is,} \qquad \forall i < \infty.\, a_\infty \prec\!\!\!\prec a_i \ \wedge\ a_{i+1} \prec\!\!\!\prec a_i,$$

and we let $\vec{A}$ be the set of such sequences. We write $\vec{\ell}$ for a list or finite subset of $\vec{A}$ (not a sequence of unrelated lists) and ℓ_∞ for the list $\{b_\infty \mid \vec{b} \in \vec{\ell}\}$. Then we define

$$a \prec\!\!\!\prec \vec{\ell} \ \equiv\ a \prec\!\!\!\prec \ell_\infty \qquad\qquad \vec{b} \prec\!\!\!\prec k \ \equiv\ \exists i < \infty.\ b_i \prec\!\!\!\prec k$$

and $$\vec{a} \prec\!\!\!\prec \vec{\ell} \ \equiv\ \exists i < \infty.\, a_i \prec\!\!\!\prec \ell_\infty \ \equiv\ \exists i < \infty.\, a_i \prec\!\!\!\prec \vec{\ell} \ \equiv\ \vec{a} \prec\!\!\!\prec \ell_\infty.$$

As in Lemma 3.6, $\vec{a} \sqsubseteq \vec{b}$ is given by $(\vec{a} \prec\!\!\!\prec \vec{b}) \vee (\vec{a} = \vec{b})$.

Lemma 11.2 Lawson sequences may be interpolated between individual basis elements and between lists of them:

$$a \prec\!\!\!\prec k \iff \exists \vec{b}.\, a \prec\!\!\!\prec \vec{b} \prec\!\!\!\prec k \qquad \text{and} \qquad k \prec\!\!\!\prec^1 \ell \implies \exists \vec{h}.\, k \prec\!\!\!\prec^1 \vec{h} \prec\!\!\!\prec^1 \ell.$$

Proof By repeated single interpolation, as in Lemma 3.10, given $a \prec\!\!\!\prec k$ there are

$$a \prec\!\!\!\prec b_\infty \prec\!\!\!\prec \cdots \prec\!\!\!\prec b_2 \prec\!\!\!\prec b_1 \prec\!\!\!\prec b_0 \prec\!\!\!\prec k,$$

so $a \prec\!\!\!\prec b_\infty$ and $\exists i < \infty.\, b_i \prec\!\!\!\prec k$, and conversely. Since $(k \prec\!\!\!\prec^1 \ell) \equiv \forall b \in k.\, \exists c \in \ell.\, b \prec\!\!\!\prec c$, the second part iterates the first over the list k. □

Lemma 11.3 Transitivity, single interpolation and boundedness hold:

$$\vec{a} \prec\!\!\!\prec \vec{k} \prec\!\!\!\prec \vec{\ell} \implies \vec{a} \prec\!\!\!\prec \vec{\ell} \implies \exists \vec{b}.\, \vec{a} \prec\!\!\!\prec \vec{b} \prec\!\!\!\prec \vec{\ell} \qquad \text{and} \qquad \exists \vec{b}.\, \vec{a} \prec\!\!\!\prec \vec{b}.$$

Proof By interpolation, using the previous result,

$$\begin{aligned}
\vec{a} \prec\!\!\!\prec \vec{k} \prec\!\!\!\prec \vec{\ell} \quad &\equiv \quad \exists i.\, a_i \prec\!\!\!\prec k_\infty \ \wedge\ \forall \vec{b} \in \vec{\ell}.\, \exists j.\ b_\infty \prec\!\!\!\prec b_j \prec\!\!\!\prec \ell_\infty \\
&\Rightarrow \quad \exists i.\, a_i \prec\!\!\!\prec k_\infty \prec\!\!\!\prec \ell_\infty \implies \vec{a} \prec\!\!\!\prec \vec{\ell} \\
\vec{a} \prec\!\!\!\prec \vec{\ell} \quad &\equiv \quad \exists i.\ a_{i+1} \prec\!\!\!\prec a_i \prec\!\!\!\prec \ell_\infty \\
&\Rightarrow \quad \exists \vec{b} i.\ a_{i+1} \prec\!\!\!\prec \vec{b} \prec\!\!\!\prec a_i \prec\!\!\!\prec \ell_\infty \implies \exists \vec{b}.\ \vec{a} \prec\!\!\!\prec \vec{b} \prec\!\!\!\prec \vec{\ell}
\end{aligned}$$

and for any $\vec{a}$, interpolate $a_2 \prec\!\!\prec a_1 \prec\!\!\prec \vec{b} \prec\!\!\prec a_0$, so that $\vec{a} \prec\!\!\prec \vec{b}$. □

Lemma 11.4 The Wilker and strong intersection rules hold in the form

$$(\vec{a} \prec\!\!\prec \vec{k}) \;\wedge\; (\vec{a} \prec\!\!\prec \vec{\ell}) \implies \exists\vec{h}.\, (\vec{a} \prec\!\!\prec \vec{h} \prec\!\!\prec^1 \vec{k}) \;\wedge\; (\vec{h} \prec\!\!\prec^1 \vec{\ell}).$$

Proof We use the greater of i and j, the weak intersection and Wilker rules in A (*cf.* Lemma 3.6) and the list form of Lawson interpolation:

$$\begin{aligned}
(\vec{a} \prec\!\!\prec \vec{k}) \;\wedge\; (\vec{a} \prec\!\!\prec \vec{\ell}) \;&\equiv\; (\exists i < \infty.\, a_i \prec\!\!\prec k_\infty) \;\wedge\; (\exists j < \infty.\, a_j \prec\!\!\prec \ell_\infty) \\
&\Rightarrow\; \exists i < \infty.\;\, a_{i+1} \prec\!\!\prec a_i \prec\!\!\prec k_\infty,\; \ell_\infty \\
&\Rightarrow\; \exists i < \infty.\, \exists h'h''.\;\, a_{i+1} \prec\!\!\prec h'' \prec\!\!\prec^1 h' \sqsubseteq k_\infty,\; \ell_\infty \\
&\Rightarrow\; \exists h'h''\vec{h}.\;\, \vec{a} \prec\!\!\prec h'' \prec\!\!\prec^1 \vec{h} \prec\!\!\prec^1 h' \sqsubseteq k_\infty,\; \ell_\infty \\
&\Rightarrow\; \exists \vec{h}.\;\, \vec{a} \prec\!\!\prec \vec{h} \prec\!\!\prec^1 \vec{k} \;\wedge\; \vec{h} \prec\!\!\prec^1 \vec{\ell}.
\end{aligned}$$
□

Lemma 11.5 If the given basis $(A, \sqsubseteq, \prec\!\!\prec)$ is bounded below, has rounded unions is positive or prime then $\vec{A}$ has the same property.

