Well founded coalgebras and recursion

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Abstract. We define well founded coalgebras and prove the recursion theorem for them: that there is a unique coalgebra-to-algebra homomorphism to any algebra for the same functor. Earlier, this was proved intuitionistically by adapting von Neumann's original argument for ordinals or well founded relations in set theory, which considers the union of partial solutions. However, that required subobjects to form a complete Heyting algebra and the functor to preserve inverse images. Our new argument for functors that just preserve monos exploits Pataraia's fixed point theorem and only uses directed unions of subobjects.

1 Introduction

Although Georg Cantor introduced well-*ordered* sets in 1883 [3, 4], it was 45 years later when John von Neumann, in his reformulation of the ordinals that became standard in set theory textbooks, gave the construction for transfinite *recursion* [15, § III]. What may now seem the obvious generalisation to well-*founded* relations was made by Ernst Zermelo in 1935, with application to proof theory [22].

Christian Mikkelsen [13, Appendix A] and Gerhard Osius [16, §§4&6] [17, §6] put these ideas in categorical form in the 1970s, representing a binary relation by a *coalgebra* for the covariant powerset functor. These works were part of the early history of elementary toposes, when it was being demonstrated that toposes can do anything that sets can do, in particular that we can use the traditional $\{\ |\ \}$ notation for objects of any topos.

In particular, Mikkelsen proved the recursion theorem in a categorical style and Osius proved that coalgebra monomorphisms capture set-theoretic inclusion. Coalgebra homomorphisms later arose as *simulations* in process algebra.

These results were extended in the 1990s to general endofunctors of a topos that preserve inverse images in [21, §6.3] and to other categories satisfying certain conditions in [20].

In response to demand from the coalgebra research community, we now only ask that the functor preserve monos, not their pullbacks, and also weaken the conditions on the category. This requires a much more subtle construction and in particular a fixed point theorem for posets with least element and *directed* joins instead of *all* of them, the intuitionistic proof of which was found by Dito Pataraia in 1996.

Classically, a binary relation $\prec$ on a carrier A is well founded if

(a) every non-empty subset $U \subset A$ has a $\prec$-minimal element; or

(b) there is no infinite descending sequence $\cdots \prec d \prec c \prec b \prec a$.

Definition 11 Intuitionistically, a binary relation $\prec$ on a carrier A is ***well founded*** if it obeys the ***induction scheme***

$$\forall\phi.\quad \frac{\forall x\colon A.\ \ (\forall y\colon A.\ \ y \prec x \Rightarrow \phi y) \Longrightarrow \phi x}{\forall x\colon A.\ \ \phi x}$$

Instead of using this general notion of well-foundedness, mathematicians often say that they are doing induction or recursion on the *length* of a string, the *height* of a tree, its *depth* in CS, or some other such numerical measure.

The general result that is being invoked is this:

Proposition 12 If $(A,\prec)$ is well founded and $f:(B,<)\to(A,\prec)$ is ***strictly monotone*** in the sense that

$$\forall b_1 b_2\colon B.\quad b_1 < b_2 \implies f b_1 \prec f b_2,$$

then $(B,<)$ is also well founded.

Proof If B has an infinite descending sequence then so does A, which is forbidden. Alternatively, if $\emptyset \neq U \subset B$ then $\emptyset \neq fU \subset A$, so there is a minimal $a \in fU$, where $a = fb$ for some $b \in U$ and this is minimal there. The more difficult intuitionistic proof is discussed in Section 4. □

Our goal is a new categorical form of von Neumann's ***General Recursion Theorem***. Adapted to intuitionistic well-*founded* relations, here is his proof, which it is essential to understand before proceeding with the rest of this paper:

Theorem 13 Let $(A,\prec)$ be a carrier with a well founded binary relation and Θ another carrier with a function $\theta:\mathcal{P}\Theta\to\Theta$ that takes an arbitrary subset of Θ as its argument and returns a single element. Then there is a unique function $f:A\to\Theta$ satisfying the ***recursion scheme***:

$$\forall x\colon A.\quad f(x) \;=\; \theta(\{fy \mid y \prec x\}).$$

Proof An ***initial segment*** of A is a subset $B \subset A$ such that

$$\forall z\colon A.\ \ z \prec y \in B \Longrightarrow z \in B$$

and an ***attempt*** is a partial function $f : A \rightharpoonup \Theta$ whose support (domain of definition) $B \subset A$ is an initial segment and

$$\forall x\colon A.\ \ x \in B \implies f(x) \;=\; \theta(\{fy \mid y \prec x\}).$$

(a) There is a unique attempt with empty support.
(b) The union of any directed family of initial segments or attempts is another such.
(c) The restriction of $\prec$ to any initial segment is well founded.
(d) Any two attempts f, g with the same support B are equal, by induction over B for the predicate $\phi x \equiv (fx = gx)$.

(e) Hence any two attempts with supports B_1 and B_2 agree on $B_1 \cap B_2$ and so may be combined into an attempt with support $B_1 \cup B_2$.

(f) For any attempt f with support B, the *successor attempt* g has support

$$C \;\equiv\; \{x \mid \forall y.\, y \prec x \Longrightarrow y \in B\} \quad \text{and is given by} \quad gx \;\equiv\; \theta\{y \mid y \prec x\}.$$

(g) In this construction, $C = B$ iff $B = A$, by induction over A for the predicate $\phi x \equiv (x \in B)$.

(h) The required solution to the recursion equation is the union of all of the attempts; this is total because it is equal to its successor. $\square$

Remark 14 In the generalisation that we consider, Proposition 12 *fails* (Section 4). Because of this, we lose steps (c) and (e) of the proof and so cannot simply form the union of all attempts in the final part.

Steps (a) and (f) provide the initial and next attempts, so by Peano recursion we can define the nth one for all $n : \mathbb{N}$. Can we not then just use step (b) at limit stages to continue this through the ordinals?

No.

First of all, ordinals are not "transfinite numbers" but require a proof to justify recursion over them: von Neumann gave this in the classical setting and we are now adapting it to our categorical one. *Using* ordinals would therefore be begging the question.

Secondly, the ordinals go on "forever" — Cesare Burali-Forti [2] showed early on that they do not form a "set". So when do we stop?

This is answered by a crucial but frequently overlooked lemma, due to Friedrich Hartogs [8], which is this: For any set X, let λ be the set of isomorphism classes of well-orderings of subsets of X. Then λ is well ordered and there is no injection $\lambda \rightarrowtail X$. In the application, we deduce that the construction reaches a fixed point at stage λ.

