Interval Analysis Without Intervals

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Real Numbers and Computers 7
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www.cs.man.ac.uk/~pt/ASD

A theorist amongst programmers

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- a logic that is complete for computably continuous functions $\mathbb{R}^{\mathbf{n}} \to \mathbb{R}$
- and some vague ideas for programming with it.

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I want you to tell me

- how you could use my ideas to extend your exact real arithmetic systems,
- what other theoretical issues (such as backtracking) emerge from your programming,
- and can you implement my language?

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But it is a strong drug — it becomes more effective when it is diluted.

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In 2004 (with Andrej Bauer) I began to apply it to the real line.

It worked very nicely.

Keeping to the original idea, it says that
the real line is Dedekind complete (NB!)
and has the Heine–Borel property ($[0, 1]$ is compact).

The language that I shall discuss today
is the fragment of the main ASD calculus for the type $\mathbb{R}$.

I have been impressed by

Intellectual diversity — many different skills applied to $\mathbb{R}$.

Theoretical issues that emerge from programming —
e.g. when and how to back-track to improve precision.

The logical content of crude arithmetic —
e.g. the Interval Newton algorithm.

I am not impressed by

Timing benchmarks.

Excessive attention to representations of real numbers.
Heavy dependency on dyadic rationals or Cauchy sequences.

Theory without insight.
Naïve and dogmatic application of naïve set theory.
This applies especially to the “theoretical foundations” of Interval Analysis.

What's in it for you?

A theoretical framework
on which to structure your programming.

Not just exact real arithmetic, but also analysis.

How to generalise interval computations
to $\mathbb{R}^n$, $\mathbb{C}$ and other (locally compact) spaces from geometry.

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Step functions and lots of other things are definable as functions to other spaces besides $\mathbb{R}$, such as the interval domain.

A very important non-Hausdorff space

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ASD exploits the analogy amongst

- (continuous) functions $X \to \Sigma$,
- programs $X \to \Sigma$,
- open subspaces $U \subset X$,
- recursively enumerable subspaces $U \subset X$,
- and observable properties of $x \in X$.

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Similar methods have been used in compiler design, where $X \to \Sigma$ is the type of continuations from X.

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Topologically, it is because $\mathbb{R}$ is Hausdorff but not discrete.

On the other hand $\mathbb{N}$ and $\mathbb{Q}$ are discrete and Hausdorff,
so we have all six relations for them.

The logic of observable properties

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Also $\exists n : \mathbb{N}.\ \phi x$, $\exists q : \mathbb{Q}.\ \phi x$, $\exists x : \mathbb{R}.\ \phi x$ and $\exists x : [0,1].\ \phi x$.

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Negation and implication are not allowed.

Because:

- this is the logic of open subspaces;
- the function $\odot \leftrightarrows \bullet$ on $\binom{\odot}{\bullet}$ is not continuous;
- the Halting Problem is not solvable.

Universal quantification

When $K \subset X$ is compact (*e.g.* $[0,1] \subset \mathbb{R}$), we can form $\forall x\colon K.\ \phi x$.

$$\frac{\ldots,\ x : K \ \vdash \ \phi x}{\ldots \ \vdash \ \forall x\colon K.\ \phi x}$$

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The useful cases of this in real analysis are

$$\begin{array}{lcl} \forall x : K. \exists \delta > 0. \phi(x, \delta) & \Leftrightarrow & \exists \delta > 0. \forall x : K. \phi(x, \delta) \\ \forall x : K. \exists n. \phi(x, n) & \Leftrightarrow & \exists n. \forall x : K. \phi(x, n) \end{array}$$

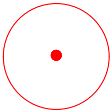
in the case where $(\delta_1 < \delta_2) \quad \wedge \quad \phi(x, \delta_2) \ \Rightarrow \ \phi(x, \delta_1)$

or $(n_1 > n_2) \quad \wedge \quad \phi(x, n_2) \ \Rightarrow \ \phi(x, n_1)$.

Recall that uniform convergence, continuity, *etc.*
involve commuting quantifiers like this.

