Cartesian Closure for Stable Categories (draft)

Paul Taylor

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1 Rigid Adjunctions

1.1 Rigid Comparisons

It is not apparent what meaning, if any, is to be attached to the group of automorphisms of a rigid comparison (perhaps this is a kind of "gauge invariance") but in view of the lemma it seems we can ignore it and make the

Definition 1.1 $\mathcal{RC}$ is the category whose (?)

- *objects* are stable categories, *i.e.*, for our purposes so far, categories with pullbacks, and
- *morphisms* are *isomorphism classes* of rigid comparisons (the right adjoints are also considered to exist only "up to isomorphism").

The symbol $\mathcal{RC}$, like the term stable category, is considered as a variable, parametric upon our specific requirements.

Exercises 1.2

(a) Composition in $\mathcal{RC}$ is well-defined by representatives.

(b) A morphism of $\mathcal{RC}$ is invertible $\iff$ it is a class of strong equivalences $\iff$ its counit is an isomorphism.

(c) Objects of $\mathcal{RC}$ are isomorphic iff they are equivalent categories.

(d) $\mathcal{RC}/\mathcal{T} \simeq \mathsf{Copt}(\mathcal{T})$ (?). □

Rigid comparisons embody an important idea from domain theory: *approximation*. This must have the property that if $\mathcal{X}'$ approximates $\mathcal{X}$ then $[\mathcal{X}' \to \mathcal{Y}]$ approximates $[\mathcal{X} \to \mathcal{Y}]$ and not *vice versa* as with precomposition with ordinary maps. Considering a stable functor to be given by its trace, we consider the diagram

$$\begin{tikzcd}
\mathcal{X} & \mathcal{T} \arrow[l, "C"'] \arrow[r, "F"] & \mathcal{Y} \\
\mathcal{X}' \arrow[u, "\Xi"] & \mathcal{T}' \arrow[l, "C'"'] \arrow[r, "F'"] \arrow[u, "\Phi"] & \mathcal{Y}' \arrow[u, "\Psi"]
\end{tikzcd}$$

in which C and C' are comparisons, F and F' isotomies and Ξ, Ψ and Φ are rigid comparisons (which are both comparisons and isotomies). As in the definition of $\mathcal{RC}$, we interpret "rigid comparison" as an isomorphism class, and the diagram is to commute up to isomorphism. The idea is that the functor $F'H' : \mathcal{X}' \to \mathcal{Y}'$ approximates $FH : \mathcal{X} \to \mathcal{Y}$, and its domain and codomain are also approximations.

This corresponds to a definition of polymorphism suggested by Eugenio Moggi: we may see **SCat** as an internal category in itself. The category-of-objects is $\mathcal{RC}$, and the category-of-morphisms has trace factorisations as objects and diagrams like this as morphisms; domain, codomain and identity are obvious forgetful functors and composition is tensor product. Moggi's definition of polymorphism is simply that **SCat** is an internal cartesian closed category which has products indexed by its category-of-objects.

1.2 Calculating Slices of Function-Spaces

We shall now calculate the category $\mathsf{Copt}(\mathcal{T}) = [\mathcal{T} \to \mathcal{T}]/\mathsf{id}$ explicitly in terms of $\mathcal{T}$, and hence show that the function-space $[\mathcal{X} \to \mathcal{Y}]$ has slices "of the same kind" as those of ($\mathcal{X}$ and) $\mathcal{Y}$. First,

Exercise 1.3 If $\mathcal{T}$ has a terminal object then $\mathsf{Copt}(\mathcal{T}) \simeq \mathcal{T}$, *i.e.* the *only* rigid comparisons into a category with finite limits are slice maps. [Hint: consider $M1$; *cf.* Lemma 2.2.4.] $\square$

Lemma 1.4 $T \mapsto \mathcal{T}/T$ induces an isotomy

$$\mathcal{T}/- : \mathcal{T} \to \mathcal{RC}$$

whose effect on morphisms, $t : T' \to T$, is given by the pair $t_! \dashv t^*$.

Proof Its restriction to $\mathcal{T}/T_0$ is the equivalence $\mathcal{T}/T_0 \simeq \mathsf{Copt}(\mathcal{T}/T_0) \simeq \mathcal{RC}/(\mathcal{T}/T_0)$. $\square$

(?) There are forgetful (pseudo)functors $\mathcal{RC} \to \mathbf{Cat}$ and $\mathcal{RC}^{\mathrm{op}} \to \mathbf{Cat}$ which take a rigid adjunction to its left and right parts respectively. Of course, we have the

Exercise 1.5 $\mathcal{T}$ is the (pseudo)colimit of the diagram $\mathcal{T} \to \mathcal{RC} \to \mathbf{Cat}$. $\square$

More interestingly,

Proposition 1.6 $\mathsf{Copt}(\mathcal{T})$ is the (pseudo)limit of the diagram $\mathcal{T}^{\mathrm{op}} \to \mathcal{RC}^{\mathrm{op}} \to \mathbf{Cat}$.

Proof What is an object of this limit? It is an assignment of an object $\kappa_T : MT \to T$ of $\mathcal{T}/T$ to each object $T \in \mathcal{T}$, consistently with the diagram, in the sense that $t^*(\kappa_T) = \kappa_{T'}$ for each $t : T' \to T$. This is exactly the content of the pullback square

$$\begin{tikzcd} MT' \arrow[r, "Mt"] \arrow[d, "\kappa_{T'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & MT \arrow[d, "\kappa_T"] \\ T' \arrow[r, "t"] & T \end{tikzcd}$$

which asserts that $\kappa : M \to \mathsf{id}$ is a cartesian transformation. This is enough to determine an object of $\mathsf{Copt}(\mathcal{T})$. A similar argument applies to morphisms. $\square$

Lemma 1.7 Any two subfunctions of the identity commute (up to canonical isomorphism), their composite being their product in $\mathsf{Copt}(\mathcal{T})$.

Proof Consider the two pullback diagrams which express naturality of κ_1 and κ_2 respectively:

$$\begin{tikzcd} M_1 \arrow[r, "\kappa_1"] & \mathsf{id} \\ M_1M_2 \arrow[u, "M_1\kappa_2"] \arrow[r, "\kappa_1 M_2"'] \arrow[ur, phantom, "\urcorner", very near start] & M_2 \arrow[u, "\kappa_2"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M_1 \arrow[r, "\kappa_1"] & \mathsf{id} \\ M_2M_1 \arrow[u, "\kappa_2 M_1"] \arrow[r, "M_2\kappa_1"'] \arrow[ur, phantom, "\urcorner", very near start] & M_2 \arrow[u, "\kappa_2"'] \end{tikzcd}$$

There is a unique mediating isomorphism $\tau : T_1T_2 \cong T_2T_1$. These diagrams are also clearly product diagrams in $\mathsf{Copt}(\mathcal{T})$. $\square$

Proposition 1.8 $\mathcal{RC}$ has pullbacks and $\mathsf{Copt}(\mathcal{T})$ has finite limits. Moreover these limits are constructed pointwise, and a pullback of rigid comparisons *quâ* rigid comparisons is also a pullback *quâ* (stable) functors.

Proof By the equivalence $\mathcal{RC}/\mathcal{T} \simeq \mathsf{Copt}(\mathcal{T})$ we have just constructed a typical pullback in $\mathcal{RC}$. Observe that objects of the pullback category, *i.e.* M_1M_2-coalgebras, correspond to pairs of objects at the other corners, *i.e.* a M_1-coalgebra together with a M_2-coalgebra whose carrier is isomorphic. This follows immediately from the fact that M_1M_2 is defined by a pullback. To construct a pullback over M in $\mathsf{Copt}(\mathcal{T})$ we find the product in $\mathsf{Copt}(\mathcal{T})/M \simeq \mathsf{Copt}(\mathsf{trace}(M))$; but $\mathsf{Copt}(\mathcal{T})$ also has a terminal object, so we can obtain all finite limits. $\square$

Exercise 1.9 Prove Propositions 2.6.4 and 2.6.6 directly for $[\mathcal{X} \to \mathcal{Y}]/S$. $\square$

Corollary 1.10 Pullbacks of stable functors and cartesian transformations exist and are calculated pointwise. $\square$

We need pullbacks to be constructed pointwise in order that they be preserved by evaluation.

Exercise 1.11 More generally if $\mathcal{T}$ has *wide* pullbacks of a given size then $\mathsf{Copt}(\mathcal{T})$ has limits and $[\mathcal{X} \to \mathcal{Y}]$ and $\mathcal{RC}$ have wide pullbacks of that size. These are calculated pointwise, and hence preserved by evaluation. $\square$

1.3 Special Adjoint Functor Theorem

We have *almost* proved that **SCat** is cartesian closed, except that we have to use the Adjoint Functor Theorem to prove that evaluation is stable, and so we have to consider questions of size. This is really a form of *spread*, which we shall discuss in the final section.