Hartogs' proof was one of the earliest formal applications of Zermelo's set theory [23] and he set out the prerequisites from that and Cantor's original work [4] very clearly. Principal amongst them is that, for any two well ordered sets, one is uniquely isomorphic to an initial segment of the other; we would now say that this is a consequence of von Neumann's (later) recursion theorem, but Cantor had actually given a proof of it.

Thirdly, the traditional theory of the ordinals depends *very heavily* on excluded middle. There are two existing intuitionistic accounts [10, 19], which show that there are several different notions. Even so, Hartogs' lemma remains irretrievably classical.

Now consider how we may use category theory to express these ideas. We have a binary relation $\prec$ on a carrier A, which could be represented in a variety of ways, but the one that we choose is as a function (morphism)

$$A \xrightarrow{\alpha} \mathcal{P}A \qquad \text{by} \qquad x \longmapsto \{y \mid y \prec x\}.$$

Definition 15 A ***coalgebra*** for an endofunctor $T : \mathcal{C} \to \mathcal{C}$ of any category is an object A of $\mathcal{C}$ together with a morphism $\alpha : A \to TA$.

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & TB \\
A \arrow[u, "\alpha"] \arrow[r, "f"] & B \arrow[u, "\beta"']
\end{tikzcd}$$

A homomorphism of coalgebras is a $\mathcal{C}$-morphism $f : A \to B$ that makes the square commute. We write $\mathbf{CoAlg}_T$ or just $\mathbf{CoAlg}$ for the category of coalgebras and homomorphisms.

To exploit this idea, we first need a full understanding of the powerset as a functor:

Notation 16 We work throughout in the logic of an elementary topos $\mathcal{S}$, but you may just take this to be $\mathbf{Set}$. That is, we do not use Excluded Middle or the Axiom of Choice, although we do use Impredicative reasoning. Then the ***covariant powerset functor*** $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ is defined on an object X by $\mathcal{P}X \equiv \Omega^X$ and a function $f : X \to Y$ by

$$\mathcal{P}fU \;\equiv\; \{fx \mid x \in U\} \;\equiv\; \{y : Y \mid \exists x{:}X.\;\; y = fx \land x \in U\} \subset Y$$

for $U \subset X$. We shall also need to define, for $V \subset Y$,

$$\begin{aligned} f^*V &\equiv \{x : X \mid fx \in V\} \\ f_*U &\equiv \{y : Y \mid \forall x{:}X.\; fx = y \Longrightarrow x \in U\}. \end{aligned}$$

These also provide the morphism parts of functors $\mathcal{S} \to \mathcal{S}$ that are respectively contravariant and covariant, since $(g;f)^*W = f^*(g^*W)$ and $(g;f)_*U = g_*(f_*U)$. More importantly for us, there are (order-)adjunctions

$$\begin{tikzcd}
U \arrow[r, tail] & X \arrow[d, "f"'] & & \mathcal{P}X \arrow[d, shift right=6, "\mathcal{P}f"'] \arrow[d, shift left=6, "f_*"] \\
V \arrow[r, tail] & Y & & \mathcal{P}Y. \arrow[u, "\dashv\; f^*\; \dashv" description]
\end{tikzcd}$$

Diagrammatically, $\mathcal{P}f$ and f^* are given by composition and pullback respectively. The logical formulae that define $\mathcal{P}fU$ and f_*U are the same except that one involves an existential and the other a universal quantifier. We shall use f_* in Section 4. $\square$

Remark 17 Let $f : B \to A$ be any function. Then $\beta\,;\mathcal{P}f \subset f\,;\alpha$ (as marked in the diagram on the left below) iff

$$\forall b, y{:}B. \quad y \prec_B b \implies fy \prec_A fb,$$

i.e. f is ***strictly monotone*** or preserves the binary relation.

$$\begin{tikzcd} b \in B \arrow[r, "\beta"] \arrow[d, "f"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}B \arrow[d, "\mathcal{P}f"] \\ A \arrow[r, "\alpha"] & \mathcal{P}A \end{tikzcd} \qquad \begin{tikzcd} \exists y \arrow[r, dotted, "\prec_B"] \arrow[d, dotted, mapsto, "f"'] & b \arrow[d, mapsto, "f"] & B \arrow[d, "f"] \\ x \arrow[r, "\prec_A"] & fb & A \end{tikzcd}$$

The reverse inclusion is

$$\forall b{:}B.\ \forall x{:}A. \quad x \prec_A fb \implies \exists y{:}B.\ x = fy \wedge y \prec_B b,$$

which is a “lifting” property similar to that defining a fibration, as illustrated on the right. In process algebra a function f with this property is known as a ***simulation*** [1].

If $f : B \rightarrowtail A$ is a subcoalgebra inclusion then the lifting is unique and being a simulation says that B is down-closed or an ***initial segment***,

$$\forall a, b{:}A. \quad a \prec b \in B \implies a \in B.$$

A coalgebra is ***extensional*** if its structure is mono. Andrzej Mostowski showed (using the recursion theorem and the axiom-scheme of replacement) that any extensional well founded relation is isomorphic to a unique set ($\in$-structure) [14, Thm 3]. Osius characterised set-theoretic inclusions as homomorphisms of extensional well founded $\mathcal{P}$-coalgebras [16, §6]. □

Here is our central concept, which we generalise from the powerset to any functor that preserves monos:

Definition 18 A coalgebra $\alpha : A \to TA$ is ***well founded*** if in any pullback diagram of the form

$$\begin{tikzcd} TU \arrow[rr, tail, "Ti"] & & TA \\ H \arrow[u] \arrow[ur, phantom, very near start, "\lrcorner"] \arrow[r, tail, "j"] & U \arrow[r, tail, "i"] & A \arrow[u, "\alpha"'] \end{tikzcd}$$

the maps i and therefore j are necessarily isomorphisms. We write $\mathbf{WfCoAlg}_T$ or just **WfCoAlg** for the category of well founded coalgebras and homomorphisms.

Essentially this “broken pullback” appears (with $T \equiv \mathcal{P}$) on page 99 of [13] and it is written symbolically as $\alpha^{-1}(\mathcal{P}U) \subset U \Longrightarrow U = A$ in [16, §4] and [17, Prop 6.1]. It was first given as the *definition* of well-foundedness in [20, 21].