Local compactness

Wherever a point a lies in the open subspace represented by ϕ, so ϕa in my logical notation,



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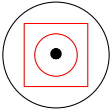

there are a compact subspace K and an open one representing β such that a is in the open set, *i.e.* βa and the open set is contained in the compact one, $\forall x \in K.\ \beta x$.

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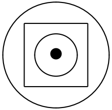

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In fact β and K come from a basis that is encoded in some way.

For example, for $\mathbb{R}$, β and K may be the open and closed intervals with dyadic rational endpoints p, q.

Then $\phi a \iff \exists p, q : \mathbb{Q}.\ a \in (p, q) \wedge \forall x \in [p, q].\ \phi x$.

Alternatively, $\phi a \iff \exists \delta > 0.\ \forall x \in [a \pm \delta].\ \phi x$.

Examples: continuity and uniform continuity

Theorem: Every definable function $f : \mathbb{R} \to \mathbb{R}$ is continuous:

$$\epsilon > 0 \;\Rightarrow\; \exists\delta > 0.\, \forall y : [x \pm \delta].\, \left(\left|fy - fx\right| < \epsilon\right)$$

Proof: Put $\phi_{x,\epsilon} y \equiv \left(\left|fy - fx\right| < \epsilon\right)$, with parameters $x, \epsilon : \mathbb{R}$.

Theorem: Every function f is uniformly continuous on any compact subspace $K \subset \mathbb{R}$:

$$\epsilon > 0 \;\Rightarrow\; \exists\delta > 0.\, \forall x : K.\, \forall y : [x \pm \delta].\, \left(\left|fy - fx\right| < \epsilon\right)$$

Proof: $\exists\delta > 0$ and $\forall x : K$ commute.

Dedekind completeness

A real number a is specified
by saying whether (real or rational) numbers d, u
are bounds for it: $d < a < u$.

Historically first example: Archimedes calculated π (the area of a circle)
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The one that isn't is called locatedness:
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Pseudo-cuts that are not (necessarily) located are called intervals.

A lambda-calculus for Dedekind cuts

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Given any pair $[\delta, \upsilon]$ of predicates for which the axioms of a Dedekind cut are provable,
we may introduce a real number:

$$\dfrac{\begin{matrix}[d : \mathbb{R}] \\ \vdots \\ \delta d : \Sigma\end{matrix} \qquad \begin{matrix}[u : \mathbb{R}] \\ \vdots \\ \upsilon u : \Sigma\end{matrix} \qquad \text{axioms for Dedekind cut}}{(\mathsf{cut}\, du.\, \delta d \wedge \upsilon u) : \mathbb{R}}$$

A λ-calculus for Dedekind cuts

The elimination rules recover the axioms.

The β-rule says that $(\mathsf{cut}\, du.\, \delta d \wedge \upsilon u)$ obeys the order relations that δ and υ specify:

$$e < (\mathsf{cut}\, du.\, \delta d \wedge \upsilon u) < t \qquad \Longleftrightarrow \qquad \delta e \wedge \upsilon t.$$

As in the λ-calculus, this simply substitutes part of the context for the bound variables.

The η-rule says that any real number a defines a Dedekind cut in the obvious way:

$$\delta d \;\equiv\; (d < a), \qquad \text{and} \qquad \upsilon u \;\equiv\; (a < u).$$

Summary of the syntax

		$\mathbb{N}$	$\mathbb{R}$	$\mathbb{N}\&\Sigma$	$\mathbb{R}\&\Sigma$	$\mathbb{N}\&?$	Σ
$\mathbb{N}$	0	succ				rec	the
$\mathbb{R}$	$0, 1$	n	$+, -, \times, \div$			rec	cut
Σ	$\top, \bot$	$=, \leqslant, \geqslant$ $<, >, \neq$	$<, >, \neq$	$\exists n$	$\exists x : \mathbb{R}$ $\forall x : [a, b]$	rec	$\wedge, \vee$

the: definition by description.
cut: Dedekind completeness.

A valuable exercise

Make a habit of trying to formulate statements in analysis according to (the restrictions of) the ASD language.

This may be easy — it may not be possible

The exercise of doing so may be 95% of solving your problem!