[This subsection is currently in a very sketchy state!]

Definition 1.12 A set $\mathcal{G}$ of objects *coseparates* a category $\mathcal{X}$ if for each pair $f, g : X \rightrightarrows Y$, if every $s : Y \to G$ with $G \in \mathcal{G}$ has $f\,; s = g\,; s$ then already $f = g$.

This is usually phrased in the negative way that if $f \neq g$ then there is some $s : X \to G$ with $f\,; s \neq g\,; s$.) For example $\{2\}$ coseparates **Set**. Obviously the whole object-set always coseparates: we are interested in the case where a *set* of objects of a large category suffices.

Conjecture 1.13 If $\mathsf{ev}_X : \mathsf{Copt}(\mathcal{X}) \to \mathcal{X}$ is stable for each $X \in \mathcal{X}$ then $\mathcal{X}$ has a set of coseparators.

Proof Here is a *pointwise*-descending class of functors whose big limit is id, but no small sub-diagram has this limit unless there is a set of coseparators. Let $\mathcal{G}$ be any set of objects of $\mathcal{X}$. Let $k_0^{\mathcal{G}}X, k_1^{\mathcal{G}}X : K^{\mathcal{G}}X \rightrightarrows X$ be the kernel pair of all arrows $s : X \to G$ with $G \in \mathcal{G}$, *i.e.* for each s we form the pullback of s against itself, and then (for each of the projections) we form the wide pullback (in which the arrows are in fact mono). Unfortunately the $k_i^{\mathcal{G}}$ are not cartesian. Observe that $k_0^{\mathcal{G}} = k_1^{\mathcal{G}} = \mathsf{id}$ iff $\mathcal{G}$ coseparates, whilst if $X \in \mathcal{G}$ then $k_0^{\mathcal{G}}X = k_1^{\mathcal{G}}X = \mathsf{id}_X$. There is a mediating map $k_i^{\mathcal{G}} \to k_i^{\mathcal{G}'}$ for $\mathcal{G}' \subset \mathcal{G}$, and the big limit is id, but this is attained by $\mathcal{G}$ iff it coseparates. $\square$

Definition 1.14 A category $\mathcal{X}$ is *well-powered* if for every object X there is a *set* of monos $X' \hookrightarrow X$ up to isomorphism in $\mathcal{X}/X$.

Exercise 1.15 Let $S : \mathcal{X} \to \mathcal{Y}$ be stable with trace $\mathcal{T}$. If $\mathcal{T}$ has a set of coseparators then $\mathsf{Spec}_S(Y)$ is a *small* groupoid, *i.e.* there is a set of components and each component group is small. In particular if $\mathcal{Y}$ has epi-mono factorisation and S is the inclusion of the monos then $\mathcal{Y}$ is co-well-powered. $\square$

Proposition 1.16 (Special Adjoint Functor Theorem) Suppose $\mathcal{X}$ is complete, well-powered and locally small and has a set of coseparators. Then every functor $H : \mathcal{X} \to \mathcal{Y}$ has a left adjoint iff it preserves limits. $\square$

[Coseparators in slices?]

Lemma 1.17 Suppose $\mathcal{X}$ has a final set $\mathcal{G}$ and $\mathcal{Y}$ is locally small and well-powered. Then $[\mathcal{X} \to \mathcal{Y}]$ is locally small and well-powered.

Proof It suffices to show that for $S_1, S_2 : \mathcal{X} \rightrightarrows \mathcal{Y}$, if $\phi, \psi : S_1 \rightrightarrows S_1$ are cartesian transformations with $\phi_G = \psi_G$ for all $G \in \mathcal{G}$ then $\phi_X = \psi_X$ for all $X \in \mathcal{X}$. Given X, let $s : X \to G$; then $\phi_X = (S_2 s)^* \phi_G = (S_2 s)^* \psi_G = \psi_X$. The same method applies to being well-powered. $\square$

1.4 The Pushout-Pullback Coincidence

The *limit-colimit coincidence* is a very famous result in domain theory which is the basis of the solution of recursive domain equations. An analogous result holds for stable domains, but its proof is of a rather different character from the results of this paper, so we do not discuss it here. However our present theory has its own limit-colimit coincidence, for wide pullbacks of rigid comparisons and the corresponding pushouts of their right adjoints. First we show that a wide pullback of rigid comparisons is also a pullback *quâ* ordinary functors.

Lemma 1.18 Let $M = \lim M_i$ be a limit diagram in $\mathsf{Copt}(\mathcal{X})$ and $C_i : \mathcal{T} \to \mathsf{Coalg}(M_i, \kappa_i)$ a cone of comparisons over the diagram $\Phi_u : \mathsf{Coalg}(M_i, \kappa_i) \to \mathsf{Coalg}(M_j, \kappa_j)$ of forgetful functors. Then the mediator $C : \mathcal{T} \to \mathsf{Coalg}(M, \kappa)$ is a comparison.

Proof An object of $\mathsf{Coalg}(M, \kappa)$ is of the form $\underline{X} = (X, (\alpha_i))$ where $\alpha_i : X \to M_i X$ is a cone over the diagram with arrows $\kappa_u : M_i \to M_j$, so that $X_i = \Phi_i \underline{X} = (X, \alpha_i)$ is an M_i-coalgebra. As usual $\Phi_i \dashv \Theta_i$ with unit ι_i and counit κ_i, *etc.* An M-homomorphism is an $\mathcal{X}$-map which is *simultaneously* a homomorphism at each i.

In particular for $T \in \mathcal{T}$, we have $CT = (C_0 T, (\beta_i))$, where $C_i T = (C_0 T, \beta_i)$, *i.e.* the cone commutes exactly. Suppose $C_i \dashv H_i$ with unit η_i and counit ϵ_i, where for simplicity we assume that the cocone also commutes exactly, *i.e.* $H_j = H_i \Theta_u$ with

$$\eta_j = \eta_i \,;\, H_i \iota_u C_i \qquad \text{and} \qquad \epsilon_j = \Phi_u \epsilon_i \Theta_u \,;\, \kappa_u \tag{1}$$

Now consider the diagram in $\mathcal{T}$ with arrows of the form

$$H_i X_i \xrightarrow{H_i \iota_u X_i} H_i \Theta_u \Phi_u X_i = H_j X_j$$

Since this diagram has a terminal vertex we may take its limit, $H\underline{X}$, with limiting cone $\gamma_i \underline{X} : H\underline{X} \to H_i X_i$, *i.e.*

$$\gamma_j \underline{X} = \gamma_i \underline{X} \,;\, H_i \iota_u X_i \tag{2}$$

It is easy to show that $\gamma_i : H \to H_i \Phi_i$ is cartesian, so if H_i is continuous and $\mathcal{T}$ is pullback-continuous then we deduce from Lemma 3.3.6 that H is also continuous.

For $T \in \mathcal{T}$ the units of the adjunctions $C_i \dashv H_i$ afford a cone $\eta_i T : T \to H_i C_i T$ over a diagram of the above form (by (1)), so let $\eta T : T \to HCT$ be its mediator; then

$$\eta_i T = \eta T \,;\, \gamma_i CT \tag{3}$$

We define ϵ by

$$\Phi_i \epsilon \underline{X} = C_i \gamma_i \underline{X} \,;\, \epsilon_i X_i \tag{4}$$

This is justified because

$$\begin{aligned} & C_j \gamma_j \underline{X} \,;\, \epsilon_j X_j \\ &= C_j(\gamma_i \underline{X} \,;\, H_i \iota_u X_i) \,;\, (\Phi_u \epsilon_i \Theta_u \Phi_u X_i \,;\, \kappa_u X_j) \qquad \text{by (1) \& (2)} \end{aligned}$$

$$
\begin{aligned}
&= C_j\gamma_i\underline{X} \,;\, \Phi_u\epsilon_i X_i \,;\, \Phi_u\iota_u X_i \,;\, \kappa_u X_j && \text{naturality of } \Phi_u\epsilon_i \\
&= C_j\gamma_i\underline{X} \,;\, \Phi_u\epsilon_i X_i && \Phi_u \dashv \Theta_u \\
&= \Phi_u(C_i\gamma_i\underline{X} \,;\, \epsilon_i X_i)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Phi_i(C\eta T \,;\, \epsilon C T) \\
&= C_i\eta T \,;\, C_i\gamma_i C T \,;\, \epsilon_i C_i T && \text{by (4)} \\
&= C_i\eta_i T \,;\, \epsilon_i C_i T && \text{by (3)} \\
&= \mathsf{id} && C_i \dashv H_i
\end{aligned}
$$

and

$$
\begin{aligned}
&(\eta H\underline{X} \,;\, H\epsilon\underline{X}) \,;\, \gamma_i\underline{X} \\
&= \eta H\underline{X} \,;\, \gamma_i C H\underline{X} \,;\, H_i\Phi_i\epsilon\underline{X} && \text{construction of } H\epsilon \\
&= \eta H\underline{X} \,;\, \gamma_i C H\underline{X} \,;\, H_i C_i\gamma_i\underline{X} \,;\, H_i\epsilon_i X_i && \text{by (4)} \\
&= \eta_i H\underline{X} \,;\, H_i C_i\gamma_i\underline{X} \,;\, H_i\epsilon_i X_i && \text{by (3)} \\
&= \gamma_i\underline{X} \,;\, \eta_i H_i X_i \,;\, H_i\epsilon_i X_i && \text{naturality of } \eta_i \\
&= \gamma_i\underline{X} && C_i \dashv H_i
\end{aligned}
$$

so $C \dashv H$. □

Theorem 1.19 Given a wide pullback diagram of rigid comparisons, the pullback of the comparisons *quâ* functors, the pushout of their right adjoints *quâ* functors and the pullback of the comparisons *quâ* rigid comparisons all exist and are equivalent.