Lemma 19 A binary relation $(A, \prec)$ is well founded in the sense of Definition 11 iff the corresponding (A, α) is a well founded $\mathcal{P}$-coalgebra.

Proof Write $U \equiv \{x \in A \mid \phi x\}$ for some predicate ϕ defined on A.

An element $(a, V) \in H \subset A \times TU$ of the pullback consists of $a \in A$ and $V \subset U \subset A$ such that

$$\alpha(a) \;\equiv\; \{x \in A \mid x \prec a\} \;=\; V.$$

Thus V is determined uniquely by a (and the structure $\alpha : A \to TA$), but for such a V to exist, a must satisfy

$$\{x \in A \mid x \prec a\} \;\subset\; U, \qquad i.e. \quad \forall x \in A.\; x \prec a \Longrightarrow \phi x.$$

The pullback H therefore corresponds to the induction *hypothesis*.

The induction *premise* is that, for every such $a \in A$ that satisfies the hypothesis, we have $a \in U$ or ϕa. In the diagram this means that $H \subset U$. The *strict* induction premise corresponds to having $H \cong U$ instead; this makes $U \subset A$ a subcoalgebra where the square is a pullback.

Well-foundedness of the coalgebra says that whenever we have a diagram of this form then $U \cong A$, just as the induction *scheme* says that whenever the premise holds then we must have $\forall x.\, \phi x$. $\square$

Our goal is to prove that well founded coalgebras admit recursion:

Definition 110 A *coalgebra* $\alpha : A \to TA$ obeys the ***recursion scheme*** if, for every *algebra* $\theta : T\Theta \to \Theta$, there is a unique map $f : A \to \Theta$ such that the square

$$\begin{tikzcd} TA \arrow[r, "Tf"] & T\Theta \arrow[d, "\theta"] \\ A \arrow[u, "\alpha"] \arrow[r, "f"] & \Theta \end{tikzcd}$$

commutes. Such f is also called a ***coalgebra-to-algebra homomorphism*** [7].

To obtain ***parametric recursion***, in which the top line is replaced by

$$Tf \times \mathsf{id} : TA \times A \longrightarrow T\Theta \times A,$$

we would just need to make Lemma 34 a bit more complicated. In fact Mikkelsen had an even more general scheme than this, although still with $T \equiv \mathcal{P}$ [13, pp 98–99] [17, Def 6.2]. Osius's account of categorical set theory [16] largely used recursion instead of well-foundedness (induction).

In a topos, well-foundedness is *necessary* for recursion [13, p100] [17, Prop 6.3] [21, Exercise 6.14]:

Proposition 111 If $\alpha : A \to TA$ obeys the recursion scheme for every algebra structure on the subobject classifier (set of truth values) $\Theta \equiv \Omega \equiv \mathcal{P}(1)$ then it is well founded. $\square$

We would like to prove the recursion theorem for a more general category $\mathcal{C}$ than just **Set** or a topos, so what do we require of it to do this? In the final step of Theorem 13, we took *unions*, so we first need to be clear what they are in a general category:

Definition 112 A ***union*** in a category is the colimit of a diagram such that
(a) the maps in the diagram are mono;
(b) the maps in the colimiting cocone are mono; and
(c) for any other cocone consisting of monos, the colimit mediator is also mono.

There is no need to introduce a corresponding special definition of an *intersection*, since if the maps in a limit diagram are monos then so (automatically) are those in the limiting cone and the mediator from another cone of monos. Dually, colimits of epis are straightforward, whilst limits of epis in, for example, **Set**, bring us into the territory of the axiom of choice. It is therefore misleading to think of monos as sub*sets* in this Definition, because in other categories (even $\mathbf{Set}^{\mathsf{op}}$) the additional conditions are not so easily satisfied.

Assumption 113 The category $\mathcal{C}$ must have
(a) an initial object, which we call $\emptyset$, all maps out of which must be mono;
(b) equalisers;
(c) ***inverse images***, *i.e.* pullbacks of monos against arbitrary maps; and
(d) "set"-indexed ***directed unions*** of subobjects.

Moreover, $\mathcal{C}$ must be
(e) ***well powered***, which means that the isomorphism classes of monos into any object of $\mathcal{C}$ form a "set".

Remark 114 The word "set" here is rather an embarrassment, given that we want to use category theory as our foundations and study $\in$-structures like ordinary mathematics. The point is that we want to use the poset of subobjects to index a union. The appropriate tools are either *indexed* or *fibred categories*, of which the existing account that comes nearest to what we need is [18]. This treats the adjoint functor theorem but not unfortunately the particular results that we require. This will be done in the extended version of this paper.

Assumption 115 The endofunctor $T : \mathcal{C} \to \mathcal{C}$ must preserve monos.

Remark 116 This is an unnaturally weak assumption, given that the proof of Theorem 13 was all about partial functions. Their composition in a category makes use of inverse images, so to define a category of coalgebras and *partial* homomorphisms, the functor T ought to *preserve* inverse image diagrams. This is true of the powerset and term algebra functors.

Moreover, the very natural Proposition 12 depends on preservation of inverse images, together with other set-like conditions on the category $\mathcal{C}$, as we shall see in Section 4.

However, there is demand from researchers who use coalgebras in theoretic computer science to prove our main theorem on weaker assumptions. Even though I do not presently know of a mathematically compelling application, we have taken up this challenge in this paper.

As we noted after Theorem 13, the effect of weakening the assumption on T on the proof of our main result is that we cannot form binary unions of attempts. We therefore need a fixed point theorem that only demands a least element and *directed* joins. This was proved intuitionistically by Dito Pataraia in 1996, but he didn't publish it before his death in 2011. Recall that a ***dcpo*** is a partial order that has all directed joins and an ***ipo*** also has a least element.

Proposition 117 Any dcpo $(X, \leq)$ has a greatest inflationary monotone endofunction, $t : X \to X$. This is idempotent and its fixed points are exactly the points that are fixed by *all* inflationary monotone endofunctions.

Proof Consider the set

$$Y \quad \equiv \quad \{s : X \to X \mid (\forall x{:}X.\ \ x \leq sx) \wedge (\forall xy{:}X\ .\ x \leq y \Longrightarrow sx \leq sy)\}$$

of inflationary monotone endofunctions of X. In the pointwise order, this inherits directed joins from (the values in) X, and id_X is the least element, so Y is an ipo.