Real numbers and representable intervals

The language that we have described

- has continuous variables and terms

$$a, \quad b, \quad c, \quad x, \quad y, \quad z \qquad \text{(in \textit{italic})}$$

  that denote real numbers, or maybe vectors,
- about which we reason using pure mathematics, using ideas of real analysis.

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We need another language

- with discrete variables and terms

$$\mathsf{a}, \quad \mathsf{b}, \quad \mathsf{c}, \quad \mathsf{x}, \quad \mathsf{y}, \quad \mathsf{z} \qquad \text{(in \textsf{sans serif})}$$

  that denote machine-representable intervals or cells,
- with which we compute directly.

Cells for locally compact spaces

For computation on the real line, the interval $\mathbf{x}$ has machine representable endpoints $\underline{\mathbf{x}} \equiv d$ and $\overline{\mathbf{x}} \equiv u$.

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A basis for a locally compact space is a family of cells.

A cell $\mathbf{x}$ is a pair $U \subset K$ of spaces
with $(\mathbf{x}) \equiv U$ open and $[\mathbf{x}] \equiv K$ compact.
For example, $U \equiv (p, q)$ and $K \equiv [p, q]$ in $\mathbb{R}^{\mathbf{1}}$.

The cell $\mathbf{x}$ is encoded in some machine-representable way.
For example, p and q are dyadic rationals.

Theory and practice

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Suppose that you want to generalise interval computations to $\mathbb{R}^{\mathbf{2}}$, $\mathbb{R}^{\mathbf{n}}$, $\mathbb{C}$, the sphere S^2 or some other space.

Its natural cells may be respectively
hexagons, close-packed spheres or circular discs.

The geometry and computation of sphere packing in many dimensions is well known amongst group theorists.

Theory and practice

The theory of locally compact spaces tells us what we need to know about the system of cells:

- How are arbitrary open subspaces expressed as unions of basic ones?
- When is the compact subspace $[\mathbf{x}]$ of one cell contained in the open subspace $(\mathbf{y})$ of another?
  We write $\mathbf{x} \Subset \mathbf{y}$ for this observable relation.
- How are any finite intersections of basic compact subspaces
  covered by finite unions of basic open subspaces?

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From the theory we derive a plan for the programming:

- how are (finite unions of) cells to be represented?
- how are the arithmetic operations and relations to be computed?
- how are finite intersections covered by finite unions?

Logic for the representation of cells

Cells are ultimately represented in the machine as integers.

These are finite but arbitrarily large.

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Programming $\forall x \in [a, b]$ is based on the Heine–Borel theorem.

Some deliberately ambiguous notation

$$x \in \mathbf{a} \qquad \text{means} \qquad x \in (\mathbf{x}) \qquad \text{or} \qquad \underline{\mathbf{x}} < x < \overline{\mathbf{x}}.$$

$$\forall x \in \mathbf{x} \qquad \text{means} \qquad \forall x \in [\mathbf{x}] \qquad \text{or} \qquad \forall x \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}].$$

$$\exists x \in \mathbf{x} \qquad \text{means} \qquad \text{both } \exists x \in (\mathbf{x}) \text{ and } \exists x \in [\mathbf{x}]$$

because these are equivalent, so long as $\mathbf{x}$ is not empty, so $\underline{\mathbf{x}} < \overline{\mathbf{x}}$.

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These define a natural direction

$$\underset{\longrightarrow}{a \;\in\; \mathbf{b}} \quad \text{and} \quad \underset{\longrightarrow}{\mathbf{a} \;\Subset\; \mathbf{b}} \quad \text{but} \quad \underset{\longleftarrow}{\forall x \in \;\mathbf{a}}$$

which also goes up arithmetic expression trees,
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$\mathbf{a} \Subset \mathbf{y}$ is like the constraint $\mathbf{y}$ `is` $\mathbf{a}$ in some versions of Prolog.
This transfers the value of $\mathbf{a}$ to $\mathbf{y}$ and (unlike "=" considered as unification) not *vice versa*.

Another constraint, on the output precision

A lazy logic programming interpretation of this would be very lazy.

To make it do anything, we also need a way to specify the precision that we require of the output.