Proof We have to show that a pullback of rigid comparisons is the pushout of their right adjoints; by the equivalence $\mathcal{RC}/\mathcal{X} \simeq \mathsf{Copt}(\mathcal{X})$ we may assume that the diagram is equivalent to the special form in the lemma. Let $S_i : \mathcal{X}_i \to \mathcal{Y}$ be a cocone of stable functors over the diagram Θ_u. Applying the trace factorisation to each of them, we find that they share the same isotomy part (up to equivalence), so they factor through $H_i : \mathcal{X}_i \to \mathcal{T}$ and $F : \mathcal{T} \to \mathcal{Y}$ with $C_i \dashv H_i$. Then $C_i : \mathcal{T} \to \mathcal{X}_i$ is a cone of comparisons over the (pointwise) limit diagram, and so there is a mediator $C : \mathcal{T} \to \mathcal{X}$, which by the lemma has a (continuous) right adjoint H. This mediates for the cocone H_i and so $S = FH$ mediates for S_i. Given another mediator, we may take its trace factorisation, of which the isotomy part must be F, whilst the comparison part mediates for the limit diagram and so must be C. □

The true significance of this result is domain-theoretic, so we shall postpone discussion to the next section.

Lemma 1.20 Let $\mathcal{Y}$ be a stable category and $\mathcal{X}$ and $\mathcal{X}_i$ be as in Lemma 2.8.1. Then the mediator

$$[\operatorname*{\mathsf{colim}}_i \mathcal{X}_i \to \mathcal{Y}] \to \lim_i\, [\mathcal{X}_i \to \mathcal{Y}]$$

is an equivalence.

Proof The functor is simply the correspondence between mediators $S = FH : \mathcal{X} \to \mathcal{T} \to \mathcal{Y}$ and cocones $S_i = FH_i : \mathcal{X}_i \to \mathcal{T} \to \mathcal{Y}$. This is essentially surjective by definition of colim and we have to show that it is full and faithful. Cartesian transformations $\phi : S' \to S$ correspond to diagonals $\Phi : \mathcal{T}' \to \mathcal{T}$. On the other hand, if $\phi_i : S'_i \to S_i$ is a natural system of cartesian transformations, their diagonals $\Phi_i : \mathcal{T}' \to \mathcal{T}$ must coincide. □

The coincidence does not hold for pushouts and coequalisers of rigid comparisons. That leaves equalisers:

Exercise 1.21 Find a parallel pair of rigid comparisons which has no equaliser, although the corresponding pair of right adjoints has a coequaliser. [Hint: a five-point poset will suffice.] □

1.5 Rigid Comparisons — Notes

However this representation would appear to introduce 3-cells, namely cartesian natural transformations between rigid comparisons, into the 2-category **SCat**, and, by iteration, worse. However we already know from Definition 1.7.1 that these have no effect.

Lemma 1.22 Let $\Phi_1, \Phi_2 : \mathcal{T} \rightrightarrows \mathcal{T}_0$ be rigid comparisons and $\tau : \Phi_1 \to \Phi_2$ a cartesian natural transformation between them. Then τ is invertible (the same is true for isotomies and functors with left adjoints, but, of course, not for arbitrary stable functors).

Proof Φ_1 and Φ_2 are isotomies, and so are the second parts of their own (non-standard) trace factorisations, the first parts being identities. In order to make the triangles commute, any diagonal between them must be a (strong) equivalence. Hence the corresponding cartesian natural transformation τ is invertible, but (in the case where $\Phi_1 = \Phi_2$) not necessarily the identity. The same argument applies to functors with left adjoints and isotomies. □

Lemma 1.23 Equivalent diagonals correspond to isomorphic subfunctions.

Proof We interpret "equivalent" in the weakest possible sense. Let $S_0, S_1, S_2 : \mathcal{X} \to \mathcal{Y}$ be stable functors with traces $\mathcal{T}_0$,, $\mathcal{T}_1$ and $\mathcal{T}_2$, and $\phi_1 : S_1 \to S_0$ and $\phi_2 : S_2 \to S_0$ be cartesian natural transformations with diagonals $\Phi_1 : \mathcal{T}_1 \to \mathcal{T}_0$ and $\Phi_2 : \mathcal{T}_2 \to \mathcal{T}_0$. Suppose that there is a strong equivalence $I_2 : \mathcal{T}_2 \to \mathcal{T}_1$ with pseudoinverse I_1 such that $\Phi_1 I_2 \cong \Phi_2$; then $I_1\Theta_1 \cong \Theta_2$ and $M = \Phi_1\Theta_1 \cong \Phi_2 I_1 I_2 \Theta_2 \cong \Phi_2\Theta_2$. Let $\kappa_2' : M \cong \Phi_2\Theta_2 \to \mathsf{id}$; then by Exercise 3.5.9 $\kappa_2' \cong \kappa_1 : M \to \mathsf{id}$ and we can recover ϕ_1 and ϕ_2 using Lemma 3.4.8, whence $\phi_2' \cong \phi_1$. □

In most of the examples of categories of stable domains which have been introduced the domains are posets, so the stronger property holds that *any two rigid comparisons which are Berry-comparable are uniquely isomorphic*. Unfortunately this simplification is not available in general, essentially because of the

Examples 1.24

(a) Let $\mathcal{X}$ be a group considered as a category with one object ($\bullet$) and $\phi_\bullet$ a nontrivial element of the centre of the group.

(b) Let $\mathcal{X} = \mathbf{VSp}$, the the category of (real) vector spaces, and ϕ_X be scalar multiplication by some fixed nonzero real number.

Then $\mathsf{id} : \mathcal{X} \to \mathcal{X}$ is a rigid comparison with a nontrivial cartesian automorphism ϕ. □

2 Cartesian Closed Categories

2.1 The Limit-Colimit Coincidence

In this final section we shall conclude our investigation of cartesian closure by constructing filtered colimits in function-spaces, and then turn to the interpretation of first-order (dependent-type) polymorphism. In the terminology of [T85] we shall show that **SCat** is *relatively* cartesian closed with respect to rigid bifibrations. Second-order (impredicative) polymorphism may be treated in the same way by considering $\mathcal{RC}$ as a domain; unfortunately it is too big (in fact, too "wide") for this to work unless we restrict attention to smaller domains. This is the subject of the final part of the paper: a systematic survey of what restrictions we may put on our domains whilst still being able to perform the constructions of interest. In this way we reproduce the previous work on specific subcategories of **SCat** and suggest new ones of potential interest.

Firstly, we must say something more about continuity (preservation of filtered colimits), which is the essential technical feature underlying the construction of fixed points and hence the modelling of recursion. We have made only passing reference to this topic, because stability and continuity are orthogonal concepts: each is more clearly understood in the absence of the other and (although this is not obvious) they may be combined with almost no interaction. For the most part, then, we have little to add to the traditional theory; in particular recursive domain equations such as $X \simeq [X \to X]_\perp$ may be solved in the usual way. The other ingredient needed for fixed points is an initial object ($\perp$), but this is just a special case of the general topic of *spread*, *i.e.* the complexity of polycolimits, which we shall discuss later.

The solution of recursive domain equations (in particular those involving the function-space functor) depends on a property called the *limit-colimit coincidence*, which was first noticed by Scott for complete lattices and is now well-known for sequences of embedding-projection pairs between domains of various kinds. It can be proved in general for **FCCat** (the 2-category of categories with small filtered colimits, continuous functors and natural transformations), although I know of no reference to a proof in the literature. Although it contains no surprises, the proof is notationally complicated and has nothing to do with stability, so we shall state the theorem without proof.

