A lambda calculus for real analysis

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This lecture

It takes me more than 20 mins to introduce my research to my own community, so this lecture will be a classical translation of some recent results. It will require First Year Undergraduate Real Analysis.

Please contact me this week by mobile (077 604 625 87) or later by email (pt@cs.man.ac.uk) to learn more.

(Maybe even invite me to your university to give a full seminar.)
Intellectual pedigree

mine:
Mathematical Tripos 1979–83
PhD in Category Theory 1983–6
A mathematician in exile in Computer Science

of this general area of research:
(compact–open) topologies on function-spaces, topological lattice theory,
semantics of programming languages, formal correctness of programs.

of the Abstract Stone Duality programme:
locale theory ("point-less topology", i.e. only using open sets)
category theory, domain theory.

The journey that Abstract Stone Duality has made so far:
from an abstract hypothesis from category theory
to computably based locally compact spaces (not just $\mathbb{R}^n$)
to constructive analysis.
Constructive analysis

The classics, although I don’t myself belong to this tradition.

A “can do” attitude to constructivity,
entirely compatible with the classical results:
we just have to be a lot more careful.


The subject is based on **metrical** $(\epsilon-\delta)$ methods.

$S \subseteq \mathbb{R}$ is **totally bounded** if it has an $\epsilon$-net — needed to define its supremum,
$S \subseteq \mathbb{R}$ is **located** if $d(x, S) \equiv \inf \{d(x, y) \mid y \in S\}$ is definable.
Recursive analysis — the bad news

Cantor space $2^\mathbb{N}$ and the closed real interval $I \equiv [0, 1] \subset \mathbb{R}$ are not compact.

Basic problem: definable/computable/recursive values can be enumerated (like the rationals — it’s just a bit more complicated).
Richard’s Paradox 1900, Turing’s Computable Numbers 1937, Specker Sequences 1949.

Let $(u_n)$ be such an enumeration of the definable elements of $[0, 1]$.
Cover each $u_n$ with the open interval $(u_n \pm \epsilon \cdot 2^{-n})$.
These intervals have total length $2\epsilon$.
With $\epsilon \equiv \frac{1}{2}$, no finite sub-collection can cover.

Another way: König’s Lemma fails: there is an infinite binary (Kleene) tree with no infinite computable path.

In addition to the metrical ($\epsilon-\delta$) methods, everything has to be coded using Gödel numbers.
Recursive analysis — the good news

This doesn’t happen in Abstract Stone Duality.

Cantor space $2^\mathbb{N}$ and the closed real interval $I \equiv [0, 1] \subset \mathbb{R}$ are compact.

$\forall$ in ASD doesn’t mean “for every definable element” — it’s defined to satisfy the formal rules of predicate calculus.

A categorical construction ensures that subspaces always have the subspace topology. (But I’m not going to talk about this in this lecture.)

The mathematical arguments are topological, not metrical.

Programming languages can be translated naturally into ASD (denotational semantics).

Conversely, every ASD term has a natural computational interpretation (as a parallel, non-deterministic, higher-order logic program).
The Classical Intermediate Value Theorem

Any continuous \( f : [0, 1] \rightarrow \mathbb{R} \) with \( f(0) \leq 0 \leq f(1) \) has a zero.

Indeed, \( f(x_0) = 0 \) where \( x_0 \equiv \sup \{x \mid f(x) \leq 0\} \).

A so-called “closed formula”.
A program: interval halving

Let $a_0 \equiv 0$ and $e_0 \equiv 1$.

By recursion, consider $c_n \equiv \frac{1}{2}(a_n + e_n)$ and

$$a_{n+1}, e_{n+1} \equiv \begin{cases} a_n, c_n & \text{if } f(c_n) > 0 \\ c_n, e_n & \text{if } f(c_n) \leq 0, \end{cases}$$

so by induction $f(a_n) \leq 0 \leq f(e_n)$.

But $a_n$ and $e_n$ are respectively (non-strictly) increasing and decreasing sequences, whose differences tend to 0.

So they converge to a common value $c$.

By continuity, $f(c) = 0$. 
Where is the zero?

For $-1 \leq p \leq +1$ and $0 \leq x \leq 3$ consider

$$f_p(x) \equiv \min(x - 1, \max(p, x - 2))$$

Here is the graph of $f_p(x)$ against $x$ for $p \approx 0$. 