We squeeze the width $\|\mathbf{x}\| \equiv (\overline{\mathbf{x}} - \underline{\mathbf{x}})$ of an interval by the constraint

$$\|\mathbf{x}\| < \epsilon \quad \equiv \quad \forall x, y \in \mathbf{x}.\, \left|x - y\right| < \epsilon.$$

This is syntactic sugar — it is already definable as a predicate in our calculus.

Failure of this constraint (as of others) causes back-tracking. This is one of the cases of back-tracking that has already emerged from programming multiple-precision arithmetic.

Moore arithmetic

Returning specifically to $\mathbb{R}$, we write $\oplus, \ominus, \otimes$ for Moore's arithmetical operations on intervals:

$$
\begin{aligned}
\mathbf{a} \oplus \mathbf{b} &\equiv [\underline{\mathbf{a}} + \underline{\mathbf{b}},\ \overline{\mathbf{a}} + \overline{\mathbf{b}}] \\
\ominus \mathbf{a} &\equiv [-\overline{\mathbf{a}},\ -\underline{\mathbf{a}}] \\
\mathbf{a} \otimes \mathbf{b} &\equiv [\mathsf{min}(\underline{\mathbf{a}} \times \underline{\mathbf{b}}, \underline{\mathbf{a}} \times \overline{\mathbf{b}}, \overline{\mathbf{a}} \times \underline{\mathbf{b}}, \overline{\mathbf{a}} \times \overline{\mathbf{b}}), \\
&\qquad \mathsf{max}(\underline{\mathbf{a}} \times \underline{\mathbf{b}}, \underline{\mathbf{a}} \times \overline{\mathbf{b}}, \overline{\mathbf{a}} \times \underline{\mathbf{b}}, \overline{\mathbf{a}} \times \overline{\mathbf{b}})],
\end{aligned}
$$

and $\lessdot, \gtrdot, \pitchfork, \Subset$ for the computationally observable relations

$$
\begin{aligned}
\mathbf{x} \lessdot \mathbf{y} &\equiv \overline{\mathbf{x}} < \underline{\mathbf{y}} \quad \equiv \quad \mathbf{y} \gtrdot \mathbf{x} \\
\mathbf{x} \pitchfork \mathbf{y} &\equiv [\mathbf{x}] \cap [\mathbf{y}] = \emptyset \quad \text{or} \quad (\overline{\mathbf{x}} < \underline{\mathbf{y}}) \vee (\overline{\mathbf{y}} < \underline{\mathbf{x}}), \\
\mathbf{x} \Subset \mathbf{y} &\equiv \underline{\mathbf{x}} < \underline{\mathbf{y}} < \overline{\mathbf{x}} < \overline{\mathbf{y}}.
\end{aligned}
$$

NB: in $\mathbf{a} \lessdot \mathbf{b}$, $\mathbf{a} \gtrdot \mathbf{b}$ and $\mathbf{a} \pitchfork \mathbf{b}$, the intervals $\mathbf{a}$ and $\mathbf{b}$ are disjoint.

Extending the Moore operations to expressions

By structural recursion on syntax, we may extend the Moore operations from symbols to expressions.

Essentially, we just

$$\begin{array}{llllllllll} \text{replace} & x & + & - & \times & < & > & \neq & \in & \exists x \\ \text{by} & \mathbf{x} & \oplus & \ominus & \otimes & \olessthan & \ogreaterthan & \pitchfork & \Subset & \exists \mathbf{x} \end{array}$$

other variables, constants, $n : \mathbb{N}$, $\wedge$, $\vee$, $\exists n$, rec, the stay the same.

(We can't translate $\forall x \in [a, b]$ — yet.)

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other variables, constants, $n : \mathbb{N}$, $\wedge$, $\vee$, $\exists n$, rec, the stay the same.

(We can't translate $\forall x \in [a, b]$ — yet.)

This extends the meaning of arithmetic expressions fx and logical formulae ϕx in such a way that

- substituting $\mathbf{x} \equiv [x, x]$ recovers the original value,
- the dependence on the interval argument $\mathbf{x}$ is monotone,
- and substitution is preserved.

Of course, the laws of arithmetic are not preserved.