Proof Given $\vec{a}$, we have $c \prec\!\!\prec a_\infty$ since A is bounded below and then $c \prec\!\!\prec \vec{b} \prec\!\!\prec a_\infty$, so $\vec{b} \prec\!\!\prec \vec{a}$.

Similarly, if $\vec{b}, \vec{c} \prec\!\!\prec \vec{a}$ then $b_i \prec\!\!\prec a_\infty$ and $c_j \prec\!\!\prec a_\infty$, so there are d and $\vec{e}$ with $b_i \prec\!\!\prec d \prec\!\!\prec \vec{e} \prec\!\!\prec a_\infty$ and $c_j \prec\!\!\prec b$. Hence $\vec{b}, \vec{c} \prec\!\!\prec \vec{e} \prec\!\!\prec \vec{a}$.

For positivity, $\vec{a} \prec\!\!\prec \vec{\circ} \iff \exists i.\, a_i \prec\!\!\prec \circ$.

For primality, $\vec{a} \prec\!\!\prec \vec{\ell} \equiv \exists i.\, a_i \prec\!\!\prec \ell_\infty \implies \exists ib.\, a_i \prec\!\!\prec b \in \ell_\infty \implies \exists \vec{b}.\, \vec{a} \prec\!\!\prec \vec{b} \in \vec{\ell}$,
where $\vec{b} \in \vec{\ell}$ is the member for which $b_\infty = b \in \ell_\infty$. □

Theorem 11.6 Any abstract basis that satisfies the single interpolation rule is isomorphic to one that also satisfies boundedness above and the strong intersection rule, by

$$\langle\, a \,|\, f \,|\, \vec{b} \,\rangle \;\equiv\; a \prec\!\!\prec \vec{b} \qquad \text{and} \qquad \langle\, \vec{b} \,|\, g \,|\, a \,\rangle \;\equiv\; \vec{b} \prec\!\!\prec a.$$

Proof Lemma 3.6 provides $\sqsubseteq$ for the new basis, all of the properties of which we have proved. We may show that these matrices have the required properties and are inverse by similar methods. In particular, they are both uniformly bounded, $\langle\, a \,|\, f \,|\, \vec{b} \,\rangle$ is uniformly weakly filtered and $\langle\, \vec{b} \,|\, g \,|\, a \,\rangle$ is strongly but non-uniformly filtered. □

Notation 11.7 Now let $(U_a, \mathcal{K}_a)$ be a concrete basis for a space X using Scott-open families that is indexed by $(A, \sqsubseteq, \prec\!\!\prec)$. We define an $\vec{A}$-indexed basis for the same space by

$$U_{\vec{a}} \;\equiv\; U_{a_\infty} \qquad \text{and} \qquad \mathcal{K}_{\vec{a}} \;\equiv\; \bigcup^{\uparrow}\{\mathcal{K}_{a_i} \mid i < \infty\}.$$

Lemma 11.8 These have the variance properties and agree with the way-below relation.

Proof By Lemma 3.6, if $\vec{a} \prec\!\!\prec \vec{b} \equiv \exists i.\, a_\infty \prec\!\!\prec a_i \prec\!\!\prec b_\infty$ then $U_{\vec{a}} = U_{a_\infty} \subset U_{b_\infty} = U_{\vec{b}}$ and

$$\vec{a} \prec\!\!\prec \vec{b} \implies \exists i.\, \forall j.\, a_i \prec\!\!\prec b_\infty \prec\!\!\prec b_j \implies \forall j < \infty.\, \exists i < \infty.\, \mathcal{K}_{a_i} \supset \mathcal{K}_{b_j}$$

so $$\mathcal{K}_{\vec{a}} \equiv \bigcup\{\mathcal{K}_{a_i} \mid i < \infty\} \;\supset\; \bigcup\{\mathcal{K}_{b_j} \mid j < \infty\} \equiv \mathcal{K}_{\vec{b}}\,.$$

Also $$\mathcal{K}_{\vec{a}} \ni U_{\vec{\ell}} \iff \exists i < \infty.\, \mathcal{K}_{a_i} \ni U_{\ell_\infty} \iff \exists i < \infty.\, a_i \prec\!\!\prec \ell_\infty \;\equiv\; \vec{a} \prec\!\!\prec \vec{\ell}.$$
□

Lemma 11.9 The filter property for concrete bases is satisfied.

Proof If $x \in U_{\vec{a}} \equiv U_{a_\infty}$ and $x \in U_{\vec{b}} \equiv U_{b_\infty}$ then there is some c with $x \in U_c$ and $a_\infty \sqsupseteq c \sqsubseteq b_\infty$. By the basis expansion of U_c and Lemma 11.2 there are $e \prec\!\!\!\prec \vec{d} \prec\!\!\!\prec c$ with $x \in U_e \subset U_{d_\infty} = U_{\vec{d}} \subset U_c$, so $\vec{a} \sqsupseteq \vec{d} \sqsubseteq \vec{b}$. □

Lemma 11.10 The basis expansion is satisfied.

Proof We use the basis expansion in A twice and then interpolate $b \prec\!\!\!\prec \vec{a} \prec\!\!\!\prec c$:

$$\begin{aligned} x \in U \quad &\Rightarrow \quad \exists b.\, x \in U_b \ \wedge\ \mathcal{K}_b \ni U \\ &\Rightarrow \quad \exists bc.\, x \in U_b \ \wedge\ \mathcal{K}_b \ni U_c \ \wedge\ \mathcal{K}_c \ni U \\ &\Rightarrow \quad \exists \vec{a}.\, x \in U_{a_\infty} \ \wedge\ \mathcal{K}_{a_\infty} \supset \mathcal{K}_{\vec{a}} \equiv \bigcup_{i<\infty} \mathcal{K}_{a_i} \supset \mathcal{K}_{a_0} \ni U \\ &\Rightarrow \quad \exists \vec{a}.\, x \in U_{\vec{a}} \ \wedge\ \mathcal{K}_{\vec{a}} \ni U \\ &\Rightarrow \quad \exists \vec{a}.\, x \in U_{a_\infty} \ \wedge\ \mathcal{K}_{a_\infty} \ni U \\ &\Rightarrow \quad \exists b.\, x \in U_b \ \wedge\ \mathcal{K}_b \ni U \implies x \in U. \end{aligned}$$

□

Theorem 11.11 Every locally compact sober topological space has a basis using compact subspaces.

Proof We have constructed an abstract basis that satisfies the strong intersection rule and a concrete one whose Scott-open families are filters by Lemma 3.11. Hence these are the neighbourhood filters of compact subspaces by Proposition 3.14. □

Remark 11.12 The other parts of the theory of abstract bases *per se* can be carried out in a world that has nothing more powerful than the ability to manipulate the type of all *finite* subsets or lists. This setting is called an ***arithmetic universe***.