For any $r, s \in Y$, the composites $r\,;s$ and $s\,;r$ both lie above both r and s in Y, because

$$\forall x{:}X. \quad x \quad \leq \quad rx,\ sx \quad \leq \quad r(sx),\ s(rx),$$

using both the inflationary and monotone properties. So the whole ipo Y is directed. Since it is also directed-*complete*, it has a greatest element, $t : X \to X$.

For any $s \in Y$, the composites $s\,;t$ and $t\,;s$ are in Y too, so $s\,;t \geq t \leq t\,;s$ by the previous argument, but also $s\,;t \leq t \geq t\,;s$ since t is the greatest element of Y. Hence $s\,;t = t = t\,;s$ and in particular $t = t\,;t$.

Finally, if $a = ta$ then $sa = s(ta) = ta = a$. $\square$

Theorem 118 Any monotone endofunction $s : X \to X$ of an ipo has a least fixed point.

Proof Consider the subset $X_0 \subset X$ that is generated by $\bot$, s and directed joins. Since the subset $\{x : X \mid x \leq sx\} \subset X$ is closed under these operations, it contains X_0 and so s restricts to an inflationary monotone function $X_0 \to X_0$. Applying the previous result to X_0, there is a greatest of these, $t : X_0 \to X_0$. Then with $a \equiv t\bot = tt\bot = ta$, we have $a = sa \in X_0$. $\square$

We will need not only the least fixed point of s itself, but also a principle that we call ***Pataraia induction***:

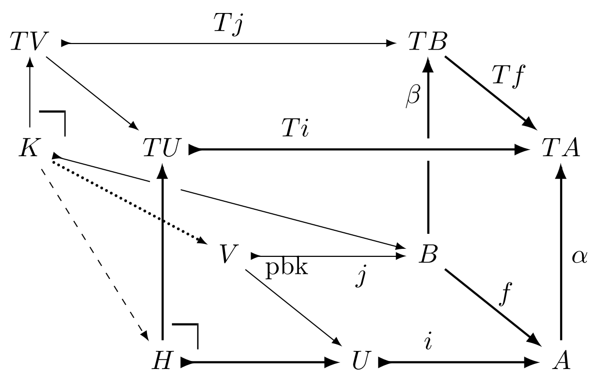
Corollary 119 Let $s : X \to X$ be a monotone endofunction of an ipo and $U \subset X$ a subset containing $\bot$ and closed under s and directed joins. Then U also contains the (same) least fixed point of s in X.

Proof The set U has the same properties as X in the Theorem, so it contains a fixed point of s, but this must be the same as the one for the whole of X. $\square$

2 Generating well founded coalgebras

Before tackling the recursion theorem itself, we study how well founded coalgebras are built up. Proposition 12 is another way of constructing them, but we postpone it to Section 4 because it depends on hypotheses that we have chosen to avoid. The first lemma is about initial segments or simulations, *cf.* Remark 17.

Lemma 21 The induction premise (broken pullback) is stable under pullback against coalgebra homomorphisms.



Proof The thick lines show the given induction premise for $i : U \rightarrowtail A$ and the homomorphism $f : B \to A$.

Let $j : V \rightarrowtail B$ be the inverse image of i along f. Apply T to this pullback square to give the parallelogram at the top, although this need not be a pullback.

Form the inverse image K of Tj along β, so that K is the induction hypothesis for $V \rightarrowtail B$.

The top, back and right quadrilaterals commute (from K to TA), so there is a pullback mediator $K \to H$ that makes the left and bottom quadrilaterals commute.

Because of the latter, there is a pullback mediator $K \to V$ that makes everything commute, in particular from K to A. This is the required induction premise. $\square$

Von Neumann's proof of the recursion theorem for ordinals (Theorem 13) forms the union of attempts, so we consider colimits next. Note, however, that we are merely *enhancing* the properties of those that *already* exist in the category $\mathcal{C}$, not asking for any more of them.

Lemma 22 The initial object $\emptyset$ of $\mathcal{C}$ carries a unique T-coalgebra structure and this is well founded. It is a *sub*coalgebra of any coalgebra.

Proof Any mono $U \rightarrowtail \emptyset$ splits, so it is an isomorphism and we have assumed that all maps $\emptyset \to X$ are mono. $\square$

Proposition 23 The forgetful functors $\mathbf{WfCoAlg} \longrightarrow \mathbf{CoAlg} \longrightarrow \mathcal{C}$ create colimits. That is, the colimit of any diagram of coalgebras and homomorphisms is given by the colimit of their carriers, if this exists. If the individual coalgebras are well founded then so is their colimit.

Proof The structure map α on the colimit is the colimit mediator, as shown in the diagram on the left, where the colimiting cocone consists of coalgebra homomorphisms, *i.e.* the parallelograms from A_i to TA commute.

$$
\begin{tikzcd}
& & TA \\
TA_i \arrow[r] \arrow[urr] & TA_j \arrow[ur] & \\
& & A \arrow[uu, dotted, "\alpha"'] \\
A_i \arrow[uu, "\alpha_i"] \arrow[r] \arrow[urr] & A_j \arrow[uu, "\alpha_j"] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& TU \arrow[rrr, "Ti"] & & & TA \\
TU_i \arrow[ur, dotted] \arrow[rrr, dotted, "Tk_i"] & & & TA_i \arrow[ur] & \\
& H \arrow[uu] \arrow[r] \arrow[dr, phantom, "\lrcorner"] & U \arrow[rr, tail, "k"] & & A \arrow[uu, "\alpha"'] \\
K_i \arrow[uu, dotted] \arrow[ur, dotted] \arrow[r, dotted] & U_i \arrow[ur, dotted] \arrow[rr, dotted, "k_i"', "\text{pbk}"] & & A_i \arrow[uu, "\alpha_i"'] \arrow[ur] &
\end{tikzcd}
$$

Now suppose that the α_i are well founded and let $k : U \rightarrowtail A$ be a predicate satisfying the induction premise for the colimit α (the upper rectangle, from H to TA).

Form the inverse images K_i of this induction premise against the homomorphisms $A_i \to A$ of the colimiting cocone, using Lemma 21.

Since each A_i is well founded, $k_i : U_i \cong A_i$.

Now U is the vertex of a cocone over the diagram A_i, so it has a mediator from the colimit A, and $U \cong A$ as required. $\square$

The next four results study notions of "successor" for attempts. We leave the proof of the first as an exercise because it is a special case of Lemma 27 with $c \equiv \mathsf{id}$ and we don't actually use it.