Extending the Moore operations to expressions

We shall write $\rlap{\forall}\mathsf{M}\, x \in \mathbf{x}.\, fx$ or $\rlap{\forall}\mathsf{M}\, x \in \mathbf{x}.\, \phi x$ for the translation of the arithmetical expression fx or logical formula ϕx.

The symbol $\rlap{\forall}\mathsf{M}$ is a cross between $\forall$ and M (for Moore).

Remember that it is a syntactic translation (like substitution). So the continuous variable x does not occur in $\rlap{\forall}\mathsf{M}\, x \in \mathbf{x}.\, fx$ or $\rlap{\forall}\mathsf{M}\, x \in \mathbf{x}.\, \phi x$.

$\rlap{\forall}\mathsf{M}$ is not a quantifier.

But there is a reason why it looks like one...

The fundamental theorem of interval analysis

Interval computation is reliable in the sense that
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More generally, bounding cells for computations in $\mathbb{R}^{n}$.

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any arbitrary open bounding cell that exists mathematically.

This is an ϵ–δ statement:
$\forall\epsilon > 0$ (the required output precision),
$\exists\delta > 0$ (the necessary size of the working intervals).

Locally compact spaces again

Recall the fundamental property of locally compact spaces:

$$\phi a \iff \exists \mathbf{x}.\, a \in \mathbf{x} \wedge \forall x \in \mathbf{x}.\, \phi x,$$

which means:

- if a satisfies the observable predicate ϕ
  (or a belongs to the open subspace that corresponds to ϕ),
- then a is in the interior of some cell $\mathbf{x}$
- throughout which ϕ holds
  (or which is contained in the open subspace that corresponds to ϕ).

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For example, $fa \in \mathbf{b} \iff \exists \mathbf{x}.\, a \in \mathbf{x} \wedge (|\!\!\forall\!\!| x \in \mathbf{x}.\, fx) \Subset \mathbf{b}$.

So we obtain arbitrary precision $\|\mathbf{b}\|$
by choosing the working interval $\mathbf{x}$ to be sufficiently small.

Solving equations

How do we find a zero of a function, x such that $0 = f(x)$?

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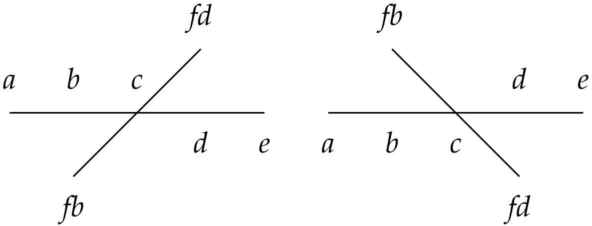
Any zero c that we can find numerically
is stable in the sense that,
arbitrarily closely to c, there are b, d with $b < c < d$
and either $f(b) < 0 < f(d)$ or vice versa.



Solving equations

The definition of a stable zero may be written in the calculus for continuous variables, and translated into intervals.

Write $\mathbf{x}$ for the outer interval $[a, e]$.

There are $b \in \mathbf{b}$, $c \in \mathbf{c}$ and $d \in \mathbf{d}$ with
$\mathbf{b} \olessthan \mathbf{c} \olessthan \mathbf{d}$ and $f(\mathbf{b}) \olessthan 0 \olessthan f(\mathbf{d})$.

So if the interval $\mathbf{x}$ contains a stable zero,
$0 \in \mathfrak{f}(\mathbf{x}) \equiv \forall x \in \mathbf{x}.\, f(x)$.
Remember that $\in$ means "in the interior".

This is how $\in f(\mathbf{x})$ and $\Subset f(\mathbf{x})$ arise
with an expression on the right of $\Subset$.

Logic programming with intervals

Remember that the continuous variable x does not occur in the translation $\forall\!\!\!\mid x \in \mathbf{x}.\ \phi x$ of ϕx. Of course, we eliminate the other continuous variables y, z, ... in the same way.

This leaves a predicate involving cellular variables like $\mathbf{x}$.

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This leaves a predicate involving cellular variables like $\mathbf{x}$.

