A Lambda Calculus for Real Analysis

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$\mathbb{N}$ and $\mathbb{Q}$ are discrete and Hausdorff.

So we have all six relations for them.

Geometric, not Intuitionistic, logic

A term $\sigma : \Sigma$ is called a proposition.

A term $\phi : \Sigma^X$ is called a predicate or open subspace.

We can form $\phi \wedge \psi$ and $\phi \vee \psi$.

Also $\exists n : \mathbb{N}.\ \phi x$, $\exists q : \mathbb{Q}.\ \phi x$, $\exists x : \mathbb{R}.\ \phi x$ and $\exists x : [0,1].\ \phi x$.

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But not $\exists x : X.\ \phi x$ for arbitrary X — it must be overt.

Negation and implication are not allowed.

Because:

- this is the logic of open subspaces;
- the function $\odot \leftrightarrows \bullet$ on $\binom{\odot}{\bullet}$ is not continuous;
- the Halting Problem is not solvable.

Universal quantification

When $K \subset X$ is compact (*e.g.* $[0,1] \subset \mathbb{R}$), we can form $\forall x\colon K.\ \phi x$.

$$\frac{x : K \ \vdash\ \top \Leftrightarrow \phi x}{\vdash\ \top \Leftrightarrow \forall x\colon K.\ \phi x}$$

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The quantifier is a (higher-type) function $\forall_K : \Sigma^K \to \Sigma$.

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Like everything else, it's Scott continuous.

The useful cases of this in real analysis are

$$\begin{array}{lcl} \forall x : K. \exists \delta > 0. \phi(x,\delta) & \Leftrightarrow & \exists \delta > 0. \forall x : K. \phi(x,\delta) \\ \forall x : K. \exists n. \phi(x,n) & \Leftrightarrow & \exists n. \forall x : K. \phi(x,n) \end{array}$$

in the case where $(\delta_1 < \delta_2) \quad \wedge \quad \phi(x,\delta_2) \Rightarrow \phi(x,\delta_1)$

or $(n_1 > n_2) \quad \wedge \quad \phi(x,n_2) \Rightarrow \phi(x,n_1)$.

Recall that uniform convergence, continuity, *etc.*
involve commuting quantifiers like this.

Propositions and statements

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For example, with $\downarrow a \equiv (\lambda d.\, d < a)$,
$\downarrow a = \downarrow b$ iff $a = b$ and $\downarrow a \Rightarrow \downarrow b$ iff $a \leq b$
in the arithmetical order.

Hence $a \leq b$, $a \geq b$ and $a = b$ are meaningful,
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in the arithmetical order.

Hence $a \leq b$, $a \geq b$ and $a = b$ are meaningful,
as statements, not as propositions.

In fact, $a \leq b$ is equivalent as a statement to $(a > b) \Rightarrow \bot$,
and $a = b$ to $(a \neq b) \Rightarrow \bot$.

We deal with $\forall \epsilon > 0$ by allowing ϵ as parameter or free variable.

Examples: continuity and uniform continuity

Recall that, from local compactness of $\mathbb{R}$,

$$\phi x \iff \exists \delta > 0.\ \forall y : [x \pm \delta].\ \phi y$$

Theorem: Every definable function $f : \mathbb{R} \to \mathbb{R}$ is continuous:

$$\epsilon > 0 \implies \exists \delta > 0.\ \forall y : [x \pm \delta].\ \left(|fy - fx| < \epsilon \right)$$

Proof: Put $\phi_{x,\epsilon} y \equiv \left(|fy - fx| < \epsilon \right)$, with parameters $x, \epsilon : \mathbb{R}$.

Theorem: Every function f is uniformly continuous
on any compact subspace $K \subset \mathbb{R}$:

$$\epsilon > 0 \implies \exists \delta > 0.\ \forall x : K.\ \forall y : [x \pm \delta].\ \left(|fy - fx| < \epsilon \right)$$

Proof: $\exists \delta > 0$ and $\forall x : K$ commute.