Theorem 2.1 Let $\mathcal{T}_{(-)} : \mathcal{I} \to \mathbf{FCCat}$ be a filtered diagram whose arrows $\Phi_u : \mathcal{T}_i \to \mathcal{T}_j$ are comparisons. Write $\Phi_u \dashv \Theta^u$ and $\mathcal{T}^i = \mathcal{T}_i$, so that the Θ^u are the arrows of a cofiltered diagram $\mathcal{T}^{(-)} : \mathcal{I}^{\mathrm{op}} \to \mathbf{FCCat}$. Then

(a) the (pseudo)limit of the cofiltered diagram $\mathcal{T}^{(-)}$ and the (pseudo)colimit of the filtered diagram $\mathcal{T}_{(-)}$ both exist and are equivalent;

(b) corresponding functors Φ_i in the colimiting cocone and Θ^i in the limiting cone are adjoint;

(c) $\mathsf{colim}_i\, \Phi_i \Theta_i \cong \mathsf{id}$ in the common limit and colimit;

(d) this is also the (pseudo)colimit in the 2-category $\mathbf{FCCat}^{\mathrm{cp}}$ whose morphisms are comparisons. □

We write $\mathsf{bilim}_i\, \mathcal{T}_i$ for the common limit and colimit of the diagram. When we write "$\mathcal{X} = \mathsf{colim}\, \mathcal{X}_i$" we mean that the context determines a diagram with typical vertex $\mathcal{X}_i$ and a cocone $\mathcal{X} \to \mathcal{X}_i$, and that this cocone is (pseudo)colimiting; similarly "$\mathcal{Y} = \lim \mathcal{Y}_j$". "$\mathcal{X} = \mathsf{bilim}\, \mathcal{X}_i$" means that both of these hold, and further that the corresponding arrows in the diagrams are adjoint, as are the cocones and cones.

Apart from the application to the solution of recursive domain equations, this has the important

Corollary 2.2 The covariant continuous function-space functor

$$(- \to -) : \mathbf{FCCat}^{\mathrm{cp}} \times \mathbf{FCCat}^{\mathrm{cp}} \to \mathbf{FCCat}^{\mathrm{cp}}$$

is continuous.

Proof The reader is invited to supply the diagrams and (co)cones in the following identities:

$$\begin{aligned}
(\mathop{\mathsf{bilim}}_i \mathcal{X}_i \to \mathop{\mathsf{bilim}}_j \mathcal{Y}_j) &= (\mathop{\mathsf{colim}}_i \mathcal{X}_i \to \lim_j \mathcal{Y}_j) && \text{the Theorem} \\
&= \lim_i (\mathcal{X}_i \to \lim_j \mathcal{Y}_j) && \text{definition of } \mathsf{colim} \\
&= \lim_i \lim_j (\mathcal{X}_i \to \mathcal{Y}_j) && \text{definition of lim} \\
&= \mathop{\mathsf{bilim}}_i \mathop{\mathsf{bilim}}_j (\mathcal{X}_i \to \mathcal{Y}_j) && \text{the Theorem}
\end{aligned}$$

The identity $(\mathsf{colim}\, \mathcal{X}_i \to \mathcal{Y}) = \lim(\mathcal{X}_i \to \mathcal{Y})$ at the level **Set** is simply the correspondence between cocones $\mathcal{X}_i \to \mathcal{Y}$ and mediators $\mathsf{colim}\, \mathcal{X}_i \to \mathcal{Y}$. This is enriched over **Cat** because natural transformations between mediators correspond bijectively to natural families of natural transformations between the cocones; we shall spell this out in Lemma 3.1.5. Finally it is enriched over **FCCat** because the forgetful functor $\mathbf{FCCat} \to \mathbf{Cat}$ creates (pseudo)limits. The same applies to $(\mathcal{X} \to \lim \mathcal{Y}_j) = \lim(\mathcal{X} \to \mathcal{Y}_j)$. The two bilimits may of course be interchanged. □

The same result will hold (for much the same reasons) in the stable case, but first we have to extend the theorem to *rigid* comparisons. We can do this by proving a separate result for isotomies.

Lemma 2.3 Let $\mathcal{T}$ be the colimit of a filtered diagram $\mathcal{T}_{(-)} : \mathcal{I} \to \mathbf{FCCat}$ whose arrows $\Phi_u : \mathcal{T}_i \to \mathcal{T}_j$ are isotomies. Then the functors $\Phi_i : \mathcal{T}_i \to \mathcal{T}$ in the colimiting cocone are also isotomies.

Proof Let 0 be an arbitrary point of $\mathcal{I}$. Since $\mathcal{I}$ is filtered, replacing $\mathcal{I}$ by the coslice $\mathcal{I}\backslash 0$ we may assume without loss of generality that this is the initial vertex, with $\hat{\imath} : 0 \to i$. Let $X^0 \in \mathcal{T}_0$; we have to show that $\Phi_0 : \mathcal{T}_0/X^0 \to \mathcal{T}/\Phi_0 X^0$ is an equivalence.

Now $\mathcal{T}$ is generated by the images of the $\mathcal{T}_i$ together with filtered colimits; by filteredness we do not even need to consider composition. Hence any map $f : Y \to \Phi_0 X^0 = X$ in $\mathcal{T}$ is at worst the mediator for a cocone over a filtered diagram, where the maps in the diagram and the cocone are images of $\mathcal{T}_i$-maps. Let the diagram have vertices $Y_i = \Phi_i Y_i^i$ and arrows $\Phi_j y_u^j$ where $y_u^j : \Phi_u Y_i^i \to Y_j^j$ for $u : i \to j$ in $\mathcal{I}$, and let the cocone be $f_i = \Phi_i(f_i^i : Y_i^i \to \Phi_{\hat{\imath}} X^0)$.

But $\Phi_{\hat{\imath}}$ is an isotomy, so $Y_i^i \cong \Phi_{\hat{\imath}} Y_i^0$ (without loss of generality equality holds) and $f_i^i = \Phi_{\hat{\imath}} f_i^0$, and then $y_u^j = \Phi_{\hat{\jmath}} y_u^0$. This reduces the diagram $(y_u^0 : Y_i^0 \to Y_j^0)$ and cocone $(f_i^0 : Y_i^0 \to X^0)$ to the level $0 \in \mathcal{I}$, and we may put $Y^0 = \operatorname{\mathsf{colim}} Y_i^0$ there, with mediator $f^0 : Y^0 \to X^0$. Then $\Phi_0 f^0$ mediates for the cocone in the colimit and hence f^0 is the required lifting of f. Lifting triangles is similar. $\square$

Lemma 2.4 Let $\mathcal{T}$ be the colimit of a filtered diagram $\mathcal{T}_{(-)} : \mathcal{I} \to \mathbf{FCCat}$ and $F_i : \mathcal{T}_i \to \mathcal{Y}$ a cocone of isotomies, where $\mathcal{Y}$ is pullback-continuous. Then the mediator $F : \mathcal{T} \to \mathcal{Y}$ is also an isotomy.

Proof Let $T = \operatorname{\mathsf{colim}} \Phi_i T_i \in \mathcal{T}$ and $y : Y \to FT$ in $\mathcal{Y}$. Form the pullback

$$\begin{tikzcd} F_i T_i' \arrow[r, "F_i t_i"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & F_i T_i \arrow[d] \\ Y \arrow[r, "y"] & FT \end{tikzcd}$$

where we may write the upper map as an F_i-image because this is an isotomy. Then the right-hand side is a typical map in a colimiting cocone in $\mathcal{Y}$ and by pullback-continuity so is the left. Put $T' = \operatorname{\mathsf{colim}} \Phi_i T_i' \in \mathcal{T}$, so that $FT' \cong \operatorname{\mathsf{colim}} F_i T_i'$, and let $t : T' \to T$ mediate the cocone. Then t' is the required F-lifting of y. Lifting triangles is similar. $\square$

Lemma 2.5 The equivalence

$$[\operatorname*{\mathsf{colim}}_i \mathcal{X}_i \to \mathcal{Y}] = \lim_i\, [\mathcal{X}_i \to \mathcal{Y}]$$

is enriched over **SCat**.

Proof We have to verify the correspondence between natural transformations between cocones and between mediators restricts to cartesian transformations. Let $\Theta_i : \mathcal{X} \to \mathcal{X}_i$ and $\Phi_i : \mathcal{X}_i \to \mathcal{X}$ be the limiting cone and colimiting cocone respectively, and $C, D : \operatorname{\mathsf{colim}}_i \mathcal{X}_i \to \mathcal{Y}$. Clearly if $\alpha : C \to D$ is cartesian then so is $\alpha_i = \alpha\Phi_i : C\Phi_i \to D\Phi_i$ for each i, so we have to prove the converse. Since any object $X \in \mathcal{X}$ can be expressed as $X = \operatorname{\mathsf{colim}} \Phi_i X_i$ with $X_i \in \mathcal{X}_i$, we may obtain α from α_i as the mediator between two colimits:

$$\begin{tikzcd} \operatorname{\mathsf{colim}} C\Phi_i X_i \cong CX \arrow[r, "\alpha X"] & DX \cong \operatorname{\mathsf{colim}} D\Phi_i X_i \\ C\Phi_i X_i \arrow[u, "C\xi_i"] \arrow[r, "\alpha_i X_i"] & D\Phi_i X_i \arrow[u, "D\xi_i"'] \end{tikzcd}$$

Morphisms $x : X' \to X$ correspond to cocones $x_j : \Phi_j X'_j \to X$; any such map is of the form $\Phi_j x_{ij} ; \Phi_i \xi_i$ with $x_{ij} : X'_j \to X_i$, and in fact corresponds to an equivalence class.