How can we accommodate Lawson sequences into this view?

In fact, we do not need them in *general*, just the ability to *interpolate* such a sequence given its endpoints $(a_\infty \prec\!\!\!\prec a_0)$. Classically, we use the interpolation property and the axiom of Dependent Choice to do this. However, if we are not going to *remember* our choices, we need some way of ensuring that we recover the same result if we repeat the selection process.

In the *free* arithmetic universe, the subobjects are recursively enumerable. Therefore, by imposing a fixed way of scheduling parallel computations, we have a deterministic way of selecting an element of any inhabited subobject.

The reason for using the formulation of abstract bases without $\sqsubseteq$ here (Lemma 3.6) is that we cannot expect this choice to respect $\sqsubseteq$. (Ideally, given $a_\infty \prec\!\!\!\prec a_i$ and $b_\infty \prec\!\!\!\prec b_j$ with $a_\infty \sqsubseteq b_\infty$ and $a_i \prec\!\!\!\prec b_j$, we would like to chose interpolants such that $a_{i+1} \prec\!\!\!\prec b_{j+1}$ too.)

This means that, instead of working with "the whole" infinite sequence $a_\infty \prec\!\!\!\prec \cdots a_2 \prec\!\!\!\prec a_1 \prec\!\!\!\prec a_0$, we may regard it as being encoded by its endpoints $(a_\infty \prec\!\!\!\prec a_0)$ along with as many iterates as we require of an interpolation operator μ that takes $a_\infty \prec\!\!\!\prec a_i$ to $a_\infty \prec\!\!\!\prec a_{i+1}$. Then we define

$$a \prec\!\!\!\prec (b_\infty \prec\!\!\!\prec b_0) \ \equiv\ a \prec\!\!\!\prec b_\infty \qquad \text{and} \qquad (b_\infty \prec\!\!\!\prec b_0) \prec\!\!\!\prec a \ \equiv\ \exists i.\, \mu^i(b_\infty \prec\!\!\!\prec b_0) \prec\!\!\!\prec a,$$

the latter being understood as $c \prec\!\!\!\prec a$ where $\mu^i(b_\infty \prec\!\!\!\prec b_0) = (b_\infty \prec\!\!\!\prec c)$. After Lemma 11.2 above, we need no further analysis than this of the meaning of the sequence $\vec{a}$.

Therefore Theorem 11.6 is valid in any arithmetic universe that has such a deterministic choice operation on inhabited subobjects, in particular in the free one. □

Presumably this is also valid in Martin-Löf Type Theory.

Is there a counterexample in a locale in a topos without Dependent Choice?

This completes the proof, which we began in Section 3, that abstract bases may be taken to satisfy all of the additional "convenient" properties in Definitions 1.10 and 1.11 as well as the primary ones in Definition 1.8. The benefit of assuming all of the roundedness rules is this:

Lemma 11.13 Let $(A, \sqsubseteq, \prec\!\!\!\prec)$ be a basis with these roundedness properties and $\phi(a)$ be a formula built from a free variable $a : A$, the way-below relation $\prec\!\!\!\prec$, the connectives $\wedge$ and $\vee$ and quantification over lists. Then ϕ is ***rounded*** in a:

$$\phi(a) \implies \exists a^- a^+.\, (a^- \prec\!\!\!\prec a \prec\!\!\!\prec a^+) \;\wedge\; \forall b.\, (a^- \prec\!\!\!\prec b \prec\!\!\!\prec a^+) \Rightarrow \phi(b). \qquad \square$$

We now turn from the general theory to some important special cases.

12 Overt spaces

Overtness has arisen independently under various names in several constructive disciplines: located subspaces in Constructive Analysis, open locales, positivity in Formal Topology and liveness in Process Algebra. It is often said to be invisible classically, but the ideas that we need in this section were actually introduced in Section 5 when we tried to construct a traditional topological space directly from an abstract basis.

It is easiest to give the initial definition of this concept using Abstract Stone Duality, but we then characterise it using abstract bases and formal covers and work with these. Finally we prove a Theorem that links Topology to Computability.

Topologically, overtness is the lattice dual of compactness, the latter being related to the universal quantifier. For example, whereas a compact subspace of a Hausdorff space is closed, so an overt subspace of a discrete space is open. Similarly, an open subspace or direct image of an overt subspace is again overt. These ideas are explored in the context of real analysis in [J].

Definition 12.1 A space X is ***overt*** if it has a term $\exists_X : \Sigma^X \to \Sigma$ that obeys the rules for ***existential quantification***:

$$\frac{\ldots,\; x : X \quad \vdash \quad \phi x \Longrightarrow \sigma}{\ldots \quad \vdash \quad \exists x.\, \phi x \Longrightarrow \sigma}$$

In classical topology, where the Sierpiński space Σ just has the two points $\top$ and $\bot$, we just have $\exists_X U \;\equiv\; (U \neq \emptyset)$ in any space. However, it is actually the points rather than Excluded Middle that make overtness trivial (Remark 12.5).

Overt locales were first studied by André Joyal, Miles Tierney [JT84] and Peter Johnstone [Joh84], who called them *open* because for them $!_X : X \to \mathbf{1}$ is an open map, *i.e.* there is a left adjoint $\exists_X \dashv !_X^*$ satisfying the Frobenius law below. The name needed to be changed because overt *sub*spaces are often *closed.*

Lemma 12.2 A space X is overt iff there is a term $\Diamond : \Sigma^X \to \Sigma$ that satisfies

$$\Diamond\,\bot \iff \bot \qquad \text{and} \qquad x : X,\; \phi : \Sigma^X \;\vdash\; \phi x \Longrightarrow \Diamond\,\phi.$$

Then $\Diamond \equiv \exists_X$ and this also preserves joins and satisfies the ***Frobenius law***

$$\Diamond\,(\sigma \wedge \phi) \iff \sigma \;\wedge\; \Diamond\,\phi$$

for any $\sigma : \Sigma$ and $\phi : \Sigma^X$, *cf.* Remark 9.8(f).

Proof These are consequences of the Phoa Principle (Remark 9.6) and the adjunction $\exists_X \dashv !_X^*$. See [J] for discussion. $\square$

Definition 12.3 More generally,

(a) an ***overt subspace*** of a (not necessarily overt) space X is by definition any operator $\lozenge : \Sigma^X \to \Sigma$ that preserves joins;

(b) it is said to be ***inhabited*** if $\lozenge \top \Leftrightarrow \top$; and

(c) a point $x : X$ is an ***accumulation point*** of $\lozenge$ if $\phi x \Longrightarrow \lozenge \phi$ for all $\phi : \Sigma^X$.

Therefore $\lozenge \equiv \exists_X$ makes any overt space X into an overt subspace of itself for which every $x : X$ is an accumulation point.