We build up arithmetical and logical expressions in this order:

- the interval arithmetical operations $\oplus, \ominus, \otimes$;
- more arithmetical operations;
- the relations $\olessthan, \ogreaterthan, \pitchfork, \Subset$;
- conjunction $\wedge$;
- cellular quantification $\exists\mathbf{x}$;
- disjunction $\vee$, integer quantification $\exists n$ and recursion;
- universal quantification $\forall x \in [a, b]$;
- more conjunction, *etc.*

Some logic programming techniques

We can manipulate $\exists \mathbf{x}$ applied to $\wedge$ using various techniques of logic programming.

- Constraint logic programming,
  essentially due to John Cleary.
  This is the closest analogue of unification for intervals.
- Symbolic differentiation, to pass the required precision of outputs back to the inputs.
- The Interval Newton algorithm for solving equations, which are expressed as $0 \in f(\mathbf{x})$.
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But adding $\exists n$ and recursion makes it Turing complete.

The universal quantifier $\forall x \in [a, b]$ applied to $\vee$ and $\exists n$,
may be turned into a recursive program using the Heine–Borel property, with $\not\forall$ as its base base.

The $\exists\mathbf{x},\wedge$ fragment

We consider the fragment of the language consisting of formulae like

$$\exists \mathbf{y}_1\mathbf{y}_2\mathbf{y}_3.\ \ \mathbf{x}_2 \oplus \mathbf{y}_1 \olessthan \mathbf{x}_3 \otimes \mathbf{x}_1 \ \wedge\ \mathbf{x}_3 \neq \mathbf{y}_3$$

$$\wedge\ \mathbf{y}_1 \otimes \mathbf{x}_3 \Subset \mathbf{z}_2 \ \wedge\ 0 \in \mathbf{z}_1 \otimes \mathbf{z}_1 \ \wedge\ \|\mathbf{z}_1\| < 2^{-40}$$

in which the variables

- $\mathbf{x}_1, \mathbf{x}_2, \ldots$ are free and occur only as plugs (on the left of $\Subset$);
- $\mathbf{y}_1, \mathbf{y}_2, \ldots$ are bound, and may occur as both plugs and sockets;
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Using convex union, each socket contains at most one plug.

Since the relevant directed graph is acyclic, bound variables that occur as both plugs and sockets may be eliminated.
So wlog bound variables occur only as plugs.

Cleary's algorithm

In the context of the rest of the problem, the free plugs $\mathbf{x}_1, \mathbf{x}_2, \ldots$ have given interval values (the arguments, to their currently known precision). The other free and bound variables are initially assigned the completely undefined value $[-\infty, +\infty]$.

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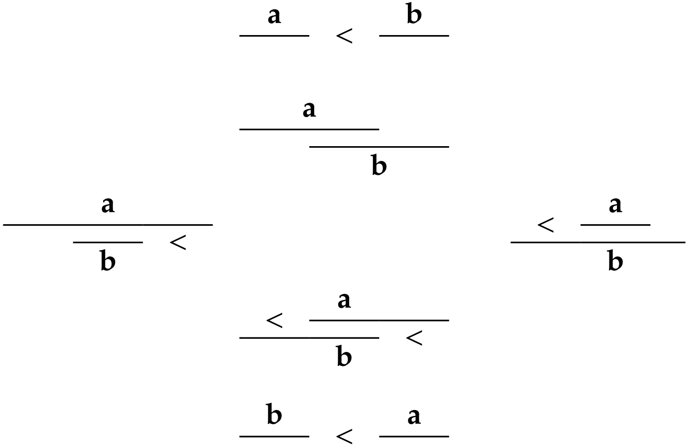
Then repeat the evaluation and test.
For this fragment, the algorithm terminates.

Cleary's "unification" rules for $\mathbf{a} \oslash \mathbf{b}$

There are six possibilities for the existing values of **a** and **b**. Remember that **a** and **b** are our current state of knowledge about certain real numbers $a \in \mathbf{a}$ and $b \in \mathbf{b}$ with $a < b$.