We make a cube out of this diagram $x : X' \to X$ in $\mathcal{X}$ and $x_{ij} : X'_j \to \Phi_u X_i$ in $\mathcal{X}_j$. The lower square is a pullback since α_j is cartesian. To show that the top is also, let $y : Y \to CX$ and $z : Y \to DX'$ and form a cube with vertex Y_{ij} by pulling back against *both* of $C\xi_i$ and $D\xi'_j$. Then by pullback-continuity we have $Y = \mathsf{colim}_{ij} Y_{ij}$ whilst by cartesianness of α_j we have $w_{ij} : Y_{ij} \to C\Phi_j X'_j$. Finally $w : Y \to CX'$ is the mediator. (?) □

Theorem 2.6 The limit-colimit coincidence holds for filtered diagrams of rigid comparisons between pullback-continuous categories. Hence $\mathcal{RC}$ has filtered colimits and the covariant stable-function-space functor $[- \to -] : \mathcal{RC} \times \mathcal{RC} \to \mathcal{RC}$ preserves them.

Proof The first part is a special case of the continuous version, except for saying that we have the filtered colimit in $\mathcal{RC}$. But Lammas 3.1.4&5 show that the colimiting cocone and mediator are isotomies as well as comparisons. Continuity of the function-space is analogous to Corollary 3.1.2, except that we need Lemma 3.1.5 to show that the colimit identity is enriched; the limit version requires nothing else to be proved (**exercise**). □

Exercise 2.7 Let $\Phi_i : \mathcal{X}_i \to \mathcal{X}$ be a cocone of rigid comparisons over a filtered diagram of them, with $\Phi_i \dashv \Theta_i$. Show that this cocone is colimiting iff $\mathsf{colim}^{\uparrow} \Phi_i \Theta_i \cong \mathsf{id}$. [Hint: rigid comparisons are comonadic.] □

2.2 Filtered Colimits of Stable Functors

In this subsection we shall use the limit-colimit coincidence to show that the function-space $[\mathcal{X} \to \mathcal{Y}]$ has filtered colimits, and that they are preserved by evaluation. Let $S_i : \mathcal{X} \to \mathcal{Y}$ be stable functors for $i \in \mathcal{I}$, with cartesian natural transformations $\phi_u : S_i \to S_j$ for $u : i \to j$ in $\mathcal{I}$. Let $S_i = F_i H^i$ with $C_i \dashv H^i$ be the standard trace factorisation *via* $\mathcal{T}_i$, and for $u : i \to j$ let $\Phi_u : \mathcal{T}_i \to \mathcal{T}_j$ be the diagonal, so that

$$\begin{tikzcd}
 & \mathcal{T}_i \arrow[dl, "C_i"'] \arrow[dr, "F_i"] \arrow[dd, "\Phi_u"] & \\
\mathcal{X} & & \mathcal{Y} \\
 & \mathcal{T}_j \arrow[ul, "C_j"] \arrow[ur, "F_j"'] &
\end{tikzcd}$$

commutes "on the nose" and $\Phi_u \dashv \Theta^u$ for $u : i \to j$.

We know from the limit-colimit coincidence, or indeed from general considerations of universal algebra, that the (pseudo)colimit, $\mathcal{T}$, of the diagram $(\mathcal{T}_i, \Phi_u)$ exists, with colimiting cocone Φ_i, and hence that there are continuous mediators $C : \mathcal{T} \to \mathcal{X}$ and $F : \mathcal{T} \to \mathcal{Y}$ so that the triangles

$$\begin{tikzcd}
 & \mathcal{T} & \\
\mathcal{T}_i \arrow[ur, "\Phi_i"] \arrow[rr, "\Phi_u"] & & \mathcal{T}_j \arrow[ul, "\Phi_j"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \mathcal{T} \arrow[dl, "C"'] \arrow[dr, "F"] & \\
\mathcal{X} & \mathcal{T}_i \arrow[u, "\Phi_i"] \arrow[l, "C_i"] \arrow[r, "F_i"'] & \mathcal{Y}
\end{tikzcd}$$

commute (up to natural isomorphism). In fact it will be possible to dispose of these irritating natural isomorphisms, because it will turn out that $\mathcal{T}$ is a trace and hence may be chosen to be standard.

Lemma 2.8 C and Φ_i are comparisons, say with $C \dashv H$ and $\Phi_i \dashv \Theta^i$, whilst F and Φ_i are isotomies. Moreover $\mathsf{colim}_{i\in\mathcal{I}}\, \Phi_i\Theta^i \cong \mathsf{id}_{\mathcal{T}}$.

Proof From the last two parts of Theorem 3.1.1, and Lemma 3.1.3. □

Corollary 2.9 $S = FH$ is a continuous stable functor and $SX = \mathsf{colim}\, S_iX$.

Proof F and H are stable. Since F and C mediate and $C \dashv H$, *etc.*, we have $F_i \cong F\Phi_i$ and $H^i \cong \Theta^i H$. Then using the lemma,

$$FHX \cong \mathsf{colim}\, F\Phi_i\Theta^i HX \cong \mathsf{colim}\, F_iH^iX = \mathsf{colim}\, S_iX$$

since F and H are continuous. □

It is easy to show that $X \mapsto \mathsf{colim}\, S_iX$ defines a continuous functor which is the colimit with respect to continuous functors and arbitrary natural transformations, but showing by brute force that this functor is stable — let alone the colimit in the stable functor category — is very difficult. It is done *in the poset case* in [T88].

Theorem 2.10 The category $[\mathcal{X} \to \mathcal{Y}]$ of stable functors and cartesian natural transformations has filtered colimits, and $\mathsf{ev} : [\mathcal{X} \to \mathcal{Y}] \times \mathcal{X} \to \mathcal{Y}$ preserves them.

Proof Let S and S_i be as constructed, so that S is the pointwise colimit. Proposition 1.4.3 gives an equivalence between the category of traces and diagonals which are adjoint pairs, and by the limit-colimit coincidence $\mathcal{T}$ is also the colimit in this category. For fixed $X \in \mathcal{X}$, $\mathsf{ev}(-, X)$ is continuous precisely because filtered colimits are constructed pointwise, whilst $\mathsf{ev}(S, -)$ is continuous because S is. But separate continuity of a function in two arguments suffices for joint continuity. □

Lemma 2.11 S is also the colimit in the category of stable functors and arbitrary natural transformations.

Proof From Proposition 1.7.3 there is an equivalence between this category and the category of traces and equivalence classes of diagonals. Since natural transformations appear in the definition of diagonals, it would appear at first sight that we are faced with a *lax* colimit problem. However composition is defined in terms of Φ alone (and in fact we only need it up to isomorphism), so in fact it is an ordinary colimit given (by construction) by $\mathcal{T}$. The F_i and (since the ϕ_u are cartesian) the C_i afford cocones, and so $(\mathcal{T}, C, F)$, and hence S, is the colimit. □

However the assumption that the natural transformations in the diagram are cartesian is necessary, because the pointwise colimit of stable functions is not in general stable.

Example 2.12 Let $\mathcal{X} = \mathbb{N} \cup \{\infty\}$ with the usual order and let $\mathcal{Y}$ be the Sierpinski space. Consider the (stable) functions $S_i : \mathcal{X} \to \mathcal{Y}$ where

$$S_i(x) = \begin{cases} \top & \text{if } i \le x \le \infty \\ \bot & \text{if } 0 \le x < i \end{cases}$$

Then $(S_i : i \in \mathbb{N})$ is a cofiltered system in the pointwise order, but

$$\lim_{i\in\mathbb{N}} S_i\Big(\mathop{\mathsf{colim}^{\uparrow}}_{x\in\mathbb{N}} x\Big) = \top \ne \bot = \mathop{\mathsf{colim}^{\uparrow}}_{x\in\mathbb{N}} \lim_{i\in\mathbb{N}} S_i(x)$$

so $\lim_i S_i$ is not continuous. By reversing the order we have a pointwise filtered system of stable functions whose colimit is not stable.

Exercise 2.13 Explain how we managed to prove that a pullback of continuous stable functors is continuous in section 2 whilst barely mentioning continuity. Also, why is our proof (using the Berry order) so radically different from the counterexample (in the pointwise order)?