Lemma 24 The functor T preserves well founded coalgebras. $\square$

Lemma 25 Let $i : (B, \beta) \rightarrowtail (A, \alpha)$ be a subcoalgebra. Then its ***relative successor*** $k : (C, \gamma) \rightarrowtail (A, \alpha)$ is given by pullback of α and Ti.

$$
\begin{tikzcd}
& A \arrow[r, "\alpha"] & TA & \\
& C \arrow[u, tail, "k"] \arrow[r, "c"] \arrow[ur, phantom, "\lrcorner", very near start] \arrow[drr, "\gamma"] & TB \arrow[u, tail, "Ti"'] & \\
B \arrow[uur, tail, "i"] \arrow[ur, tail, "j"] \arrow[urr, "\beta"'] & & & TC \arrow[ul, tail, "Tj"'] \arrow[uul, tail, "Tk"']
\end{tikzcd}
$$

The pullback mediator $j : B \to C$ makes $(B, \beta) \rightarrowtail (C, \gamma) \rightarrowtail (A, \alpha)$ as subcoalgebras when we we define $\gamma \equiv c\,;Tj$.

Proof $i = j\,;k$ and $k\,;\alpha = c\,;Ti = c\,;Tj\,;Tk = \gamma\,;Tk$ and $j\,;\gamma = j\,;c\,;Tj = \beta\,;Tj$. □

Lemma 26 If (A, α) is well founded then $j : B \cong C$ iff $i : B \cong A$.

Proof If $B \cong C$ then A, B, TB and TA form a pullback. It is the one in Definition 18 of well-foundedness, except that $K = U = B$. Therefore $B \cong A$. The converse is immediate from the pullback construction. □

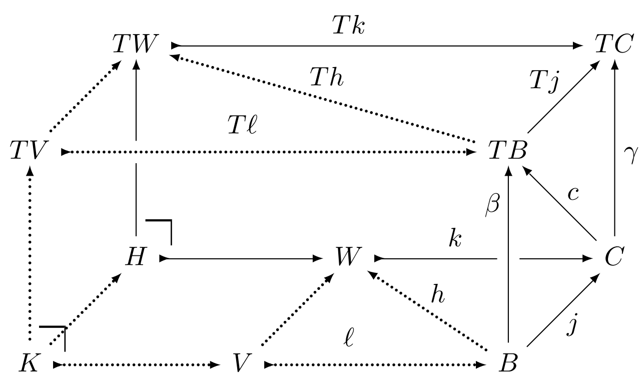
In the case of the covariant powerset, any subcoalgebra of a well founded coalgebra is again well founded. Using this, we could deduce well-foundedness of C from that of TB and hence from that of B by Lemma 24. As we have chosen to use weaker conditions in our account, we need a slightly more complicated result at this point, which we call the ***sandwich lemma***.

Lemma 27 Let (B, β) be a well founded coalgebra and $j : B \to C$ and $c : C \to TB$ maps such that $\beta = j\,;c$. Put $\gamma \equiv c\,;Tj$. Then (C, γ) is also a well founded coalgebra and j and c are homomorphisms.

Proof They are homomorphisms because

$$j\,;\gamma \;\equiv\; j\,;c\,;Tj \;=\; \beta\,;Tj \qquad \text{and} \qquad \gamma\,;Tc \;\equiv\; c\,;Tj\,;Tc \;=\; c\,;T\beta.$$

Now let $k : W \rightarrowtail C$ satisfy the induction premise given by the pullback H and form the inverse image of this along j, using Lemma 21. This gives the induction premise K for the subobject $\ell : V \rightarrowtail B$:



Since B is well founded, $\ell : V \cong B$ and so there is a map $h : B \to W$ making the triangle with C commute. The one with TB, TW and TC also commutes.

The top right triangle ($\gamma = c\,;Tj$) commutes too, so the maps $C \to TB \to TW$ and id_C form a commutative square at TC. This factors through the pullback H, splitting the inclusion $H \rightarrowtail W \rightarrowtail C$ as required. □

Proposition 28 The (isomorphism classes of) well founded subcoalgebras of any coalgebra A form an ipo, on which the relative successor operation is monotone and inflationary. This operation has a unique fixed point, which is the greatest well founded subcoalgebra of A.

Proof The first sentence sums up the results of this section, together with Assumption 113 that the category $\mathcal{C}$ be well powered (Remark 114). Notice that Lemmas 23, 25 and 27 did not assume that A or the vertex of a cocone was well founded.

The second part comes from Pataraia's Theorem 118. If we use this with the empty coalgebra as $\bot$, we get the *least* fixed point B of the successor.

However, we may apply the Theorem with *any* well founded coalgebra D as $\bot$, obtaining a fixed point A' with $D \leq A'$. Since B was least, $B \leq A'$.

Now we use Lemma 26 with A' instead of A: since B agrees with its successor, $D \leq A' \cong B$. Therefore B is the *greatest* well founded subcoalgebra of A. $\square$

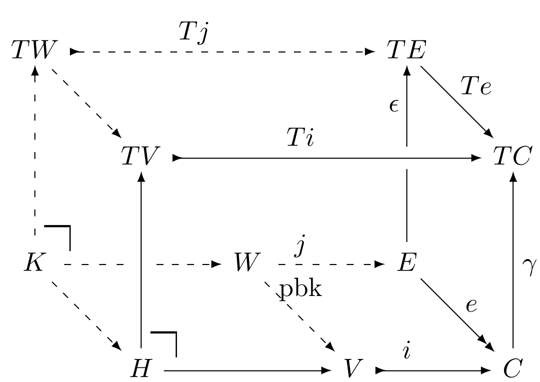
In the next section we will use Pataraia *induction* on the dcpo with $\bot$ and $\top$ of well founded subcoalgebras of a well founded coalgebra to prove the recursion theorem. But first, on an additional assumption, we can improve the "greatest subcoalgebra" to an adjoint.

Definition 29 A map $e : A \twoheadrightarrow B$ is ***extremal epi*** if it is ***orthogonal*** to all monos $i : C \rightarrowtail D$, written $e \perp i$. That is, in any commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d] & B \arrow[d] \arrow[dl, dotted] \\ C \arrow[r, tail, "i"] & D \end{tikzcd}$$

there is a unique map $B \to C$ making the two triangles commute. If, as in our case, pullbacks of monos exist, this is equivalent to saying that if $e = f\,;i$ with i mono then in fact i is invertible.