Example: Dini's theorem

Theorem: Let $f_n : K \to \mathbb{R}$ be an increasing sequence of functions

$$n : \mathbb{N},\ x : K \ \vdash\ f_n x \le f_{n+1} x : \mathbb{R}$$

that converges pointwise to $g : K \to \mathbb{R}$, so

$$\epsilon > 0,\ x : K \ \vdash\ \top \Leftrightarrow \exists n.\, gx - f_n x < \epsilon.$$

If K is compact then f_n converges to g uniformly.

Proof: Using the introduction and Scott continuity rules for $\forall$,

$$\begin{aligned} \epsilon > 0 \ \vdash\ \top \quad & \Leftrightarrow \quad \forall x : K.\ \exists n.\, gx - f_n x < \epsilon \\ & \Leftrightarrow \quad \exists n.\, \forall x : K.\ gx - f_n x < \epsilon \end{aligned}$$

Corollary: Since ASD has a computational interpretation, Dini's theorem is computationally valid.

Relative containment of open subspaces

Let σ, α, β be propositions with parameters $x_1 : X_1, ..., x_k : X_k$.

(We conventionally write Γ for this list.
Semantically, Γ is the space $X_1 \times \cdots \times X_k$.)

Then σ, α, β define open subspaces of Γ.

They satisfy a Gentzen-stle rule of inference:

$$\frac{\Gamma,\ \sigma \Leftrightarrow \top \ \vdash\ \alpha \ \Rightarrow\ \beta}{\Gamma\ \vdash \sigma \wedge \alpha \ \Rightarrow\ \beta}$$

in which the top line means

within the open subspace of Γ defined by σ,
the open subspace defined by α
is contained in the open subspace defined by β.

and the bottom line means

the intersection of the open subspaces defined by σ and α
is contained in that defined by β.

Relative containment of closed subspaces

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Semantically, Γ is the space $X_1 \times \cdots \times X_k$.)

Then σ, α, β define closed subspaces of Γ.

They satisfy a Gentzen-stle rule of inference:

$$\frac{\Gamma,\ \sigma \Leftrightarrow \bot\ \vdash\ \alpha\ \Rightarrow\ \beta}{\Gamma\ \vdash \alpha\ \Rightarrow\ \sigma \vee \beta}$$

in which the top line means

> *within the closed subspace of Γ defined by σ,*
> *the closed subspace defined by α*
> *contains the closed subspace defined by β.*

and the bottom line means

> *the intersection of the closed subspaces defined by σ and β*
> *is contained in that defined by α.*

Exercise for everyone!

Make a habit of trying to formulate statements in analysis according to (the restrictions of) the ASD language.

This may be easy — it may not be possible

The exercise of doing so may be 95% of solving your problem!

Constructive intermediate value theorem

Suppose that $f : \mathbb{R} \to \mathbb{R}$ doesn't hover, *i.e.*

$$b, d : \mathbb{R} \vdash b < d \Rightarrow \exists x. (b < x < d) \wedge (fx \neq 0),$$

and $f0 < 0 < f1$. Then $fc = 0$ for some $0 < c < 1$.

Interval trisection: Let $a_0 \equiv 0$, $e_0 \equiv 1$,

$$b_n \equiv \tfrac{1}{3}(2a_n + e_n) \quad \text{and} \quad d_n \equiv \tfrac{1}{3}(a_n + 2e_n).$$

Then $f(c_n) \neq 0$ for some $b_n < c_n < d_n$, so put

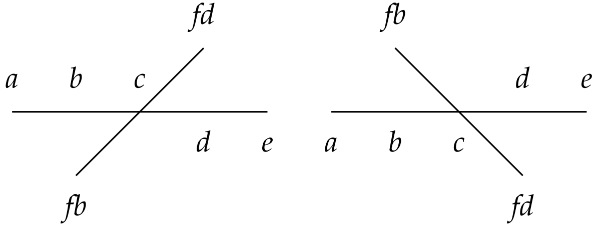
$$a_{n+1},\ e_{n+1} \equiv \begin{cases} a_n,\ c_n & \text{if } f(c_n) > 0 \\ c_n,\ e_n & \text{if } f(c_n) < 0. \end{cases}$$

Then $f(a_n) < 0 < f(e_n)$ and $a_n \to c \leftarrow e_n$.

(This isn't the ASD proof/algorithm yet!)