Exercise 2.14 Show that $[\mathcal{X} \to \mathcal{Y}]$ has and ev preserves cofiltered limits by the same technique. The above proof is based on the *obvious* forgetful functor **SCat** $\to$ **FCCat**, but we may instead take a stable category to its *opposite*, hence interchanging the notions of comparison and homomorphism.

2.3 Dependent Sums

Suppose we have a domain $\mathcal{Y}[X]$ which depends on an object $X \in \mathcal{X}$ ranging over another domain. We may quantify over the variable $\mathcal{X}$ to give the *dependent sum*,

$$\Sigma X : \mathcal{X}.\mathcal{Y}[X] = \{(X, Y) : X \in \mathcal{X}, Y \in \mathcal{Y}[X]\}$$

or the *dependent product*,

$$\Pi X : \mathcal{X}.\mathcal{Y}[X] = \big\{(SX)_{X \in \mathcal{X}} : \forall X \in \mathcal{X}.SX \in \mathcal{Y}[X]\big\}$$

This is *first order polymorphism*, which we shall discuss in the next two sections. Replacing $\mathcal{X}$ by a "type of types" such as $\mathcal{RC}$ we obtain *Girard-Reynolds second-order polymorphism*, but there are size problems involved in this.

We have to make the definition of the dependent sum more functorial, so for $x : X' \to X$ there is to be a *rigid adjunction*

$$\mathcal{Y}[x]_! \dashv \mathcal{Y}[x]^*$$

which is (pseudo)functorial in x, *i.e.*

$$\mathcal{Y}[-] : \mathcal{X} \to \mathcal{RC}$$

$\mathbf{Y} = \Sigma X : \mathcal{X}.\mathcal{Y}[X]$ is then given by the *total category* of $\mathcal{X} \to \mathcal{S}\mathit{cat} \to \mathbf{Cat}^{\mathrm{op}}$. This has *objects* the pairs (X, Y) where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}[X]$ and *morphisms* $(x, y) : (X', Y') \to (X, Y)$ where $x : X' \to X$ in $\mathcal{X}$ and either $y : \mathcal{Y}[x]_!(Y') \to Y$ in $\mathcal{Y}[X']$ or equivalently $y : Y' \to \mathcal{Y}[x]^*(Y)$ in $\mathcal{Y}[X]$. The functor $P : \mathbf{Y} \to \mathcal{X}$ given by $(X, Y) \mapsto X$ is called the *Grothendieck fibration*, and for fixed $X_0 \in \mathcal{X}$ we have *injections* $\nu_{X_0} : \mathcal{Y}[X_0] \to \mathbf{Y}$.

Definition 2.15 A functor $p : \mathbf{Y} \to \mathcal{X}$ is a *bifibration* if both it and $p : \mathbf{Y}^{\mathrm{op}} \to \mathcal{X}^{\mathrm{op}}$ are fibrations; this is equivalent to saying that substitution functors have left adjoints. (This term is due to Lamarche.) It is a *rigid bifibration* if the adjunctions are rigid.

It is well-known that this set-up interprets dependent sums: our purpose is to examine the specific results for stable categories.

Theorem 2.16 Suppose $\mathcal{X}$ and $\mathbf{Y}$ have pullbacks and $P : \mathbf{Y} \to \mathcal{X}$ is a rigid bifibration. Then the slices of $\mathbf{Y}$ are of the form $\mathcal{Y}_0 \downarrow T$ for some functor $T : \mathcal{X}_0 \to \mathcal{Y}_0$.

Proof Let $(X_0, Y_0) \in \mathbf{Y}$, *i.e.* $X_0 \in \mathcal{X}$ and $Y_0 \in \mathcal{Y}[X_0] = P^{-1}(X_0)$. Put $\mathcal{X}_0 = \mathcal{X}/X_0$ and $\mathcal{Y}_0 = \mathcal{Y}[X_0]/Y_0$. Now define the functor $T : \mathcal{X}_0 \to \mathcal{Y}_0$ on objects by

$$T(X \xrightarrow{x} X_0) = \Big(\mathcal{Y}[x]_!\big(\mathcal{Y}[x]^*(Y_0)\big) \xrightarrow{\kappa} Y_0\Big)$$

and use the horizontal liftings afforded by $\mathcal{Y}[x]^*$ to provide those for $\xi : (X \to X_0) \to (X' \to X_0)$; in other words the definition on morphisms is essentially the same.

Now let $(x, y) : (X, Y) \to (X_0, Y_0)$ in $\mathbf{Y}$, *i.e.* $y : Y \to \mathcal{Y}[x]^*(Y_0)$ in $\mathcal{Y}[X]$. But $\mathcal{Y}[x]_!$ is an isotomy, so

$$\mathcal{Y}[X]/\mathcal{Y}[x]^*(Y_0) \simeq \mathcal{Y}[X_0]/\mathcal{Y}[x]_!\big(\mathcal{Y}[x]^*(Y_0)\big) \simeq \mathcal{Y}_0/Tx$$

In other words, (X, Y) is determined up to isomorphism by x and $\mathcal{Y}[x]_!(y) : \mathcal{Y}[x]_!(Y) \to Tx$, which is an object of $\mathcal{Y}_0 \downarrow T$. The same applies to morphisms. Hence $\mathbf{Y}/(X_0, Y_0) \to \mathcal{X}/X_0$ is equivalent to the forgetful functor $\mathcal{Y}_0 \downarrow T \to \mathcal{X}_0$. □

The functor $T : \mathcal{X}/X_0 \to \mathcal{Y}[X_0]/Y_0$ is actually just the restriction of the indexation $\mathcal{Y}[-] : \mathcal{X} \to \mathcal{RC}$, as we may see by comparison with Proposition 2.6.4.

Corollary 2.17 The fibration and injections are stable.

Proof We have to show that P and ν_{X_0} have left adjoints on slices.

[$\dashv P$] On slices P is $\mathcal{Y}_0 \downarrow T \to \mathcal{X}_0$ by $(Y \xrightarrow{w} TX) \mapsto X$. For $X \in \mathcal{X}_0$ let $I \to TX$ be the polyinitial candidate below TX, *i.e.* the initial object of $\mathcal{Y}[X]/TX$; then the left adjoint is $X \mapsto (0 \to TX)$.

[$P \dashv$] There is also a *right* adjoint on slices, given by $X \mapsto (TX \to TX)$.

[$\dashv \nu_{X_0}$] On slices ν_{X_0} is $Y \mapsto (Y \to 1)$, which has left adjoint $(Y \xrightarrow{w} TX) \mapsto Y$ (but no right adjoint). $\square$

Proposition 2.18 In this notation the rigid comparisons and homomorphisms are given by composition and pullback, so for $\xi : X' \to X$ in $\mathcal{X}_0$,

$$\begin{array}{rcl} \mathcal{Y}[\xi]_!\big(X', Y' \xrightarrow{y'} TX'\big) & = & \big(X,\ (T\xi)_!\, y'\big) \\ \mathcal{Y}[\xi]^*\big(X, Y \xrightarrow{y} TX\big) & = & \big(X',\ (T\xi)^* y\big) \end{array}$$

where $Y, Y' \in \mathcal{Y}_0$, *etc.* (*cf.* Lemma 2.2.3 and Exercise 2.2.7; recall that $f_!\, y = y\,;f$). $\square$

Exercise 2.19 When does P has a left adjoint (*cf.* Exercise 1.7.7)? When is it an isotomy?

The magnitude of what we have just proved is easily overlooked. Imagine the base category as the frame of a weaving loom, with the fibres as the "warp". The substitution functors form the "weft", which may in the general case wander erratically, and so there is little redundancy in the description. However if these functors are rigid homomorphisms, once the top border has been woven (which may be arbitrary), the strands of the weft below it are *parallel*, and so the top row determines the structure of the whole tapestry. Hence *lemmas* like "a compact object of the total category is a compact object of a compact fibre" become obvious *corollaries*.

In fact it can be shown (using lemmas of this kind) that the total category of a fibration whose substitution functors are merely stable (so they have poly- rather than rigid adjoints) is, for example, locally finitely presentable iff the base category and fibres are. However one may show that whereas *parts* of the construction of the dependent product are also possible in this way, its display is not a fibration unless the given fibration is also an op-fibration, *i.e.* its substitutions have (rigid) left adjoints. Consequently the construction under weaker hypotheses, besides being far more difficult to execute (and this property possessed by finite presentability is not shared by other features) is of no interest in the study of polymorphism.

If we are interested in *continuous* stable functors we must show that the total category has and the fibration and injections preserve filtered colimits. This result again has nothing to do with stability, and so we shall omit the proof, even though this also is not to be found anywhere in the literature.