For subspaces, we may regard the accumulation points as providing the extent of $\lozenge$. However, we can only understand overtness if we regard them as secondary to the operator (or its equivalent positivity). Sometimes there is an equivalent open or closed subspace, as explained in [J], but in general the extent need not be locally compact.

Beware that being inhabited does not mean *ā priori* that the subspace has an accumulation point: we have a Theorem to prove about this.

Example 12.4 For any sequence $f : \mathbb{N} \to X$, the operator $\lozenge U \equiv \exists n.\, fn \in U$ defines an overt subspace. In this, the limit of any convergent subsequence is an accumulation point (hence the name). Any recursively enumerable subset may be used here in place of $\mathbb{N}$ itself. □

Remark 12.5 Indeed we may replace $\mathbb{N}$ here with anything that we consider to be a "set" in whichever logical foundations we are using. This means that any space that has enough points (Warning 6.12) is overt.

Indeed, for any $\lozenge$ operator in Point–Set Topology with Excluded Middle,

$$\begin{aligned} V &\equiv \{x \mid \exists U.\, x \in U \wedge \neg \lozenge U\} = \bigcup \{U \mid \neg \lozenge U\} \\ C &\equiv \{x \mid \forall U.\, x \in U \Rightarrow \lozenge U\} \end{aligned}$$

are complementary open and closed subspaces such that $\lozenge U \iff U \between C$. □

Leaving the uninteresting classical case behind, the preservation of joins invites characterisation in terms of the basis:

Proposition 12.6 Overt subspaces correspond bijectively to ***positivities***. These are subsets $r \subset A$ of the basis that are rounded and located, or equivalently upper and positive,

$$r \ni b \Longrightarrow \exists a.\, r \ni a \prec\!\!\!\prec b \qquad \text{and} \qquad r \ni a \prec\!\!\!\prec \ell \Longrightarrow \exists b.\, r \ni b \in \ell$$

or $$r \ni a \sqsubseteq b \Longrightarrow r \ni b \qquad \text{and} \qquad r \ni a \lhd u \Longrightarrow \exists b.\, r \ni b \in u,$$

where $$r \equiv \{a \mid \lozenge U_a\} \qquad \text{and} \qquad \lozenge U \iff \exists a.\ (a \in r) \wedge \mathcal{K}_a \ni U.$$

Then a formal point $p \subset A$ is an accumulation point of $\lozenge$ iff $p \subset r \subset A$.

Proof Proposition 4.13 and (the proof of) Lemma 7.11 characterised the subset r. We recover $\lozenge$ from r by the basis expansion and r from $\lozenge$ by roundedness. The containment $p \subset r$ is the restriction of the definition of an accumulation point to the basis and this is recovered for the same reason. □

Now we turn to the characterisation of overt spaces in terms of $\lhd$ and $\prec\!\!\!\prec$.

Lemma 12.7 If a space has a positive basis (with no $a \prec\!\!\!\prec \circ$) then it is overt.

Proof Let $\lozenge U \equiv \exists a.\, \mathcal{K}_a \ni U$, so by hypothesis

$$\lozenge \bot \equiv \lozenge U_\circ \equiv \exists a.\, \mathcal{K}_a \ni U_\circ \equiv \exists a.\, (a \prec\!\!\!\prec \circ) \iff \bot.$$

Then $\lozenge$ is $\exists_X$ by Lemma 12.2 because, by the basis expansion,

$$x \in U \iff \exists a.\, x \in U_a \ \wedge\ \mathcal{K}_a \ni U \implies \exists a.\, \mathcal{K}_a \ni U \ \equiv\ \lozenge U.$$

So long as the given basis has single interpolation, the positivity is $A^+ \equiv \{b \mid \exists a.\, a \prec\!\!\!\prec b\}$, *cf.* Lemma 3.5. $\square$

However, we cannot obtain a positive basis for an overt space "negatively" by just omitting the a with $a \prec\!\!\!\prec \circ$.

Notation 12.8 For any overt space X with concrete basis $(U_a, \mathcal{K}_a)$ indexed by $(A, \sqsubseteq, \prec\!\!\!\prec)$, let

$$A^+ \ \equiv\ \{a \mid \exists x.\, x \in U_a\} \subset A.$$

This is the positivity that corresponds to $\lozenge \equiv \exists_X$ by Proposition 12.6. Since it is located, we never have $A^+ \ni a \prec\!\!\!\prec \circ$.

The key result is (our version of) a lemma that Peter Johnstone discovered [Joh84, Lemma 2.5] on a ferry journey while he was investigating locale theory *without* excluded middle. He stated it as $a \ll (b+c) \implies (a \ll b) \vee (c \in A^+)$, which in our notation is $a \prec\!\!\!\prec \ell \sqcup \{c\} \implies (a \prec\!\!\!\prec \ell) \vee (c \in A^+)$.

Whilst it may appear that we are simply cutting ℓ down to its intersection with A^+, doing that need not yield a constructively finite subset [Kur20]. However, Scott openness of $\mathcal{K}_a$ does provide some suitable finite $k \subset \ell \cap A^+$.

Lemma 12.9 $\quad a \prec\!\!\!\prec \ell \iff \exists k.\, a \prec\!\!\!\prec k \sqsubseteq \ell \ \wedge\ k \subset A^+$.

Proof $\quad \mathcal{K}_a \ni U_\ell \iff \mathcal{K}_a \ni \bigcup\{U_b \mid b \in \ell \cap A^+\} \iff \exists k.\, \mathcal{K}_a \ni U_k \ \wedge\ k \subset \ell \cap A^+$. $\square$

In particular, $a \prec\!\!\!\prec c \implies (a \prec\!\!\!\prec \circ) \vee (c \in A^+)$. We use the form above to eliminate empty basic subspaces from the interpolants that are provided by the Wilker and intersection rules for the given basis:

Theorem 12.10 A space is overt iff it has a positive abstract basis.

Proof $\quad$ We may restrict the basis expansion to A^+ because

$$x \in U \iff \exists a.\, x \in U_a \ \wedge\ \mathcal{K}_a \ni U \iff \exists a.\, x \in U_a \ \wedge\ a \in A^+ \ \wedge\ \mathcal{K}_a \ni U.$$

It still obeys the filter property because in the statement

$$x \in U_a \ \wedge\ x \in U_b \implies \exists c.\, x \in U_c \ \wedge\ (a \sqsupseteq c \sqsubseteq b),$$

we have $a, b, c \in A^+$. Hence the concrete basis $(U_a, \mathcal{K}_a)$ may be cut down to A^+.