![Graph of $f_p(x)$ against $x$ for $p \approx 0$.](image)
Where is the zero?

The behaviour of $f_p(x)$ depends qualitatively on $p$ and $x$ like this:

\[
\begin{array}{c|c|c}
   & -ve & positive \\
\hline
 0 & negative & \\
-1 & & +ve \\
\end{array}
\]

\[
\begin{align*}
f(1) = 0 & \iff p \geq 0 \\
f(2) = 0 & \iff p \leq 0 \\
f(\frac{3}{2}) = 0 & \iff p = 0
\end{align*}
\]

If there is some way of finding a zero of $f_p$, as a side-effect it will decide how $p$ stands in relation to 0.
Let’s bar the monster

\[ f : \mathbb{R} \to \mathbb{R} \text{ doesn’t hover} \text{ if,} \]

for any \( e < t, \exists x. (e < x < t) \land (fx \neq 0). \]

Any nonzero polynomial doesn’t hover.
Interval halving again

Suppose that $f$ doesn’t hover.

Let $a_0 \equiv 0$ and $e_0 \equiv 1$.

By recursion, consider

$$b_n \equiv \frac{1}{3}(2a_n + e_n) \quad \text{and} \quad d_n \equiv \frac{1}{3}(a_n + 2e_n).$$

Then $f(c_n) \neq 0$ for some $b_n < c_n < d_n$, so put

$$a_{n+1}, e_{n+1} \equiv \begin{cases} a_n, c_n \quad \text{if} \quad f(c_n) > 0 \\ c_n, e_n \quad \text{if} \quad f(c_n) < 0, \end{cases}$$

so by induction $f(a_n) < 0 < f(e_n)$.

But $a_n$ and $e_n$ are respectively (non-strictly) increasing and decreasing sequences, whose differences tend to 0.

So they converge to a common value $c$.

By continuity, $f(c) = 0$. 
Stable zeroes

The revised interval halving algorithm finds zeroes with this property:

\[ a \in \mathbb{R} \text{ is a stable zero of } f \]

if, for all \( e < a < t \),

\[ \exists yz. (e < y < a < z < t) \land (fy < 0 < fz \lor fy > 0 > fz). \]

Check that a stable zero of a continuous function really is a zero.

Classically, a zero is stable iff every nearby function (in the sup or \( l_\infty \) norm) has a nearby zero.
Straddling intervals

An open subspace $U \subset \mathbb{R}$ touches $S$, i.e. contains a stable zero, $a \in U \cap S$, iff $U$ contains a straddling interval,

$$[e, t] \subset U \quad \text{with} \quad fe < 0 < ft \quad \text{or} \quad fe > 0 > ft.$$

Proof $\iff$ The straddling interval is an intermediate value problem in miniature.

If an interval $[e, t]$ straddles with respect to $f$
then it also does so with respect to any nearby function.
The possibility operator

Write $\Diamond U$ if $U$ contains a straddling interval.

By hypothesis, $\Diamond I \iff \top$ (where $I$ is some open interval containing $\Pi$).

Trivially, $\Diamond \emptyset \iff \bot$.

$\Diamond \bigcup_{i \in I} U_i \iff \exists i. \Diamond U_i$.

Consider

$V^\pm \equiv \{x | \exists y : \mathbb{R}. \exists i : I. (fy \geq 0) \land [x, y] \subset U_i\}$

so $\Pi \subset V^+ \cup V^-$. 

Then $x \in (a, c) \subset V^+ \cap V^-$ by connectedness, with $fx \neq 0$ and $[x, y] \subset U_i$. 
The Possibility Operator as a Program

Let ♦ be a property of open subspaces of $\mathbb{R}$ that preserves unions and satisfies ♦ $U_0$ for some open interval $U_0$.

Then ♦ has an “accumulation point” $c \in U_0$, i.e. one of which every open neighbourhood $c \in U \subset \mathbb{R}$ satisfies ♦ $U$.

In the example of the intermediate value theorem, any such $c$ is a stable zero.