Proposition 2.20 Let $X \in \mathbf{FCCat}$ and $\mathcal{Y}[-] : \mathcal{X} \to \mathbf{FCCat}^{\mathrm{op}}$ be continuous in the sense that filtered colimits are taken to cofiltered (pseudo)limits. Then $\Sigma X : \mathcal{X}.\mathcal{Y}[X]$ has filtered colimits and they are preserved by the fibration and injections. $\square$

It is quite essential that $\mathcal{Y}[-]$ be *continuous* in this result, and so the functor T must also be continuous.

Now let's consider the complexity of other polycolimits (the *spread*). The essence of the result already lies in Exercise 1.6.7: if $\mathcal{X}$ and each $\mathcal{Y}[X]$ are groups then since $p : \mathbf{Y} \to \mathcal{X}$ is a fibration it is (exactly) a surjective group homomorphism, in fact $\mathbf{Y} \cong \mathcal{Y}[X] : \mathcal{X}$, a *group extension* of $\mathcal{Y}[X]$ by $\mathcal{X}$ (**exercise:** what is the associated action?).

Definition 2.21 $\mathbf{B} \to \mathcal{A}$ is the *groupoid-extension of* $\mathcal{A}$ *by* $\mathcal{B}$ if this is the fibration corresponding to the indexation $\mathcal{B} : \mathcal{A} \to \mathbf{Gpd}^{\mathrm{op}}$ where $\mathcal{A}$ is a groupoid. (Of course this makes the functor $\mathbf{B} \to \mathcal{A}$ an isotomy, but we intend it to be actually a fibration.) The extension is *split* if the fibration is split, *i.e.* $\mathcal{B}$ is a functor (not a pseudofunctor).

Exercise 2.22 $(\langle x_i, y_i\rangle : \langle X_i, Y_i\rangle \to \langle X, Y\rangle)$ is a polycolimit candidate in $\Sigma X : \mathcal{X}.\mathcal{Y}[X]$ iff $(x_i : X_i \to X)$ is in $\mathcal{X}$ and $(y_i : \mathcal{Y}[x_i]_!(Y_i) \to Y)$ is in $\mathcal{Y}[X]$. $\square$

Proposition 2.23 The spread of $\Sigma X : \mathcal{X}.\mathcal{Y}[X]$ is the groupoid extension of the spread of $\mathcal{X}$ by that of $\mathcal{Y}$.

Proof Let $\mathbf{Y} \to \mathcal{X}$ be the display of a dependent sum and $\langle X_i, Y_i \rangle$ a diagram in $\mathbf{Y}$ of type $\mathcal{I}$. Let $\mathcal{A}$ be the groupoid of polycolimit candidates for the diagram X_i in $\mathcal{X}$. For a candidate $A = (x_i : X_i \to X) \in \mathcal{A}$, let $\mathcal{B}[A]$ be the groupoid of polycolimit candidates for the diagram $\mathcal{Y}[x_i]_!(Y_i)$ in $\mathcal{Y}[X]$, and for an automorphism $a : A \cong A$ of the candidate, let $\mathcal{B}[a] = \mathcal{Y}[a]^*$. Using the exercise we deduce that $\Sigma A : \mathcal{A}.\mathcal{B}[A]$ is the groupoid of polycolimits of $\mathbf{Y}$. $\square$

Exercise 2.24 Recall that in a fibration, maps of the form

$$\begin{tikzcd} \langle X', \mathcal{Y}[x]^*(Y) \rangle \arrow[r] & \langle X, Y \rangle & \text{in} & \mathbf{Y} \arrow[d, "P"] \\ X' \arrow[r, "x"] & X & \text{in} & \mathcal{X} \end{tikzcd}$$

are called *horizontal*; indeed the definition of a fibration is that there is a factorisation system (Definition 1.3.5) of $\mathbf{Y}$-maps into *vertical* maps (whose images under P are identities) followed by horizontal maps.

(a) Show that *horizontal maps are cartesian* in the sense that they give rise to a cartesian transformation $\mathcal{Y}[x]^*\nu_X \to \nu_X$.

(b) The above diagram is called the *horizontal lifting of x at $\langle X, Y \rangle$*. Show that the lifting preserves pullbacks.

(c) A pullback of a horizontal map is horizontal. $\square$

Exercise 2.25 Similarly *op-horizontal* maps are those of the form $\langle X', Y' \rangle \to \langle X', \mathcal{Y}[x]_!(Y') \rangle$. Show that for a rigid bifibration, op-horizontal maps are also cartesian. [Hint: take the vertical-horizontal factorisation of the op-horizontal maps and use the fact that $\iota : \mathsf{id} \to \mathcal{Y}[x]^*\mathcal{Y}[x]_!$ is cartesian.] $\square$

Exercise 2.26 Describe the effect of the functors

$$\Sigma_{\mathcal{X}} : (\mathcal{X} \to \mathcal{RC}) \to \mathcal{RC} \qquad \text{and} \qquad \Sigma : \big(\Sigma\mathcal{X} : \mathcal{RC}.(\mathcal{X} \to \mathcal{RC})\big) \to \mathcal{RC}$$

on morphisms, and show that they are continuous. [Hint: for naturality of $\Phi[X] \dashv \Theta[X]$ with respect to $x : X' \to X$, $\mathcal{Y}[x]_!$ and $\mathcal{Y}[x]^*$ must *both* commute with each of $\Phi[X]$, $\Theta[X]$, $\kappa[X]$ and $\iota[X]$, interchanging $X \leftrightarrow X'$ and $\mathcal{Y} \leftrightarrow \mathcal{Y}'$; *cf.* Proposition 3.4.1 below.] $\square$

2.4 The Beck Condition

Proposition 2.27 The quantifier Σ commutes with substitution:

$$\big(\Sigma Y : \mathcal{Y}[X].\mathcal{Z}[X,Y]\big)\big[X := SX'\big] = \Sigma Y : \mathcal{Y}[SX'].\mathcal{Z}[SX', Y]$$

Proof When we express dependent types with the display $\mathbf{Y} \to \mathcal{X}$ instead of the pseudofunctor $\mathcal{Y} : \mathcal{X} \to \mathcal{RC}$, substitution becomes pullback and sums become composition of displays. This is illustrated in the following diagram:

$$\Sigma(X',Y):\big(\Sigma X':\mathcal{X}'.\mathcal{Y}[SX']\big).\mathcal{Z}[SX',Y]\cong$$

$$\begin{tikzcd}
\Sigma X':\mathcal{X}'.\Sigma Y:\mathcal{Y}[SX'].\mathcal{Z}[SX',Y] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma X':\mathcal{X}'.\mathcal{Y}[SX'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}' \arrow[d, "S"] \\
\Sigma(X,Y):\big(\Sigma X:\mathcal{X}.\mathcal{Y}[X]\big).\mathcal{Z}[X,Y] \arrow[r] & \Sigma X:\mathcal{X}.\mathcal{Y}[X] \arrow[r] & \mathcal{X}
\end{tikzcd}$$

$$\cong \Sigma X:\mathcal{X}.\Sigma Y:\mathcal{Y}[X].\mathcal{Z}[X,Y]$$

where the isomorphisms are "Fubini's theorem" for calculating iterated dependent sums. The commutation of the quantifier (Σ) with substitution — known as the *Beck condition* — is the fact that the iterated sum on the top (obtained by substitution and then sum) is indeed the pullback (obtained by sum and then substitution). □

There is a similar condition for dependent products.

There are also interesting examples of indexations in universal algebra where the substitutions have left adjoints, such as the assignment of the category of modules to a ring (induced and restricted modules) and the category of models to a theory (free and underlying algebras), but these do not satisfy the Beck condition. Nor is it satisfied (by the substitution functors and their adjoints) in *continuous* domain-theoretic models of polymorphism, so one might imagine that there are simply two branches of indexed category theory: one concerned with *logic* where adjoints to substitution are called *quantifiers* and satisfy the Beck condition, and the other concerned with *model theory* where the (left) adjoint is a "free" functor and does not satisfy this condition.

However when we come to *stable* domain theory, this changes: the base type $\mathcal{X}$ has pullbacks, which we now consider algebraically significant (as they are in logic but not model theory), and so they should be preserved in the "domain" of domains $\mathcal{RC}$. Hence it is appropriate to make $\mathcal{Y}[-]:\mathcal{X}\to\mathcal{RC}$ *stable*. This arises quite naturally from Lemma 2.6.2 and when we consider second-order polymorphism, because of the

Proposition 2.28 The stable function space functor $[-\to-]:\mathcal{RC}\times\mathcal{RC}\to\mathcal{RC}$ is stable and continuous.

Proof Continuity has already been proved in Theorem 3.1.6, and preservation of wide pullbacks is exactly analogous using Theorem 2.8.2 and Lemma 2.8.3. To deduce stability we have to use the Adjoint Functor Theorem as in section 2.7. □

Lamarche has observed that stability has the result that the Beck condition holds. Since the functor T of Theorem 3.3.2 is essentially the restriction of $\mathcal{Y}[-]$ to a slice, and by construction it also preserves the terminal object, T then has a left adjoint.