Stable zeroes

The interval trisection finds zeroes with this property:



Definition: $c : \mathbb{R}$ is a stable zero of f if

$$a, e : \mathbb{R} \vdash a < c < e \Rightarrow \exists bd. \quad (a < b < c < d < e) \\ \wedge \quad (fb < 0 < fd \ \vee fb > 0 > fd).$$

The subspace $Z \subset [0,1]$ of all zeroes is compact.
The subspace $S \subset [0,1]$ of stable zeroes is overt (as we shall see...)

Straddling intervals

An open subspace $U \subset \mathbb{R}$ contains a stable zero $c \in U \cap S$ iff U also contains a straddling interval,

$$[b, d] \subset U \quad \text{with} \quad fb < 0 < fd \quad \text{or} \quad fb > 0 > fd.$$

[$\Rightarrow$] From the definitions. [$\Leftarrow$] The straddling interval is an intermediate value problem in miniature.

Straddling intervals

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[⇒] From the definitions. [⇐] The straddling interval is an intermediate value problem in miniature.

Notation: Write $\Diamond\, U$ if U contains a straddling interval.

We write this containment in ASD using the universal quantifier.

$$\begin{aligned} \Diamond\,\phi \;\equiv\; \exists bd. \quad & (\forall x : [b, d].\ \phi x) \\ \wedge \quad & (fb < 0 < fd) \vee (fb > 0 > fd). \end{aligned}$$

The possibility operator

By hypothesis, $\lozenge(0,1) \Leftrightarrow \top$, whilst $\lozenge\,\emptyset \Leftrightarrow \bot$ trivially.

$\lozenge \bigcup_{i\in I} U_i \iff \exists i.\ \lozenge\, U_i$.

If $f : \mathbb{R} \to \mathbb{R}$ is an open map, this is easy.

If $f : \mathbb{R} \to \mathbb{R}$ doesn't hover, it depends on connectedness of $\mathbb{R}$.

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Definition: A term $\Diamond : \Sigma^{\Sigma^X}$ with this property
is called an overt subspace of X.

A simpler example: For any point $a : X$,
the neighbourhood filter $\Diamond \equiv \eta a \equiv \lambda\phi.\ \phi a$ is a possibility operator.

$\Diamond$ is a point iff it also preserves $\top$ and $\wedge$.

The Possibility Operator as a Program

Theorem: Let $\lozenge$ be an overt subspace of $\mathbb{R}$ with $\lozenge\top \Leftrightarrow \top$.

Then $\lozenge$ has an accumulation point $c \in \mathbb{R}$,
i.e. one of which every open neighbourhood $c \in U \subset \mathbb{R}$ satisfies $\lozenge U$:

$$\phi : \Sigma^{\mathbb{R}} \vdash \phi c \Rightarrow \lozenge\phi$$

Example: In the intermediate value theorem, any such c is a stable zero.

Proof: Interval trisection.

Corollary: Obtain a Cauchy sequence from a Dedekind cut.

(I expect to get a representation $\mathbf{2}^{\mathbb{N}} \rightharpoonup \mathbb{R}$ in the sense of TTE by proving a result of Brattko & Hertling in ASD.)

Possibility operators classically

Define $\Diamond U$ as $U \cap S \neq \emptyset$, for *any subset* $S \subset X$ *whatever*.

Then $\Diamond \left(\bigcup_{i \in I} U_i \right)$ iff $\exists i.\ \Diamond U_i$.

Conversely, if $\Diamond$ has this property, let

$$
\begin{aligned}
S &\equiv \{a \in X \mid \text{for all open } U \subset X, \quad a \in U \Rightarrow \Diamond U\} \\
W &\equiv X \setminus S = \bigcup \{U \text{ open} \mid \neg \Diamond U\}
\end{aligned}
$$

Then W is open and S is closed.
$\neg \Diamond W$ by preservation of unions.
Hence $\Diamond U$ holds iff $U \not\subset W$, *i.e.* $U \cap S \neq \emptyset$.

If $\Diamond$ had been derived from some S' then $S = \overline{S'}$, its closure.

Classically, every (sub)space S is overt.

Necessity operators

Let $K \subset \mathbb{R}$ be any compact subspace.