Now we prove the Wilker and weak intersection rules that make $(A^+, \sqsubseteq, \prec\!\!\!\prec)$ an abstract basis. First we apply them for the given basis A and then we use Johnstone's lemma to reduce the interpolant:

$$\begin{array}{lcl}
a \prec\!\!\!\prec \ell & \Rightarrow & \exists k.\, a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell \\
 & \Rightarrow & \exists kh.\, a \prec\!\!\!\prec h \sqsubseteq k \prec\!\!\!\prec^1 \ell \ \wedge\ (h \subset A^+) \\
 & \Rightarrow & \exists h \subset A^+.\, a \prec\!\!\!\prec h \prec\!\!\!\prec^1 \ell \\
a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1 \ \wedge\ k \prec\!\!\!\prec \ell_2 & \Rightarrow & \exists \ell'.\, a \prec\!\!\!\prec \ell' \sqsubseteq \ell_1 \sqcap \ell_2 \\
 & \Rightarrow & \exists \ell' h.\, a \prec\!\!\!\prec h \sqsubseteq \ell' \sqsubseteq \ell_1 \sqcap \ell_2 \ \wedge\ (h \subset A^+) \\
 & \Rightarrow & \exists h \subset A^+.\, a \prec\!\!\!\prec h \sqsubseteq \ell_1 \sqcap \ell_2.
\end{array}$$

Finally, since A^+ is located, $A^+ \ni a \prec\!\!\!\prec \ell \Longrightarrow \exists b.\, b \in \ell \cap A^+$, so A^+ is a positive basis. □

In Formal Topology, the usual definition of overtness is this:

Theorem 12.11 A space is overt iff there is a positivity $r \subset A$ such that $b \lhd b^+ \equiv \{b\} \cap r$.

Proof Suppose that there is such a positivity and let $\lozenge$ be the corresponding operator by Proposition 12.6, so $\lozenge U \equiv \exists a \in r.\, \mathcal{K}_a \ni U$. Then

$$\lozenge \bot \;\equiv\; \exists a \in r.\, \mathcal{K}_a \ni U_\circ \;\equiv\; \exists a \in r.\, a \prec\!\!\!\prec \circ \;\Longleftrightarrow\; \bot.$$

If $x \in U_a$ then $a \in r$ since $a \lhd \{a\} \cap r$, so

$$\begin{aligned} x \in U \quad &\Leftrightarrow \quad \exists a.\, x \in U_a \;\wedge\; \mathcal{K}_a \ni U \\ &\Rightarrow \quad \exists a.\, a \in r \;\wedge\; \mathcal{K}_a \ni U \;\equiv\; \lozenge U. \end{aligned}$$

Hence $\lozenge$ is $\exists_X$ as in Lemma 12.2.

Conversely, if X is overt then A^+ (Notation 12.8) is a positivity. Also, from Section 7,

$$\begin{aligned} b \lhd b^+ \quad &\equiv \quad (\forall a.\, a \prec\!\!\!\prec b \Longrightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset b^+) \\ &\Leftrightarrow \quad \big(\forall a.\, a \prec\!\!\!\prec b \Longrightarrow (a \prec\!\!\!\prec \circ) \;\vee\; (b \in A^+)\big) \end{aligned}$$

since finite $\ell \subset b^+$ must be $\circ$ or $\{b\}$. This property is true by Johnstone's Lemma 12.9. □

Finally we characterise overt subspaces.

Definition 12.12 A space X or its basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ is called ***recursively enumerable*** if there is some bijection $A \cong R \subset \mathbb{N}$ where R and (the image of) $\prec\!\!\!\prec$ are recursively enumerable.

Every object that is definable in Abstract Stone Duality is recursively enumerable, as are the naturally occurring examples in Section 13, even if we choose to think of them classically.

We also call an overt subspace $\lozenge$ recursively enumerable if the corresponding positivity $r \equiv \{a \mid \lozenge U_a\} \subset A \cong R \subset \mathbb{N}$ is recursively enumerable. Again this happens if $\lozenge$ is definable in ASD or, we claim, naturally occurring. However, there is potentially some ambiguity in this usage, but it will be resolved by the Theorem that we aim to prove.

Lemma 12.13 An abstract basis is recursively enumerable iff there is an enumeration $k_{(-)} : \mathbb{N} \to \mathsf{Fin}(A)$ and a decidable predicate $\mathsf{WB}(j, a, k)$ such that, for all $i \in \mathbb{N}$, $a \in A$ and $k \in \mathsf{Fin}(A)$,

$$a \prec\!\!\!\prec k \;\Longleftrightarrow\; \exists j.\;\; i < j \;\wedge\; k = k_j \;\wedge\; \mathsf{WB}(j, a, k).$$

Proof Stephen Kleene's Theorem [Kle43, Section 4]. □

Remark 12.14 *Is there a similar result in Martin-Löf Type Theory, maybe where $\mathsf{WB}(j, a, k)$ says that j encodes a proof that $a \prec\!\!\!\prec k_j$?*

What is the result in locale theory?

Lemma 12.15 Let $\lozenge$ be a recursively enumerable overt subspace of a (not necessarily overt but) recursively enumerable space and suppose that $\lozenge U$ holds. Then $\lozenge$ has an accumulation point that also lies in U.

Proof It suffices to consider $U \equiv U_a$, so $a \in r$. The result is essentially Lemma 5.7: we must find a formal point p with $a \in p \subset r$, where r is rounded and located by Proposition 12.6. We use Kleene's Theorem to modify the enumeration assumption at the beginning of the proof and then the construction proceeds in the same way from $a_0 \equiv a$. That is, except that:

At the ith stage, if $\mathsf{WB}(i, a_i, k_i)$ is false (even though some later $\mathsf{WB}(j, a_i, k_i)$ and hence $a_i \prec\!\!\!\prec k_i$ may be true) then we just let $a_{i+1} \equiv a'$ for any $r \ni a' \prec\!\!\!\prec a_i$ by roundedness of r.

If $\mathsf{WB}(i, a_i, k_i)$ is true then $a' \prec\!\!\!\prec a_i \prec\!\!\!\prec k_i$ and as before $a' \prec\!\!\!\prec k' \prec\!\!\!\prec^1 a_i, k_i$ and there is some $a_{i+1} \in r \cap k'$ by locatedness of r.

Such choices can be made because the sets are recursively enumerable, as is the resulting $p \equiv \{b \mid \exists i.\, a_i \prec\!\!\!\prec b\}$. This is also a $\prec\!\!\!\prec$-filter as before.

For locatedness, if $a_i \prec\!\!\!\prec a' \prec\!\!\!\prec k$ then, by assumption on the enumeration of $\mathsf{Fin}\,(A)$, we have $k \equiv k_j$ and $\mathsf{WB}(j, a, k)$ for some j with $i < j$. This means that $a_j \prec\!\!\!\prec a_i \prec\!\!\!\prec a' \prec\!\!\!\prec k \equiv k_j$ and then $a_{j+1} \prec\!\!\!\prec b \in k_j$, so $b \in k \cap p$.