Lemma 210 Let E be a well founded coalgebra and $e : E \twoheadrightarrow C$ a homomorphism that is extremal epi as a $\mathcal{C}$-map. Then C is also well founded.



Proof Let $i : V \rightarrowtail C$ be a subobject that satisfies the induction premise that is given by the broken pullback from H to TC (at the front).

Pull this back along the homomorphism $e : E \twoheadrightarrow C$, using Lemma 21.

By well-foundedness of E, we have $j : W \cong E$.

Since $e : E \twoheadrightarrow C$ is extremal epi and it factors through the mono $i : V \rightarrowtail C$, the latter is also an isomorphism. $\square$

Theorem 211 If $\mathcal{C}$ has factorisation into extremal epis and monos then the inclusion **WfCoAlg** $\rightarrowtail$ **CoAlg** has a right adjoint.

$$
\begin{tikzcd}
TE \arrow[rrr, "Tf"] \arrow[dr, "Te"'] & & & TA \\
 & TC \arrow[urr, tail, "Tj"] \arrow[r, tail, dotted, "Tk"'] & TD \arrow[ur, tail, "Ti"'] & \\
E \arrow[uu, "\epsilon"] \arrow[rrr, "f"] \arrow[dr, "e"'] & & & A \arrow[uu, "\alpha"'] \\
 & C \arrow[uu, dotted, "\gamma"] \arrow[urr, tail, "j"] \arrow[r, tail, dotted, "k"'] & D \arrow[uu, "\delta"'] \arrow[ur, tail, "i"'] &
\end{tikzcd}
$$

Proof We claim that the largest well founded subcoalgebra $i : D \rightarrowtail A$ from Proposition 28 provides the adjoint. This means that any coalgebra homomorphism $f : E \to A$ with E well founded factors uniquely through i.

Let $E \twoheadrightarrow C \rightarrowtail A$ be the factorisation of f as an extremal epi followed by a mono. Applying T gives $Tf = Te\,; Tj$ with Tj mono. Using $E \twoheadrightarrow C$ and $TC \rightarrowtail TA$ in Definition 29, there is a unique map $\gamma : C \to TC$ making the two parallelograms commute.

Then (C, γ) is a well founded coalgebra by Lemma 210 and it is a subcoalgebra of (A, α) by construction. It is therefore a subcoalgebra of (D, δ) by Proposition 28. The map $E \to D$ is unique since $i : D \rightarrowtail A$ is mono. $\square$

3 The recursion theorem

The proof of the recursion theorem has similar components to the constructions in the previous section. The ipo to which we apply Pataraia's Theorem now consists of partial functions instead of subobjects. However, there is no need to modify the "well powered" assumption, because it suffices to consider the collection of subobjects of $A \times \Theta$ instead of those of A.

Remark 31 An *attempt* from a coalgebra $\alpha : A \to TA$ to an algebra $\theta : T\Theta \to \Theta$ is intended to be a partial map $f : A \rightharpoonup \Theta$ with well founded support that is a *sub*homomorphism in the sense that *if* the composite *via* TA is *defined* then so is that *via* Θ and then they are equal.

$$\begin{tikzcd}
TA \arrow[r, harpoon, "Tf"] & T\Theta \arrow[d, "\theta"] \\
A \arrow[u, "\alpha"] \arrow[r, harpoon, "f"'] \arrow[ur, phantom, "\sqsubseteq"] & \Theta
\end{tikzcd}$$

Despite our concerns in Remark 116, we can avoid the need to define a category of coalgebras and partial homomorphisms, because the notion of attempt has a simple equivalent form:

Definition 32 An ***attempt*** is a diagram of the form

$$\begin{tikzcd}
TA & TB \arrow[l, tail, "Ti"'] \arrow[r, "Tf"] & T\Theta \arrow[d, "\theta"] \\
A \arrow[u, "\alpha"] & B \arrow[l, tail, "i"] \arrow[u, "\beta"'] \arrow[r, "f"'] & \Theta
\end{tikzcd}$$

with B well founded. That is, a subcoalgebra inclusion (or initial segment, *cf.* Remark 17) $B \rightarrowtail A$ together with a coalgebra-to-algebra homomorphism $B \to \Theta$.

A map f satisfies the recursion scheme (Definition 110) exactly when it is a ***total attempt***, *i.e.* with $i : B \cong A$.

Lemma 33 Let A be a well founded coalgebra, Θ an algebra and $f, g : A \rightrightarrows \Theta$ be total attempts. Then $f = g$.

Proof The two parallel squares on the right commute since f and g are total attempts. Let $i : E \rightarrowtail A \rightrightarrows V$ be the equaliser in $\mathcal{C}$.

$$\begin{tikzcd}
TE \arrow[rr, tail, "Ti"] & & TA \arrow[r, shift left, "Tg"] \arrow[r, shift right, "Tf"'] & T\Theta \arrow[d, "\theta"] \\
& E \arrow[dr, tail, "i"] & & \\
H \arrow[uu] \arrow[ur, dotted] \arrow[rr, tail] & & A \arrow[uu, "\alpha"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \Theta
\end{tikzcd}$$

Form the pullback H of $A \to TA \leftarrow TE$; the composites $H \rightrightarrows T\Theta$ are equal by construction, so $H \rightarrowtail A \rightrightarrows \Theta$ are equal. Then $H \rightarrowtail A$ factors through the equaliser, so $H \rightarrowtail E \rightarrowtail A$. Hence $i : E \cong A$ by well-foundedness of A and so $f = g$ [13, page 99] [16, Prop 6.5] [17, Prop 6.3] [19, 2.5] [21, Prop 6.3.9]. $\square$

Next we lift attempts to the relative successor (C, γ) that we constructed in Lemma 25.

Lemma 34 Let $(A, \alpha) \xleftarrow{i} (B, \beta) \xrightarrow{f} (\Theta, \theta)$ be an attempt. Then the diagram

$$\begin{tikzcd}
 & A \arrow[r, "\alpha"] & TA & \\
 & C \arrow[u, "k"'] \arrow[ur, phantom, very near start, "\urcorner"] \arrow[r, "c"] \arrow[d, "g"] \arrow[drr, "\gamma"] & TB \arrow[u, "Ti"] \arrow[d, "Tf"] & \\
B \arrow[uur, "i"] \arrow[ur, "j"] \arrow[urr, "\beta"] \arrow[r, "f"] & \Theta & T\Theta \arrow[l, "\theta"'] & TC \arrow[uul, "Tk"'] \arrow[ul, "Tj"'] \arrow[l, "Tg"']
\end{tikzcd}$$

defines another one, extending f, where $\gamma \equiv c \,; Tj$ and $g \equiv c \,; Tf \,; \theta$.