Interval halving again: let $a_0 \equiv 0$, $e_0 \equiv 1$
and, by recursion, $b_n \equiv \frac{1}{3}(2a_n + e_n)$ and $d_n \equiv \frac{1}{3}(a_n + 2e_n)$, so

$$\Diamond(a_n, e_n) \equiv \Diamond((a_n, d_n) \cup (b_n, e_n)) \Leftrightarrow \Diamond(a_n, d_n) \lor \Diamond(b_n, e_n).$$

Then at least one of the disjuncts is true,
so let $(a_{n+1}, e_{n+1})$ be either $(a_n, d_n)$ or $(b_n, e_n)$.

Hence $a_n$ and $e_n$ converge from above and below respectively to $c$.

If $c \in U$ then $c \in (a_n, e_n) \subset (c \pm \epsilon) \subset U$ for some $\epsilon > 0$ and $n$,
but ♦$(a_n, e_n)$ is true by construction,
so ♦$U$ also holds, since ♦ takes $\subset$ to $\Rightarrow$. 
Enclosing cells of higher dimensions

Straddling intervals can be generalised.

Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) with \( n \geq m \).

Let \( C \subset \mathbb{R}^n \) be a sphere, cube, etc.

\( C \) is an **enclosing cell** if

\( 0 \in \mathbb{R}^m \) lies in the interior of the image \( f(C) \subset \mathbb{R}^m \).

(There is a definition for locally compact spaces too.)

Write \( \Diamond U \) if \( U \subset \mathbb{R}^n \) contains an enclosing cell.

If \( \Diamond (\bigcup_{i \in I} U_i) \iff \exists i. \Diamond U_i \) then

cell halving finds stable zeroes of \( f \).
Modal operators, separately

\[ Z \equiv \{ x \in \mathbb{I} \mid fx = 0 \} \text{ is closed and compact.} \]
\[ W \equiv \{ x \mid fx \neq 0 \} \text{ is open.} \]
\[ S \text{ is the subspace of stable zeroes.} \]

For \( U \subset \mathbb{R} \) open, write \( \Box U \) if \( Z \subset U \) (or \( U \cup W = \mathbb{R} \)).

\( \Box X \) is true and \( \Box U \land \Box V \Rightarrow \Box (U \cap V) \)
\( \Diamond \emptyset \) is false and \( \Diamond (U \cup V) \Rightarrow \Diamond U \lor \Diamond V. \)

\( (Z \neq \emptyset) \iff \Box \emptyset \) is false
\( (S \neq \emptyset) \iff \Diamond \mathbb{R} \) is true

Both operators are Scott continuous.
Modal operators, together

The modal operators ♦ and □ for the subspaces $S \subset Z$ are related in general by:

\[
\Box U \land ♦ V \Rightarrow ♦ (U \cap V)
\]

\[
\Box U \iff (U \cup W = X)
\]

\[
♦ V \Rightarrow (V \not\subset W)
\]

$S$ is dense in $Z$ iff

\[
\Box (U \cup V) \Rightarrow \Box U \lor ♦ V
\]

\[
♦ V \Leftarrow (V \not\subset W)
\]

In the intermediate value theorem for functions that don’t hover (e.g. polynomials):

$S = Z$ in the **non-singular** case

$S \subset Z$ in the **singular** case (e.g. double zeroes).
Open maps

For continuous $f : X \to Y$,
if $V \subseteq Y$ is open, so is $f^{-1}(V) \subseteq X$
if $V \subseteq Y$ is closed, so is $f^{-1}(V) \subseteq X$
if $U \subseteq X$ is compact, so is $f(U) \subseteq Y$
(if $U \subseteq X$ is overt, so is $f(U) \subseteq Y$)

$f : X \to Y$ is **open** if,
whenever $U \subseteq X$ is open, so is $f(U) \subseteq Y$.

If $f : X \to Y$ is open then
if $V \subseteq Y$ is overt, so is $f^{-1}(V) \subseteq X$.

If $f : X \to Y$ is open then all zeroes are stable.
Examples of open maps

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, and $\det \left( \frac{\partial f_j}{\partial x_i} \right) \neq 0$.

If $f : \mathbb{C} \to \mathbb{C}$ is analytic and not constant — even if it has coincident zeroes.

Cauchy’s integral formula:

A disc $C \subset \mathbb{C}$ is enclosing iff $\oint_{\partial C} \frac{dz}{f(z)} \neq 0$.