(For example, all zeroes in a bounded interval.)

$U \mapsto (K \subset U)$ is Scott continuous.

Notation: Write $\Box\,\phi$ for $\forall x : K.\ \phi x$.

Modal operators, separately

$\Box$ encodes the compact subspace $Z \equiv \{x \in \mathbb{I} \mid fx = 0\}$ of all zeroes.
$\Diamond$ encodes the overt subspace S of stable zeroes.

$$\Box X \text{ is true} \quad \text{and} \quad \Box U \wedge \Box V \Rightarrow \Box(U \cap V)$$
$$\Diamond \emptyset \text{ is false} \quad \text{and} \quad \Diamond(U \cup V) \Rightarrow \Diamond U \vee \Diamond V.$$

$$(Z \neq \emptyset) \quad \text{iff} \quad \Box \emptyset \text{ is false}$$
$$(S \neq \emptyset) \quad \text{iff} \quad \Diamond \mathbb{R} \text{ is true}$$

Modal operators, together

$\lozenge$ and $\square$ for the subspaces $S \subset Z$ are related in general by:

$$\square U \land \lozenge V \;\Rightarrow\; \lozenge(U \cap V)$$

$$\square U \iff (U \cup W = X)$$

$$\lozenge V \;\Rightarrow\; (V \not\subset W)$$

S is dense in Z iff

$$\square(U \cup V) \;\Rightarrow\; \square U \lor \lozenge V$$

$$\lozenge V \;\Leftarrow\; (V \not\subset W)$$

In the intermediate value theorem
for functions that don't hover (*e.g.* polynomials):

- $S = Z$ in the non-singular case
- $S \subset Z$ in the singular case (*e.g.* double zeroes).

Modal laws in ASD notation

Overt subspace	Compact subspace
$\Diamond\bot \iff \bot$	$\Box\top \iff \top$
$\Diamond(\phi \vee \psi) \iff \Diamond\phi \vee \Diamond\psi$	$\Box(\phi \wedge \psi) \iff \Box\phi \wedge \Box\psi$
$\sigma \wedge \Diamond\phi \iff \Diamond(\sigma \wedge \phi)$	$\sigma \vee \Box\phi \iff \Box(\lambda x.\, \sigma \vee \phi x)$

Commutative laws:

$$\Diamond\big(\lambda x.\, \blacklozenge(\lambda y.\, \phi xy)\big) \iff \blacklozenge\big(\lambda y.\, \Diamond(\lambda x.\, \phi xy)\big)$$

$$\Box\big(\lambda x.\, \blacksquare(\lambda y.\, \phi xy)\big) \iff \blacksquare\big(\lambda y.\, \Box(\lambda x.\, \phi xy)\big)$$

Mixed modal laws for a compact overt subspace.

$$\Box\phi \vee \Diamond\psi \Leftarrow \Box(\phi \vee \psi) \quad \text{and} \quad \Box\phi \wedge \Diamond\psi \Rightarrow \Diamond(\phi \wedge \psi)$$

Empty/inhabited is decidable

Theorem: Any compact overt subspace $(\Box, \Diamond)$ is either empty $(\Box\,\bot)$ or non-empty $(\Diamond\,\top)$.

Proof:

$$\begin{array}{ccc} \Diamond\,\top \;\Leftrightarrow\; \bot & \text{empty} & \Box\,\bot \;\Leftrightarrow\; \top \\ \Diamond\,\top \;\Leftrightarrow\; \top & \text{inhabited} & \Box\,\bot \;\Leftrightarrow\; \bot \\ \Box\,\bot \vee \Diamond\,\top \;\Leftarrow & \text{complementary} & \Box\,\bot \wedge \Diamond\,\top \;\Rightarrow \\ \Box(\bot \vee \top) \;\Leftrightarrow\; \Box\,\top \;\Leftrightarrow\; \top & \text{(mixed)} & \Diamond(\bot \wedge \bot) \;\Leftrightarrow\; \Diamond\,\bot \;\Leftrightarrow\; \bot \end{array}$$

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Proof:

$\Diamond\,\top \;\Leftrightarrow\; \bot$	empty	$\Box\,\bot \;\Leftrightarrow\; \top$
$\Diamond\,\top \;\Leftrightarrow\; \top$	inhabited	$\Box\,\bot \;\Leftrightarrow\; \bot$
$\Box\,\bot \vee \Diamond\,\top \;\Leftarrow$	complementary	$\Box\,\bot \wedge \Diamond\,\top \;\Rightarrow$
$\Box(\bot \vee \top) \;\Leftrightarrow\; \Box\,\top \;\Leftrightarrow\; \top$	(mixed)	$\Diamond(\bot \wedge \bot) \;\Leftrightarrow\; \Diamond\,\bot \;\Leftrightarrow\; \bot$

The dichotomy (either $\Box\,\bot$ or $\Diamond\,\top$) means that the parameter space Γ is a disjoint union.

So, if it is connected, like $\mathbb{R}^n$, something must break at singularities.

It is the modal law $\Box(\phi \vee \psi) \Rightarrow \Box\,\phi \vee \Diamond\,\psi$.

Non-empty compact overt subspace of $\mathbb{R}$ has a maximum

Theorem: Any overt compact subspace $K \subset \mathbb{R}$ is

- either empty
- or has a **greatest** element, $\max K \in K$.

Definition: $\max K$ satisfies, for $x : \mathbb{R}$,

$$
\begin{array}{rcl}
(x < \max K) & \Leftrightarrow & (\exists k : K.\ x < k) \\
(\max K < x) & \Leftrightarrow & (\forall k : K.\ k < x) \\
k : K & \vdash & k \le \max K
\end{array}
$$

$$
\frac{\Gamma,\ k : K \ \vdash\ k \le x}{\Gamma \ \vdash\ \max K \le x}
$$

Compact overt subspace of ℝ has a maximum

Proof: Define a Dedekind cut (next slide)

$$\delta d \;\equiv\; \exists k : K.\ d < k \quad \text{and} \quad \upsilon u \;\equiv\; \forall k : K.\ k < u$$

Hence there is some $a : \mathbb{R}$ with

$$\delta d \;\Leftrightarrow\; (d < a) \quad \text{and} \quad \upsilon u \;\Leftrightarrow\; (a < u)$$

Moreover, $a \in K$.

K is also the closed subspace
co-classified by $\omega x \equiv \Box(\lambda k.\, x \neq k)$,
so we must show that $\omega a \Leftrightarrow \bot$.

$$\begin{aligned}
\omega a \;\equiv\; \Box(\lambda k.\, a \neq k) \;&\Leftrightarrow\; \Box(\lambda k.\, a < k) \;\vee\; (k < a) \\
&\Rightarrow\; \Diamond(\lambda k.\, a < k) \;\vee\; \Box(\lambda k.\, k < a) \\
&\equiv\; \delta a \vee \upsilon a \\
&\Leftrightarrow\; (a < a) \vee (a < a) \;\Leftrightarrow\; \bot.
\end{aligned}$$

Compact overt subspace of $\mathbb{R}$ defines a Dedekind cut

Overt subspace $\Diamond$		Compact subspace $\Box$
$\bot, \vee, \bigvee^{\uparrow}$ and so $\exists_{\mathbb{R}}$	commutes with	$\top, \wedge$ and $\bigvee^{\uparrow}$
$\delta d \;\equiv\; \Diamond(\lambda k.\, d < k)$	Dedekind cut	$\upsilon u \;\equiv\; \Box(\lambda k.\, k < u)$
$(d < e) \wedge \delta e \;\equiv$	lower/upper	$\upsilon t \wedge (t < u) \;\equiv$
$(d < e) \wedge \Diamond(\lambda k.\, e < k)$		$\Box(\lambda k.\, k < t) \wedge (t < u)$
$\Leftrightarrow\; \Diamond(\lambda k.\, d < e < k)$	(Frobenius/$\Box\top$)	$\Leftrightarrow\; \Box(\lambda k.\, k < t < u)$
$\Rightarrow\; \Diamond(\lambda k.\, d < k) \;\equiv\; \delta d$	(transitivity)	$\Rightarrow\; \Box(\lambda k.\, k < u) \;\equiv\; \upsilon u$
$\Leftarrow$	rounded (interpolation)	$\Leftarrow$
$\exists d.\, \delta d \;\equiv\; \exists d.\; \Diamond(\lambda k.\, d < k)$	inhabited	$\exists u.\, \upsilon u \;\equiv\; \exists u.\; \Box(\lambda k.\, k < u)$
$\Leftrightarrow\; \Diamond(\lambda k.\, \exists d.\, d < k)$	(directed joins)	$\Leftrightarrow\; \Box(\lambda k.\, \exists u.\, k < u)$
$\Leftrightarrow\; \Diamond\top \;\Leftrightarrow\; \top$ (inhabited)	(extrapolation)	$\Leftrightarrow\; \Box\top \;\Leftrightarrow\; \top$