Hence we have $a \equiv a_0 \in p \subset r$ as required. □

We regard this proof as defining a *function* that takes the starting point a and (deterministically) yields a formal point p_a. This is justified in the same way as in Remark 11.12.

Theorem 12.16 Every recursively enumerable overt subspace is the image of some (non-unique) sequence $f : r \to X$, where r is the corresponding positivity, as in Example 12.4.

Proof For each $a \in r$, let p_a be the formal point that is constructed in the Lemma starting from $a \in r$. Then

$$\langle\, a \,|\, f \,|\, b \,\rangle \;\equiv\; (b \in p_a)$$

defines a matrix for $a \in r \subset A$ and $b \in A$ because

(a) it is trivially contravariant, rounded and saturated in a because we regard $r \subset A$ as a space whose basis has trivial $\sqsubseteq$, $\prec\!\!\!\prec$ and $\lhd$;

(b) it has the partition property because p_a is located with respect to $\prec\!\!\!\prec$;

(c) it is rounded, bounded and strongly filtered in b because p_a is a $\prec\!\!\!\prec$-filter;

(d) $a \in r \Longrightarrow \langle\, a \,|\, f \,|\, a \,\rangle$ because $a \in p_a$; and

(e) $a \in r \wedge \langle\, a \,|\, f \,|\, b \,\rangle \Longrightarrow b \in r$ because $p_a \subset r$ by construction.

Then Theorem 4.22 defines a continuous function $f : r \to X$ and Example 12.4 gives an overt subspace $\blacklozenge$ where

$$\blacklozenge U_b \;\equiv\; \exists a \in r.\, fa \in U_b \;\equiv\; \exists a \in r.\, \langle\, a \,|\, f \,|\, b \,\rangle \iff b \in r,$$

so $\blacklozenge$ agrees with the given operator $\lozenge$ by Proposition 12.6. □

Remark 12.17 We claim that this result makes overtness the gateway between topology and computability. Any program that takes (necessarily discrete) input data and yields (approximations to) a point of a space X is of the form in Example 12.4. Conversely, by Lemma 12.15, every definable inhabited overt subspace has a computable point. Whilst the former may be trite and the latter spectacularly infeasible as they stand, they do at least establish a purely topological characterisation of what can be done computationally.

This becomes a little less far-fetched when we restrict attention to $\mathbb{R}^{\mathsf{n}}$ and its usual basis with $U_{\langle x,r\rangle} \equiv B(x, r) \equiv \{y \mid |\, x - y\,| < r\}$. It turns out that $d(x) < r$ is a reasonable notation for $\lozenge\, B(x, r)$ because it says how far x is from the nearest accumulation point. This relates overtness to ***locatedness*** in Constructive Analysis [Spi10], but familiar numeral algorithms such as Newton–Raphson iteration are also very similar to this [work in progress].

Therefore we may think about problems such as solving equations *mathematically* by adding this concept to our usual topological repertoire. Then we may hand over the resulting λ-term to a *computational* proof-theorist, who may be able to discover the accumulation points in a more efficient way.

13 Examples

I need more examples. What are the applications of local compactness in other mathematical disciplines, besides real manifolds? I intend to re-write these in traditional notation instead of that of ASD.

Example 13.1 The integers provide a basis for themselves in which $n \sqsubseteq m$ iff $n = m$, whilst $n \prec\!\!\!\prec \ell$ iff $n \in \ell$ and $n \lhd u$ iff $n \in u$. Then any ξ is trivially covariant, rounded and located. It is bounded and filtered iff

$$\exists n.\, \xi n \qquad \text{and} \qquad \xi n \wedge \xi m \Longrightarrow (n = m)$$

respectively, so it is a formal point iff it is a ***description*** [Pea97] [A]. □

There are similar representations for numerals (finite sets) **0**, **1**, **2**, ..., and indeed any set with decidable equality or any overt discrete Hausdorff space.

The previous incarnation of this work included two sections on prime bases and the generalised interval domain. I don't intend to include the whole of these, but some précis would be appropriate.

Example 13.2 A topological space is a ***continuous dcpo*** iff it has a ***prime*** basis, which is one that satisfies the strong intersection rule and

$$a \prec\!\!\!\prec \ell \implies \exists b.\, a \prec\!\!\!\prec b \in \ell.$$

Example 13.3 Given any abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ that satisfies the strong intersection rule, $(A, \sqsubseteq, \prec\!\!\!\prec^1)$ is a prime basis. We call this the ***(generalised) interval domain*** of the space X whose basis was given.

The ASD nucleus corresponding to $\prec\!\!\!\prec^1$ is

$$\mathcal{E}^1\Phi\xi \;\equiv\; \exists ab.\, \xi a \;\wedge\; (a \prec\!\!\!\prec b) \;\wedge\; \Phi(\lambda c.\, b \sqsubseteq c).$$

We saw this in Lemma 10.15 and [I].

Beware that this definition depends on the basis: isomorphic spaces with different bases may have non-isomorphic interval domains.

This is because a continuous function $f : X \to Y$ need not lift to their interval domains. In particular, the matrix $\langle\, a \mid f \mid b \,\rangle$ is bounded and filtered, but not necessarily uniformly, whilst any matrix for a function between continuous dcpos is uniformly bounded and filtered. □

Example 13.4 If however the given basis is directed then its interval domain is the ***Smyth powerdomain*** and its points correspond to the saturated compact subspaces of the given space. □

By way of a more powerful worked example, here is

Proposition 13.5 The exponential $\mathbf{2}^{\mathbb{N}}$ (***Cantor space***) exists as an ASD object and as a locally compact space.

Proof The basis A consists of finite strings of 0s and 1s, with $a \sqsubseteq b$ if b is an initial segment of a. Some examples of the way below relation are

$$a \prec\!\!\!\prec a, \qquad a \prec\!\!\!\prec \{a0, a1\}, \qquad a \prec\!\!\!\prec \{a00, a010, a011, a10, a11\},$$

so interpolation, Wilker and roundedness are trivial, but, as in Example 1.6, we leave the interested reader to formulate $a \prec\!\!\prec \ell$ explicitly. Also, $(a \sqsupseteq c \sqsubseteq b) \Longrightarrow (a = c) \vee (c = b)$, so if ξ is a filter with $\xi a \wedge \xi b$ then $(a \sqsubseteq b) \vee (b \sqsubseteq a)$. This makes ξ essentially a finite or infinite sequence of 0s and 1s.