Stokes’s theorem!
Possibility operators classically

Define ♦ \( U \) as \( U \cap S \neq \emptyset \),
for any subset \( S \subset \mathbb{R} \) whatever.

Then \( ♦ \left( \bigcup_{i \in I} U_i \right) \) iff \( \exists i. ♦ U_i \).

Conversely, if ♦ has this property, let
\[ S \equiv \{ a \in \mathbb{R} \mid \text{for all open } U \subset \mathbb{R}, \quad a \in U \Rightarrow ♦ U \}. \]
\[ W \equiv \mathbb{R} \setminus S = \bigcup \{ U \text{ open} \mid ¬ ♦ U \} \]
Then \( W \) is open and \( S \) is closed.

\( ¬ ♦ W \) by preservation of unions.
Hence ♦ \( U \) holds iff \( U \not\in W \), i.e. \( U \cap S \neq \emptyset \).

If ♦ had been derived from some \( S' \)
then \( S = \overline{S'} \), its closure.
Possibility operators: summary

◊ is defined, like compactness, in terms of unions of open subspaces, so it is a concept of **general topology**

The proof that ◊ preserves joins uses ideas from **geometric topology**, like connectedness and sub-division of cells.

◊ is like a bounded existential quantifier, so it’s **logic**.

A very general **algorithm** uses ◊ to find solutions of problems.

But classical point-set topology is too clumsy to take advantage of this.
A lambda calculus for topology — predicates

Only use predicates \((\phi, \psi)\) that denote open subspaces, equivalently, which are computably observable.

On \(\mathbb{N}, \mathbb{Z}, \mathbb{Q}\): \(n = m, n \neq m, n < m, n \leq m, n > m\) and \(n \geq m\).

On \(\mathbb{R}\): \(a \neq b, a < b\) and \(a > b\), but not \(a = b, a \leq b\) or \(a \geq b\).
(This is entirely familiar in numerical computation.)

Logically: \(\top\) (true), \(\bot\) (false), \(\phi \land \psi, \phi \lor \psi\) and \(\exists n : \mathbb{N}. \phi n\)
but not \(\neg \phi\) (not), \(\phi \Rightarrow \psi\) or \(\forall n : \mathbb{N}. \phi n\).

We shall also allow \(\exists x : \mathbb{R}. \phi x\) and \(\forall x : \mathbb{I}. \phi x\),
but not \(\forall \epsilon > 0. \phi \epsilon\).
Statements — comparing predicates

You can’t say very much in the language of predicates.

A statement is an equality $\phi \iff \psi$ or inequality $\phi \Rightarrow \psi$
where $\phi$ and $\psi$ are predicates (technical distinction).

Predicates can be existentially quantified ($\exists n : \mathbb{N}.\phi n$),
statements cannot.

Nested implication ($\phi \Rightarrow \psi \Rightarrow \theta$ is not allowed (in the current version).

Examples: $\phi \Rightarrow \bot$ (not $\phi$), $(a < b) \Rightarrow \bot (a \geq b)$, $(a \neq b) \Rightarrow \bot (a = b)$.
Open and closed subspaces

If \( \phi(x) \) is a predicate with a free variable (argument) \( x : \mathbb{R} \) then
\[ \{x \mid \phi(x)\} \subset \mathbb{R} \] is an open subspace and
\[ \{x \mid \phi(x) \leftrightarrow \bot\} \subset \mathbb{R} \] is a closed subspace.

We can think of \( \phi : \mathbb{R} \to (\odot) \) as a continuous function
whose target is the Sierpiński space.

\[ \{x \mid \phi(x)\} \subset \mathbb{R} \text{ is } \phi^{-1}(\top) \text{ and} \]
\[ \{x \mid \phi(x) \leftrightarrow \bot\} \subset \mathbb{R} \text{ is } \phi^{-1}(\bot). \]

The Sierpiński space \((\odot)\) has two points (classically)
one (called \(\odot\) or \(\top\)) is open, the other (\(\bullet\) or \(\bot\)) is closed.

It is not Hausdorff.
It appears in many textbooks as a pathetic counterexample.

It is the key to understanding:
topologies of function-spaces,
semantics of programming languages,
Abstract Stone Duality.
Compact subspaces

The **neighbourhoods** of a compact subspace are more important than its **points**.