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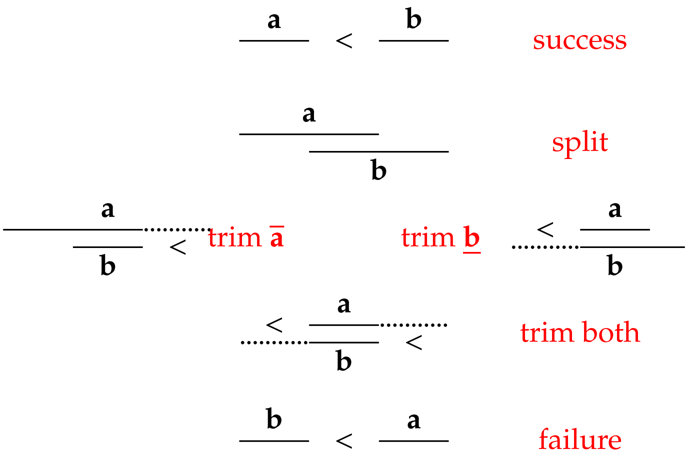

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Then we have to trim

- $\overline{\mathbf{a}}$ to $\overline{\mathbf{c}} - \underline{\mathbf{b}}$, and
- $\overline{\mathbf{b}}$ to $\overline{\mathbf{c}} - \underline{\mathbf{a}}$.

There are similar (but more complicated) rules for $\otimes$.

Moore's Interval Newton algorithm (my version)

Given a function f and and interval $\mathbf{x}$,

Evaluate

- the function f at a point x_0 in the middle of $\mathbf{x}$
- and the derivative f' on the whole interval: $\forall x \in \mathbf{x}.\, f'(x)$.

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This bounds the values of the function throughout the interval:

$$f(x) \quad \in \quad f(x_0) \oplus (x - x_0) \otimes \forall\!\!\!\!|\, x \in \mathbf{x}.\, f'(x)$$

This is a two-term Taylor series.
It's how we should define derivatives of interval-valued functions.

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Slogan: Crude arithmetic gives subtle logical information.

Translating the universal quantifier

Applying the translation to ϕx, we need to simplify

$$\forall x \in \mathbf{a}.\, \phi x \;\equiv\; \forall x \in \mathbf{a}.\, \exists \mathbf{x}.\, x \in \mathbf{x} \wedge \forall\!\!\!\mid x' \in \mathbf{x}.\, \phi x'.$$

This says that the compact (closed bounded) interval $\mathbf{a}$
is covered by the open interiors of cells $\mathbf{x}$
each of which satisfies the translation $\forall\!\!\!\mid x' \in \mathbf{x}.\, \phi x'$.

The Heine–Borel property (classical theorem, axiom of ASD)
says that there is a finite sub-cover,
so wlog $\|\mathbf{x}\| = 2^{-k}$ for some k.

Translating $\forall$ with $\vee$ and $\exists n$

It's natural to include ($\vee$ and) $\exists n$ in the Heine–Borel property:

$$\forall x \in [0,1].\, \exists n.\, \phi n x \qquad \Longleftrightarrow$$

$$\exists k.\, \bigwedge_{j=0}^{2^k-1} \exists n.\, \forall\!\!\!\!|\, x \in [j \cdot 2^{-k},\ (j+1)\cdot 2^{-k}].\, \phi_n x.$$

We can read this as a recursive program for

$$\theta[a,b] \ \equiv\ \forall x \in [a,b].\, \exists n.\, \phi_n x$$

that splits $[a,b]$ into subintervals. When these get smaller than $2^{-k}(b-a)$, use $\forall\!\!\!\!|$ instead of deeper recursion.

$$\begin{aligned}\theta[a,b] \quad \Longleftrightarrow \quad & \exists k.\, \left(\exists n.\, \forall\!\!\!\!|\, x \in [a,\ a+2^{-k}(b-a)].\, \phi_n x\right) \\ & \wedge\ \theta[a+2^{-k}(b-a),\ b]\end{aligned}$$

Conclusion: some programming projects

(Logic) programming environment together with multiple precision arithmetic.

Use this to implement:

- Cleary's algorithm, Interval Newton, ...
- Cellular computation for $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{C}$, ...
- Heine–Borel translation of $\forall$.

Syntactic stuff:

- Simple front end to translate the continuous language into the interval methods.
- Proof assistant for the deduction rules of ASD.