The Bishop-style proof

Definition: K is totally bounded if, for each $\epsilon > 0$, there's a finite subset $S_\epsilon \subset K$ such that $\forall x : K.\ \exists y \in S_\epsilon.\ \left|x - y\right| < \epsilon$.

Proof: If K is closed and totally bounded,

- either the set S_1 is empty, in which case K is empty too,
- or $x_n \equiv \max S_{2^{-n}}$ defines a Cauchy sequence that converges to $\max K$.

But K is also overt, with $\Diamond \phi \ \equiv\ \exists \epsilon > 0.\ \exists y \in S_\epsilon.\ \phi y$.

Definition: K is located if, for each $x \in X$, $\inf\{|x - k| \mid k \in K\}$ is defined.
(A different usage of the word "located".)

closed, totally bounded $\Rightarrow$ compact and overt $\Rightarrow$ located
(in TTE) also r.e. closed

- Total boundedness and locatedness are metrical concepts.
- Compactness and overtness are topological.

The real interval is connected (usual proof)

Any closed subspace of a compact space is compact.
Any open subspace of an overt space is overt.

Any clopen subspace of an overt compact space is overt compact, so it's either empty or has a maximum.

Since the clopen subspace is open, its elements are interior, so the maximum can only be the right endpoint of the interval.

Any clopen subspace has a clopen complement.

- They can't both be empty, but
- in the interval they can't both have maxima (the right endpoint).

Hence one is empty and the other is the whole interval.

Connectedness in modal notation

We have just proved

$$\Diamond(\phi \land \psi) \Leftrightarrow \bot,\ \Box(\phi \lor \psi) \Leftrightarrow \top \ \vdash\ \Box\phi \lor \Box\psi \Leftrightarrow \top$$

where $\Box\,\theta \ \equiv\ \forall x : [0,1].\ \theta x$ and $\Diamond\,\theta \ \equiv\ \exists x : [0,1].\ \theta x$.

Using the mixed modal law $\Diamond\phi \land \Box\psi \ \Rightarrow\ \Diamond(\phi \land \psi)$
and the Gentzen-style rules

$$\frac{\sigma \Leftrightarrow \top \ \vdash\ \alpha \ \Rightarrow\ \beta}{\vdash \sigma \land \alpha \ \Rightarrow\ \beta} \qquad \frac{\sigma \Leftrightarrow \bot \ \vdash\ \alpha \ \Rightarrow\ \beta}{\vdash \alpha \ \Rightarrow\ \beta \lor \sigma}$$

connectedness may be expressed in other ways:

$$\begin{array}{lcl}
\Diamond(\phi \land \psi) \Leftrightarrow \bot & \vdash & \Box(\phi \lor \psi) \ \Rightarrow\ \Box\phi \lor \Box\psi \\
\Diamond(\phi \land \psi) \Leftrightarrow \bot & \vdash & \Box(\phi \lor \psi) \land \Diamond\phi \land \Diamond\psi \ \Rightarrow\ \bot \\
\Box(\phi \lor \psi) & \Rightarrow & \Box\phi \lor \Box\psi \lor \Diamond(\phi \lor \psi) \\
\Box(\phi \lor \psi) \land \Diamond\phi \land \Diamond\psi & \Rightarrow & \Diamond(\phi \land \psi)
\end{array}$$

Weak intermediate value theorems

Let $f : [0,1] \to \mathbb{R}$, and use two of these forms of connectedness.