This had emerged by about 1970 in the study of topologies of function-spaces.

A compact subspace $K$ (at least, of a Hausdorff space $H$) is determined by which open subspaces $U \subset H$ **cover** it — $K \subset U$.

We write $\Box U$ or $\Box \phi$ when this happens.

$\Box$ satisfies the modal laws, in particular $\Box H \Leftrightarrow \top$ and $\Box(U \cap V) \Leftrightarrow \Box U \land \Box V$. 
Directed unions

By the "finite open sub-cover" definition of compactness,
if \( \square \bigcup_{i \in I} U_i \) then \( \square \bigcup_{i \in F} U_i \) for some finite \( F \subset I \).

This definition can be simplified by assuming something about the system \( \{U_i \mid i \in I\} \).

We call it directed if \( I \) is non-empty (better, inhabited) and, for each \( U_i, U_j \) there's some \( U_k \) with \( U_i \cup U_j \subset U_k \).

Then the "finite open sub-cover" definition becomes:
\( \square \) preserves directed unions.
Scott continuity

A function between complete lattices that preserves directed unions is called **Scott continuous**.

Dana Scott (1972) defined the corresponding topology on function-spaces $X \rightarrow (\odot)$.

It is a special case (in fact, the critical one) of Ralph Fox’s **compact–open topology** (1945).

The function-space $X \rightarrow (\odot)$ is the topology (lattice of opens) of $X$, itself equipped with the Scott topology.

If $X$ is locally compact, so is $X \rightarrow (\odot)$.

In our language, **all functions are continuous** in the traditional Weierstrass “$\epsilon–\delta$” sense for $f : X \rightarrow \mathbb{R}$ in Scott’s “directed joins” sense for function-spaces.
Any compact subspace $K \subset H$ of a Hausdorff space is closed.

In the ambient Hausdorff space $H$, $x \neq y$ is an open predicate (since $H \subset H \times H$ is closed).

The compact subspace $K \subset H$ has a $\Box$ operator.

The closed subspace is defined by its open/observable non-membership predicate.

This is $\omega x \equiv \Box(\lambda y. x \neq y)$.

It says that $x \notin C$ iff $C \subset \{y \mid x \neq y\} \equiv \overline{\{x\}}$. 
Any closed subspace $C \subset K$ of a compact space is compact.

The ambient compact space $H$ has a $\Box$ operator that we call $\forall_K$
with $\forall_K \phi \iff \top$ iff $\phi \iff \top$.

The closed subspace has an open non-membership predicate $\omega$.

As a compact subspace, it has a $\Box$ operator given by

\[ \Box \phi \equiv \forall_K (\omega \lor \phi) \]

which says that $\phi$ and the complement $\omega$ of $C$ together cover $K$. 
The Intermediate Value Theorem in ASD

Develop □ as a finitary theory of compact subspaces using the modal laws.

Develop ♦ as a theory of overt subspaces using the modal laws in an entirely lattice ("de Morgan") dual way.

Use the modal laws for compact overt $K \subset \mathbb{R}$ to define a Dedekind cut, which is $\max K$. (Bishop-style constructive analysis uses total boundedness to do this.)

$[0, 1]$ is connected — by the usual argument.

♦ (defined using straddling intervals) preserves joins.

Use interval-halving with ♦ to find stable zeroes.
The subspaces $S$ and $Z$ again

In the non-singular case, $\square$ and $\Diamond$ make the zero-set compact overt. It therefore has a maximum element!

The operators $\square$ and $\Diamond$ are Scott-continuous throughout the parameter space (eg for $x^3 - 3px - 2q = 0$), unlike $Z$ and $S$ considered as sets.

In the possibly singular case (eg double zeroes) $Z$ (all zeroes) is closed and compact, but not necessarily overt $S$ (stable zeroes) is overt, but not necessarily closed.
Midlands Graduate School

Next week at Birmingham University.
www.cs.bham.ac.uk/~pbl/mgs

Four (hour) lectures on Abstract Stone Duality.

Others on Category Theory, Lambda Calculus, Denotational Semantics, Functional Programming, Quantum Programming, Game Semantics, etc.

Accommodation still available: email A.Jung@cs.bham.ac.uk

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