Proof

$f = \beta : Tf \,; \theta \;=\; j \,; c \,; Tf \,; \theta \;=\; j \,; g$

$g \equiv c \,; Tf \,; \theta \;=\; c \,; Tj \,; Tc \,; TTf \,; T\theta \,; \theta \;=\; c \,; Tj \,; Tg \,; T\theta \;\equiv\; \gamma \,; Tg \,; T\theta$ □

Recall from Lemma 26 that if B is well founded then so is C, whilst if A is well founded and $B \cong C$ then $B \cong A$.

Lemma 35 The initial object is the support of a unique attempt. The union of any directed family of attempts with well founded supports is another such.

Proof By Lemma 22, Proposition 23 and the universal property of colimits. □

We can now achieve our principal goal, the ***Recursion Theorem***.

Theorem 36 From any well founded coalgebra to any algebra there is a unique total attempt.

Proof Let $\mathcal{A}$ be the collection of attempts $A \rightharpoonup \Theta$ with well founded support, ordered by inclusion. Since each of these can be expressed as a subobject of $A \times \Theta$ and the category $\mathcal{C}$ is well powered, this collection is a set. This means that $\mathcal{C}$ has directed unions indexed by it, whence $\mathcal{A}$ is a dcpo (Lemma 35). It also has a least element, given by the attempt whose support is the initial object of $\mathcal{C}$ (Lemma 35). The relative successor (Lemma 34) defines a monotone inflationary endofunction of $\mathcal{A}$.

Similarly, Proposition 28 said that the well founded subcoalgebras of A also form an ipo $\mathcal{B}$. But $\mathcal{B}$ has a top element, given by A itself, which is the unique fixed point of the relative successor.

Moreover there is a function $\mathsf{supp} : \mathcal{A} \to \mathcal{B}$ that forgets the values of the attempts but preserves the least element, directed joins and the successor.

By Lemma 33, supp is injective, so we may consider it as an inclusion $\mathcal{A} \subset \mathcal{B}$ that is closed under $\bot$, successor and directed joins. Therefore, by Pataraia induction (Corollary 119), $\mathcal{A}$ contains the unique fixed point of the successor, which is the unique total attempt $A \to \Theta$ [13, pp 101–4] [17, Prop 6.5]. $\square$

From this we deduce the relationship between well founded coalgebras and the initial algebra. Of course, the latter need not exist, as in the case of the powerset, in which case well founded coalgebras provide a "good enough" substitute. Two of the steps in the circular equivalence below were identified by Joachim Lambek [11] and Daniel Lehmann and Michael Smyth [12]:

Lemma 37 The structure maps of the initial algebra, final coalgebra and final well founded coalgebra, if they exist, are isomorphisms.

$$
\begin{tikzcd}
T\Theta \arrow[d, "\theta"'] \arrow[r, dotted, shift right, "T\alpha"'] & TT\Theta \arrow[l, shift right, "T\theta"'] \arrow[d, "T\theta"] \\
\Theta \arrow[r, dotted, shift right, "\alpha"'] & T\Theta \arrow[l, shift right, "\theta"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TA \arrow[r, shift left, "T\alpha"] & TTA \arrow[l, dotted, shift left, "T\theta"] \\
A \arrow[u, "\alpha"] \arrow[r, shift left, "\alpha"] & TA \arrow[u, "T\alpha"'] \arrow[l, dotted, shift left, "\theta"]
\end{tikzcd}
$$

These objects are therefore both algebras and coalgebras for T, so coalgebra-to-algebra homomorphisms from or to them are respectively the same as plain algebra or coalgebra homomorphisms.

Proof This is illustrated by the diagrams. It also applies to the final *well founded* coalgebra because the functor T preserves them (Lemma 24). $\square$

Lemma 38 The initial algebra has no proper subalgebra. $\square$

Corollary 39 If any of the following exists then it satisfies the other properties too:

(a) a final well founded coalgebra;

(b) a well founded coalgebra whose structure map is an isomorphism;

(c) an initial fixed point ($A \cong TA$);

(d) an initial algebra.

Moreover, it is unique up to unique isomorphism.

Proof The Recursion Theorem says that the final well founded coalgebra has the universal property of the initial algebra.

Conversely, let A be the initial algebra *qua* coalgebra and suppose that $i : U \rightarrowtail A$ satisfies the induction premise. This makes U into a subalgebra, so by Smyth's lemma i is split. Hence A is well founded.

Again by the Recursion Theorem, from any other well founded coalgebra B there is a unique coalgebra-to-algebra homomorphism $B \to A$. But this is the same as a coalgebra homomorphism, so A is the final well founded coalgebra. $\square$

4 Reflecting well-foundedness

We mentioned Proposition 12 as a very important result about well founded relations that is lost if we only require the functor to preserve monos and not inverse images. In this section we impose the stronger condition and give the categorical proof of the result. See [21, Prop 2.6.2] for a box-style proof in natural deduction for well founded relations.

The essential logical tool in this proof is the *universal quantifier* in the subobject $f_* V$, as Gerhard Osius noted in [16, Prop 6.3(a)]. Any topos has this, but we are considering more general categories, so this is a further assumption on the subobjects.

In order to show that these two extra conditions are *necessary* (and as an aid to understanding the categorical proof), we briefly consider what our main definition means when the category $\mathcal{C}$ is merely a partial order.

Definition 41 Let $(X, \leq, \wedge)$ be a poset with binary meets and $s : X \to X$ be a monotone endofunction. We say that $x : X$ is a ***well founded element*** if

$$x \ \leq\ sx \qquad \text{and} \qquad \forall u{:}X.\ \ (su \wedge x\ \leq\ u)\ \Longrightarrow\ x\ \leq\ u.$$

Lemma 42 Joins (such as they exist) and s preserve well-foundedness. □

Definition 43 A ***Heyting semilattice*** $(X, \leq, \wedge, \to)$ is a poset with meets and another binary operation, called ***implication***, that satisfies

$$(a\ \wedge\ b)\ \leq\ c \iff a\ \leq\ (b \to c), \qquad \text{so} \qquad b \wedge (b \to c)\ \leq\ c.$$

Lemma 44 In a Heyting semilattice, $(-) \wedge b$ preserves (distributes over) all joins that exist. If this holds in a complete lattice then $b \to (-)$ exists. □

Proposition 45 Let X be a Heyting semilattice, $s : X \to X$ preserve binary meets and $x, y : X$. If x is well founded and $x \geq y \leq sy$ then y is well founded too.