It is located iff it is an infinite such sequence, because

$$\Gamma \ \vdash \xi a \ \wedge \ a \prec\!\!\prec \{a0, a1\} \ \Longrightarrow \ \xi(a0) \ \vee \ \xi(a1).$$

The Γ emphasises that parameters are allowed, so the disjunction says that $\Gamma = \Gamma_0 + \Gamma_1$ (as a disjoint union of spaces) where $\Gamma_0 \vdash \xi(a0)$ and $\Gamma_1 \vdash \xi(a1)$.

We have therefore described morphisms $\xi : \Gamma \to X$, where X is the space defined by the basis, and shown that these are bijective with $f : \Gamma \times \mathbb{N} \to \mathbf{2}$, as required for the universal property of $X \cong \mathbf{2}^{\mathbb{N}}$.

For the interval domain, the relations $\prec\!\!\prec^1$ and $\sqsubseteq$ are the same, so roundedness is trivial. Hence a generalised interval is encoded by a sequence that may be finite or infinite, but it must be defined on an *initial segment* of $\mathbb{N}$. These intervals are *some* of the compact subspaces. □

Now we turn to (Dedekind) real intervals and numbers:

Proposition 13.6 A term $\xi : \Sigma^A$ is rounded, bounded and filtered with respect to the basis for $\mathbb{R}$ defined by intervals $(d, u) \subset [d, u]$ iff there are disjoint lower and upper reals (δ, υ) with

$$\xi(d, u) \iff \delta d \ \wedge \ \upsilon u.$$

Therefore X^1 is the (usual) interval domain [Sco72, ES98], as presented in Section I 7.

Proof Let A be the set of pairs with $d < u$ rational. Without imposing any conditions on $\xi : \Sigma^A$ and $\delta, \upsilon : \Sigma^{\mathbb{Q}}$, they satisfy the adjunction

$$\frac{\xi(d, u) \Longrightarrow \delta d \ \wedge \ \upsilon u}{\exists u'. \, \xi(d, u') \Longrightarrow \delta d \quad \wedge \quad \exists d'. \, \xi(d', u) \Longrightarrow \upsilon u.}$$

One of these transformations takes rounded ξ as above to rounded (δ, υ) in the sense of Section I 6, and the other does the converse. Likewise for boundedness. If ξ is filtered then

$$\begin{aligned} \delta d \wedge \upsilon u \quad & \equiv \quad \exists et. \, \xi(d, t) \ \wedge \ \xi(e, u) \\ & \Rightarrow \quad \exists efst. \, \xi(f, s) \ \wedge \ d \leq f < s \leq u \ \Longrightarrow \ d < u, \end{aligned}$$

so δ and υ are disjoint. Conversely,

$$\xi(d, t) \wedge \xi(e, u) \ \equiv \ \delta d \wedge \upsilon t \wedge \delta e \wedge \upsilon u \Longrightarrow \delta c \wedge (c < v) \wedge \upsilon v \ \equiv \ \xi(c, v),$$

where $c \equiv \mathsf{max}(d, e)$, $v \equiv \mathsf{min}(t, u)$ and so $(c, v) = (d, t) \cap (e, u)$.

When ξ is filtered we recover $\xi \mapsto (\delta, \upsilon) \mapsto \xi$, whilst if δ and υ are bounded then $(\delta, \upsilon) \mapsto \xi \mapsto (\delta, \upsilon)$. Hence when all of these conditions hold on either side we have a bijection and (δ, υ) is an interval or Dedekind pseudo-cut. □

Proposition 13.7 An interval given in this way by ξ is located in the sense of this paper iff the corresponding (δ, υ) is located in the sense of [I]. In this case they define a real number in the form of a Dedekind cut.

Proof Given $e < t$, there are $c < d < e < t < u < v$ with $\xi(d, u)$ and $[d, u] \subset (e, v) \cup (c, t)$, so by locatedness $\xi(e, v) \vee \xi(c, t)$, whence $\delta e \vee \upsilon t$. The converse is more complicated, involving the reorganisation of a conjunction (over the list ℓ) of binary disjunctions, but is essentially Lemma I 6.16. □

14 Conclusion

We have proved several *weak equivalences of categories.*

Definition 14.1 In the ***category of weak abstract bases and matrices***,
(a) an *object* is an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ that satisfies the principal axioms of Definition 1.8 (co- and contravariance, Wilker and weak intersection) and the roundedness properties of Definition 1.10 (single interpolation, rounded union and boundedness above and below);
(b) a *morphism* $\langle\, | f |\, \rangle : (A, \sqsubseteq, \prec\!\!\!\prec) \to (B, \sqsubseteq, \prec\!\!\!\prec)$ is a matrix that satisfies Definition 1.13 (co- and contravariance, roundedness on both sides, partition, boundedness, weak filteredness and saturation);
(c) the *identity map* on $(A, \sqsubseteq, \prec\!\!\!\prec)$ is the way-below relation, $\langle\, a \,|\, \mathsf{id}_X \,|\, b \,\rangle \equiv (a \prec\!\!\!\prec_X b)$; and
(d) morphisms are *composed* using the saturated composition operation in Notation 4.6:

$$\langle\, a \,|\, f \,;\, g \,|\, c \,\rangle \;\equiv\; \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \,|\, f \,|\, b \,\rangle \;\wedge\; \langle\, b \,|\, g \,|\, c \,\rangle.$$

Definition 14.2 The ***category of strong abstract bases and matrices*** is the full subcategory of the previous one consisting of bases that also obey the strong or rounded intersection rule. By Lemma 3.3 or 4.20, the matrices are strongly filtered.

The concrete category of "locally compact spaces and continuous maps" is weakly equivalent to one or both of these abstract categories. This is the case for each of the four formulations of topology that we have considered, in the mathematical foundations that are appropriate to that subject. We begin with Formal Topology because it is the most similar to our abstract bases.

Theorem 14.3 The category of locally compact formal topologies and continuous functions is weakly equivalent to the strong abstract category, in Martin-Löf type theory.

Proof Definition 6.4, Lemma 6.17 and Remark 7.14 discussed how locally compact formal covers are defined. Proposition 7.3 and its preceding two lemmas derived an abstract basis $\ll$ from a locally compact cover $\triangleleft$ and Lemmas 7.5 to 7.7 did the converse.

Proposition 7.17 translated between matrices for $\prec\!\!\!\prec$ and $\triangleleft$, the latter being the definition of continuous functions between covers that the Formal Topologists use.

The results of Sections 3 and 11, regarded solely as operations on abstract bases, show how to add the extra properties to them; we may assume Dependent Choice in doing this because it is a feature of Martin-Löf Type Theory. □

Theorem 14.4 The category of locally compact locales and continuous functions is weakly equivalent to the category of weak abstract bases and matrices, in the logic of an elementary topos. If the topos satisfies the axiom of Dependent Choice then the category is also equivalent to the strong one.