Put $\phi x \equiv (0 < fx)$ and $\psi x \equiv (fx < 0)$.
Use $\Diamond(\phi \wedge \psi) = \bot \ \vdash \ \Box(\phi \vee \psi) \wedge \Diamond\phi \wedge \Diamond\psi \ \Rightarrow \ \bot$.
$\Diamond(\phi \wedge \psi) \ \Leftrightarrow \ \bot$ by disjointness.
Then $(f0 < 0 < f1) \ \wedge \ \left(\forall x : [0,1].\, fx \neq 0\right) \ \Leftrightarrow \ \bot$.
So the closed, compact subspace $Z \equiv \{x : \mathbb{I} \mid fx = 0\}$ is not empty.

Put $\phi x \equiv (e < fx)$ and $\psi x \equiv (fx < t)$.
Use $\Box(\phi \vee \psi) \wedge \Diamond\phi \wedge \Diamond\psi \ \Rightarrow \ \Diamond(\phi \wedge \psi)$.
$\Box(\phi \vee \psi)$ by locatedness.
Then $(f0 < e < t < f1) \ \Rightarrow \ \left(\exists x : [0,1].\, e < fx < t\right)$.
or $\epsilon > 0 \ \vdash \ \exists x.\, \left|fx\right| < \epsilon$.
So the open, overt subspace $\{x \mid e < fx < t\}$ is inhabited.

Straddling intervals in ASD

Let $f : [0,1] \to \mathbb{R}$ be a function that doesn't hover.

Proposition: $\Diamond$ preserves joins, $\Diamond(\exists n.\, \theta_n) \iff \exists n.\, \Diamond\, \theta_n$.

Proof: Consider
$\phi^{\pm} x \;\equiv\; \exists n.\, \exists y.\, (x < y < u) \wedge (fy \gtrless 0) \wedge \forall z : [x,y].\, \theta_n z.$
Then $\exists x.\, \phi^+ x \wedge \phi^- x$ by connectness.

Lemma: $0 < a < 1$ is a stable zero of f iff
it is an accumulation point of $\Diamond$, *i.e.* $\phi a \;\Rightarrow\; \Diamond\,\phi$.

Theorem: $\Diamond$ and $\Box$ obey $\Box\,\phi \wedge \Diamond\,\psi \;\Rightarrow\; \Diamond(\phi \wedge \psi)$.

They also obey $\Box(\phi \vee \psi) \;\Rightarrow\; \Box\,\phi \vee \Diamond\,\phi$
iff f doesn't touch the axis without crossing it.

When f is a polynomial, this is the non-singular case, where f has no zeroes of even multiplicity.

Solving equations in ASD

In the non-singular case, all zeroes are stable,
$\lozenge$ and $\Box$ define a non-empty overt compact subspace,
which has a maximum.

So the classical textbook proof of IVT,

$$a \equiv \ \mathsf{sup}\,\{x : [0,1] \mid fx \le 0\},$$

is computationally meaningful!

The set of zeroes varies discontinuously at singularities in the parameters.

The modal operators $\Box$ and $\lozenge$ are Scott-continuous throughout the parameter space.

The interval trisection algorithm for $\lozenge$ finds some zero,
even in the singular case,
but it behaves non-deterministically and catastrophically.

Differentiation

Define $(fx, f'x)$ together by a Dedekind cross-hair.

Characterise $(e_0 < fx < t_0) \wedge (e_1 < f'x < t_1)$ by

$$\begin{aligned} \exists\delta.\ \forall h : [0,\delta]. \qquad & e_1 + e_1 h < f(x+h) < t_0 + t_1 h \\ \wedge \quad & e_1 - t_1 h < f(x-h) < t_0 - e_1 h \end{aligned}$$

This is a Dedekind cut in (e_0, t_0) since $f : \mathbb{R} \to \mathbb{R}$ is a function.

It is bounded in (e_1, t_1) if f is Lipschitz at x.

It is a Dedekind cut in (e_1, t_1) if f is differentiable at x.

I need help!

I'm a categorist, not an analyst.

I last did real analysis as a second year undergraduate.

I need a real analyst to set an agenda for me.

I also need a job from September 2006.