Proof Suppose that $v : X$ satisfies $sv \wedge y \leq v$ and put $u \equiv (y \to v)$. Then

$$su \wedge y\ =\ s(y \to v) \wedge sy \wedge y\ =\ s\big((y \to v) \wedge y\big) \wedge y\ \leq\ sv \wedge y\ \leq\ v,$$

since $y = sy \wedge y$, s preserves meets and $(y \to v) \wedge y \leq v$. Hence, by definition of $(y \to v)$,

$$su \wedge x\ \leq\ su\ \leq\ (y \to v)\ \equiv\ u.$$

Then $y \leq x \leq u \equiv (y \to v)$ by well-foundedness of x, and $y \leq (y \to v) \wedge y \leq v$. □

Examples 46 The additional hypotheses are necessary.

$$\begin{array}{ccc} y & \leq & sy = ss\bot \\ \begin{sideset}{}{}\end{sideset}\text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ \bot & \leq & s\bot \end{array} \qquad\qquad \begin{array}{ccccccccccc} y & \leq & sy & \leq & ssy & \leq & sssy & \leq & \cdots & \leq & s^\omega y \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & & & \| \\ \bot & \leq & s\bot & \leq & ss\bot & \leq & sss\bot & \leq & \cdots & \leq & s^\omega \bot \end{array}$$

In both cases, the elements $s^n\bot$ and $s^\omega\bot$ are well founded by Lemma 42, but y is not, because $s\bot \wedge y \le \bot$ but $y \not\le \bot$.

The first is a Heyting semilattice, but s does not preserve the meet $y \wedge s\bot = \bot$.

The second is also distributive but it is not a Heyting semilattice, since $y \wedge (-)$ does not preserve the directed join $\bigvee^{\uparrow} s^n\bot$. However, s preserves meets because, for $n < \omega$ and $m \le \omega$,

$$s^n\bot \;\wedge\; s^m y \quad = \quad s^{\mathsf{min}(n,m)}\bot. \qquad \square$$

Assumption 47 In addition to Assumption 113,

(a) the category $\mathcal{C}$ must now have inverse images (pullbacks) of monos against any map f, and so an order-preserving operation f^* on subobjects;

(b) the functor $T : \mathcal{C} \to \mathcal{C}$ must preserve inverse image diagrams; and

(c) each operation f^* must have a right adjoint f_* on subobjects.

See Notation 16 for f^* and f_* in a topos.

Theorem 48 Let $f : (B, \beta) \to (A, \alpha)$ be a coalgebra homomorphism, where (A, α) is well founded. Then (B, β) is also well founded.

Proof Given the diagram marked in thick lines, apply the right adjoint f_* to $j : V \rightarrowtail B$, to get $i : f_*V \rightarrowtail A$. The counit of this adjunction is $\epsilon : f^*f_*V \to V$ and makes the little triangle $(*)$ commute, where f^* is given by pullback (inverse image) along f. The upper part of the diagram is the T-image of the lower part, including the pullback but not K. Let $H \equiv \alpha^*T(f_*V)$ be the pullback of Ti and α and f^*H its pullback along f.

$$\begin{tikzcd}
 & (Tf)^*T(f_*V) = T(f^*f_*V) \arrow[rr, "\text{pbk}"'] \arrow[dl, "T\epsilon"'] \arrow[dr, tail] & & T(f_*V) \arrow[dr, tail, "Ti"] & \\
TV \arrow[rr, tail, "Tj"] & & TB \arrow[rr, "Tf"] & & TA \\
 & f^*H \arrow[rr, "\text{pbk}"'] \arrow[uu, dotted] \arrow[dr, tail] \arrow[dl, dotted] & & H \arrow[uu] \arrow[dr, tail] \arrow[dd, dotted] & \\
K \arrow[uu] \arrow[r, tail] & V \arrow[r, tail, "j", "(*)"'] & B \arrow[uu, "\beta"'] \arrow[rr, "f"] & & A \arrow[uu, "\alpha"'] \\
 & f^*f_*V \arrow[u, "\epsilon"] \arrow[ur, tail] \arrow[rr, "\text{pbk}"] & & f_*V \arrow[ur, tail, "i"'] &
\end{tikzcd}$$

By construction, the whole diagram of solid lines commutes from f^*H to TA. In particular, $f^*H \rightarrowtail B \to TB$ and $f^*H \to H \to T(f_*V)$ agree at TA, so there is a pullback mediator $f^*H \to T(f^*f_*V)$. Then $f^*H \to T(f^*f_*V) \to TV$ agrees with $f^*H \rightarrowtail B$ at TB, so there is also a pullback mediator $f^*H \to K$.

This shows that $f^*H \subset V$ as $\mathcal{C}$-subobjects of B. Therefore, by the adjunction $f^* \dashv f_*$, we have $H \subset f_*V$ as subobjects of A.

That is, there is a map $H \to f_*V$ that makes the right-hand part of the diagram into a broken pullback. Now, since A is well founded, $i : f_*V \cong A$, so $f^*f_*V \cong B$ and $j : V \cong B$. □

Further work

An extended version of this paper may be found at

`www.Paul Taylor.EU/ordinals`

I began this work in the 1990s in the hope of including intuitionistic, categorical versions of techniques in set theory, particularly recursion over the ordinals, in [21]. A particular goal was Theorem 118, but Dito Pataraia not only got there first but did it by much simpler methods; indeed it does not seem to be possible to use ordinals to prove this result intuitionistically.

The two fruitful applications of well founded coalgebras were not for sets but posets, where using different notions of formal monos for the structure maps of extensional coalgebras yields many kinds of ordinals [21, §6.7]. Then working in categories of fibrations provides a way of *defining* (not constructing) transfinite iteration of functors [21, §9.5] and maybe a categorical version of the axiom scheme of replacement.

However, the simpler excursions into different categories and functors tend to reduce to well founded *relations*, although the functors serve to relate these to other structure.

I also envisaged that characterising well founded coalgebras might give a way of describing free algebras for complicated functors, such as partial models of type theories. The generalisation to functors that only preserve monos rather than inverse images might help with the construction of free algebras for infinitary equational theories.

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