Proof Definitions 6.1 and 6.2, Lemma 6.14 and Corollary 6.15 explained what locally compact locales and continuous lattices are and Proposition 6.16 obtained an abstract basis from them.

The converse construction turns the formal cover in the previous result into a frame or locale using Lemma 6.6 and Theorem 6.7; Proposition 7.8 characterised this using $\prec\!\!\!\prec$. Then Lemma 7.9 provides the Scott-open family $(\mathcal{K}_a)$ such that $\mathcal{K}_a \ni U_\ell \iff a \prec\!\!\!\prec \ell$.

Continuous functions, which are defined as reverse frame homomorphisms, correspond to matrices by the arguments in Section 4, with $\bigcup$, $\cap$ and $K_a\subset$ replaced by $\bigvee$, $\wedge$ and $\mathcal{K}_a\ni$. Bases may be improved to obey the single interpolation and rounded union rules by a similar translation of

Lemma 3.4. If Dependent Choice is available, Section 11 showed how use it to impose the strong intersection rule. □

Theorem 14.5 The category of locally compact sober topological spaces and continuous functions is weakly equivalent to the strong category of abstract bases and matrices, in a set theory with excluded middle and the Axiom of Choice.

Proof Sections 1, 2, 3 and 11 showed how concrete bases using compact subspaces or Scott-open families yield abstract bases and can be improved to have all of the additional properties. Conversely, Section 5 defined a locally compact sober space from any *countable* abstract basis.

For the general case, we turn the locale in the previous result into a sober topological space. Lemma 7.11 showed that formal points for the abstract basis (Definition 5.1) agree with those for the locale and formal cover (Proposition 6.10). By Theorem 7.12 there are enough of them to make the extent (Proposition 6.11) an isomorphism between the abstract frame and the lattice of open sets of formal points (Definition 5.2). Then the Scott-open families in Lemmas 5.4 and 7.9 agree and satisfy the basis expansion. We also obtain $\mathcal{K}_a \subset U_\ell \iff a \prec\!\!\!\prec \ell$ from Lemma 7.9 instead of Lemma 5.8 and its preceding results. The space is sober by Lemma 5.9 without the countability restriction and in the strong case Theorem 5.10 describes the basic compact subspaces.

Section 4 showed how matrices correspond bijectively to continuous functions between sober spaces and deduced the saturated composition operation. □

Remark 14.6 Our development in Point–Set Topology in Section 5 was interrupted by the need to find enough formal points to characterise the way-below relation. We eventually proved this in Theorem 7.13, once we had the *benefit* of the concept, structure and properties of the $\triangleleft$ relation. In particular, we now see that we needed to apply Lemma 3.12 about maximal filters, not in the concrete frame of open sets of points (*cf.* Lemma 5.9), but in the abstract one that is defined directly from the abstract basis (Theorem 7.12). Only after doing so can we deduce that these two frame are isomorphic and hence prove the Theorem.

Remark 14.7 In Abstract Stone Duality, Proposition 9.10 showed that every concrete basis using λ-terms defines an abstract one. Conversely, the results of Section 10 constructed a nucleus $\mathcal{E}$ from any abstract basis.

Our introduction to ASD relied on the equivalence with the other formulations of topology, whereas the appropriate notion of "set" for ASD is either an object of an arithmetic universe or an overt discrete space in ASD. The construction of the strong abstract category really belongs in this much weaker logic. However, the axioms of both the topology and the foundations are then *so* weak that we have a whole paper's [work in progress] worth of work to do to construct the category, its products and its exponentials, but the outcome of this is that it is a model of ASD.

Remark 14.8 The main outcome of this lengthy investigation is that *the same structure*, at least as far as its *topological* description is concerned, is equivalent to the category of locally compact spaces in all four formulations, whereas each of those accounts has its own *ad hoc* features.

This is possible because, in the four kinds of abstract basis, *the words "set" and "relation" are understood in different ways*, since we are working in different logical foundations.

Consequently, the meaning of the notion of "continuous function" varies with logical strength. Indeed, we have a precise way of saying this: a continuous function in Point–Set Topology is a matrix (a certain kind of logical predicate on sets) that is *definable* in set theory with Excluded Middle and Choice, whereas a continuous function in Formal Topology is a matrix that is definable in Martin–Löf Type Theory, *etc.*

This is an observation that is already logically relevant for familiar spaces such as $\mathbb{N}$ and $\mathbb{R}$ that have homes in all four worlds. There are, for example, faster growing continuous real-valued functions in traditional topology than in the other subjects.

Remark 14.9 Cutting the full power of Section 4 down to just Proposition 4.13, we have a weak equivalence between the categories of

(a) locally compact spaces and operators that preserve all joins (but not necessarily meets) and

(b) bases and matrices that are co- and contravariant, rounded, saturated and have the partition property (but need not be bounded or filtered).

Again, there are results for each of the four kinds of topology. There are also further generalisations to (not necessarily distributive) continuous lattices and to bases and covers without the intersection rules.

Remark 14.10 In particular, by Proposition 12.6 overt subspace operators $\lozenge$ are in bijection with positivities (certain subset of the basis) in each of the four forms of topology. It is in this application that we see the most dramatic differences amongst the four logical settings, ranging from the classical one, where overtness is useless, to ASD, where in principle it provides an algorithm for solving a problem.

Remark 14.11 This range of different logics has a bearing on what constitutes "constructive" mathematics. Unfortunately, there is a tendency of mathematicians working on one camp to claim a monopoly on this word to the exclusion of the others. In this paper we have seen three approaches to topology that live in "constructive" worlds, by which we mean not the classical one.

If we are going to forbid excluded middle and the Axiom of Choice, why allow impredicativity?

But if you are going to adopt that position, how do you justify the infinite subsets that are needed in Formal Topology?

Our $\prec\!\!\!\!\prec$ has the advantage that its theory only uses *finite* subsets and ***coherent logic***: entailments between existentially quantified formulae. Further work will show that matrices or ASD terms that are definable in our weakest logic are *computable*. According to the Church–Turing thesis, there is only one notion of computability, whereas the question of which axioms and arguments count as "constructive" is open to debate.

After that, we can try to do *computation* with matrices for continuous functions between locally compact spaces.

Remark 14.12 In a different direction, we may see the axiomatisation of abstract bases as the notion of local compactness stripped of the cultural baggage of the different approaches to topology. We simply have relations between sets.

They're not just sets. We have used lists or finite sets, whilst $\mathsf{Fin}\,(A)$ is the free algebra (semilattice) for a functor on sets. The categorical mind will be able to ring many changes on this idea. In fact, this is the reason for keeping the preorder $\sqsubseteq$ even though Lemma 3.6 showed that it is redundant: it is a *clue* to possibly more general structure, such as a category.

Maybe the notion of locally compact space will be even more of a *discovery* than our already diverse opening diagram suggests.

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