Proposition 2.29 Let $\mathcal{Y}[-]:\mathcal{X}\to\mathcal{RC}$ be stable and suppose the square on the left is a pullback in $\mathcal{X}$:

$$\begin{tikzcd}
X_1\times_{X_0}X_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "x_1"] \\
X_2 \arrow[r, "x_2"] & X_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Y}[X_1\times_{X_0}X_2] \arrow[d, "\mathcal{Y}[\pi_2]_!"'] & \mathcal{Y}[X_1] \arrow[l, "\mathcal{Y}[\pi_1]^*"'] \arrow[d, "\mathcal{Y}[x_1]_!"] \\
\mathcal{Y}[X_2] \arrow[ur, phantom, "\stackrel{\delta}{\Rightarrow}"] & \mathcal{Y}[X_0] \arrow[l, "\mathcal{Y}[x_2]^*"]
\end{tikzcd}$$

Then the right-hand square commutes up to isomorphism δ.

Proof Let $Y_1\in\mathcal{Y}[X_1]$ and put $Y_0=\mathcal{Y}[x_1]_!(Y_1)$; we shall work in the slice of $\Sigma X:\mathcal{X}.\mathcal{Y}[X]$ by (X_0,Y_0), so to simplify notation assume that this is terminal. The object Y_1 is now given by

$(X_1, 1 \xrightarrow{y_1} TX_1)$, where $y_1 = \mathcal{Y}[x_1]_!(\iota_{Y_1})$. Forming the pullback on the left in $\mathcal{Y}_0$,

$$\begin{tikzcd}
Y \arrow[r, "y_3"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T(X_1 \times X_2) \arrow[r, "T\pi_2"] \arrow[d, "T\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & TX_2 \arrow[d, "!_{TX_2}"'] \\
1 \arrow[r, "y_1"] & TX_1 \arrow[r, "!_{TX_1}"] & TX_0{=}1
\end{tikzcd}$$

we represent

	Y_1	by	(X_1, y_1)	
	$\mathcal{Y}[\pi_1]^*(Y_1)$	by	$(X_1 \times X_2,\ y_3)$	
	$\mathcal{Y}[\pi_2]_!\big(\mathcal{Y}[\pi_1]^*(Y_1)\big)$	by	(X_2, y_2)	where $y_2 = y_3\,;T\pi_2$
	$Y_0 = \mathcal{Y}[x_1]_!(Y_1)$	by	(X_0, id_1)	
and	$\mathcal{Y}[x_2]^*\big(\mathcal{Y}[x_1]_!(Y_1)\big)$	by	$(X_2, \mathsf{id}_{TX_2})$	

so the mediator is $\delta = y_2$. However the bottom composite is the identity and the right-hand square is a pullback by the hypothesis that the indexation is stable, so the mediator is an isomorphism. Observe that we used Theorem 3.3.2 to give the representation of $\mathcal{Y}[f]^*$ and hence to make the left-hand square a pullback. □

Exercise 2.30 Let $\mathcal{Y}[-] : \mathcal{X} \to \mathcal{RC}$ be stable. Show that

(a) $\mathcal{Y}_0 \downarrow T \cong G \downarrow \mathcal{X}_0$ where $G \dashv T$;

(b) $\nu_{X_0} : \mathcal{Y}[X_0]/Y_0 \to \mathbf{Y}/\langle X_0, Y_0\rangle$ has a second left adjoint given by the unit $Y \mapsto (Y \to TGY)$;

(c) op-horizontal lifting preserves pullbacks;

(d) the pullback of an op-horizontal map is op-horizontal. □

Exercise 2.31 Show that $\Sigma : \big(\Sigma\mathcal{X} : \mathcal{RC}.[\mathcal{X} \to \mathcal{RC}]\big) \to \mathcal{RC}$ is stable (and continuous). □

2.5 Dependent Products

Dependent products are a generalisation of function-spaces, and we shall use the same technique to calculate their slices. As with dependent sums, this turns out to be much simpler than in the usual case. Dependent products over types ($\mathcal{RC}$) we shall consider separately, since they raise a number of questions of a different kind.

It is not difficult to see that the dependent product $\Pi X : \mathcal{X}.\mathcal{Y}[X]$ must be *constructed* as the domain $\mathsf{Sect}(P)$ of *sections* of the fibration $P : \mathbf{Y} = \Sigma X : \mathcal{X}.\mathcal{Y}[X] \to \mathcal{X}$. However although this exists irrespective of the properties of P (since

$$\mathsf{Sect}(P) \hookrightarrow [\mathcal{X} \to \mathbf{Y}] \underset{S \mapsto \mathsf{id}}{\overset{S \mapsto S\,;P}{\rightrightarrows}} [\mathcal{X} \to \mathcal{X}]$$

is an equaliser in **SCat**) it does not give the dependent product unless P is a fibration whose substitutions are rigid homomorphisms.

Suppose that the base domain is itself dependent upon a variable $Z : \mathcal{Z}$, so that $\mathcal{X} : \mathcal{Z} \to \mathcal{RC}$. The Beck condition requires that the quantifier Π commute with substitution for Z, either of a determinate value $Z_0 \in \mathcal{Z}$ or an expression SZ' with $Z' \in \mathcal{Z}'$. This means that $\Pi X : \mathcal{X}[Z].\mathcal{Y}[Z,X]$ must be functorial in Z, and in fact be a dependent type $\mathcal{Z} \to \mathcal{RC}$.

Conjecture 2.32 Let $S : \mathbf{Y} \to \mathcal{X}$ be a (stable) functor between (stable) categories. Suppose that S^*, pullback against S in $(\mathbf{S})\mathbf{Cat}$, has a right adjoint. Then S is a (rigid) bifibration.

Theorem 2.33 Let $\mathcal{X} : \mathcal{Z} \to \mathcal{RC}$ and $\mathcal{Y} : (\Sigma Z : \mathcal{Z}.\mathcal{X}[Z]) \to \mathcal{RC}$ be dependent types. Then $\Pi X : \mathcal{X}[Z].\mathcal{Y}[Z,X]$, given by the domain of sections of $P[X] : \Sigma X : \mathcal{X}[Z].\mathcal{Y}[Z,X] \to \mathcal{X}[Z]$, is also a dependent type. Moreover evaluation is stable and the Beck condition holds.

Proof The substitution functors and their adjoints are given by pre- and post-composition with those of $\mathcal{X}$ and $\mathcal{Y}$ in a standard way. The adjunctions are rigid because the units and counits are obtained by applying stable functors to cartesian transformations. □

Theorem 2.34 The functor $\Pi : \big(\Sigma\mathcal{X} : \mathcal{RC}.[\mathcal{X} \to \mathcal{RC}]\big) \to \mathcal{RC}$ is stable and continuous.

Proof From Exercise 3.4.6, the construction of $\mathbf{Y} = \Sigma X : \mathcal{X}.\mathcal{Y}[X]$ is continuous and stable, and by the same technique so is that of $P : \mathbf{Y} \to \mathcal{X}$, so we have only to show that $P \mapsto \mathsf{Sect}(P)$ is stable and continuous. For stability, we use the pushout-pullback coincidence (Theorem 2.8.2) to express the diagram as a wide pullback of rigid comparisons; similarly for continuity we use the limit-colimit coincidence (Theorem 3.1.6) and a cofiltered limit of rigid homomorphisms. Then these limits commute with equalisers. □

These results are standard: what interests us here is how to calculate dependent products, in particular what their slices and polycolimits look like.

Proposition 2.35 Let $S : \mathcal{X} \to \mathbf{Y}$ be a section of the fibration P corresponding to $\mathcal{Y} : \mathcal{X} \to \mathcal{RC}$. Then there is a (stable) functor $\mathcal{X} \to \mathcal{RC}$ such that $\mathsf{Sect}(P)/S$ is given by the pseudolimit of a diagram $\mathcal{X}^{\mathrm{op}} \to \mathcal{RC}^{\mathrm{op}} \to \mathbf{Cat}$.

Proof Let $\phi : S' \to S$ in $\mathsf{Sect}(P)$ and $x : X' \to X$ in $\mathcal{X}$. Write the vertical-horizontal factorisation of Sx as $|Sx|\,;\underline{Sx}$. Then in the diagram

$$\begin{tikzcd} S'X' \arrow[r, "|S'x|"] \arrow[d, "\phi_{x'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y}[x]^*(S'X) \arrow[r, "\overline{S'x}"] \arrow[d, "\mathcal{Y}[x]^*(\phi_X)"'] \arrow[dr, phantom, "\lrcorner", very near start] & S'X \arrow[d, "\phi_X"'] \\ SX' \arrow[r, "|Sx|"'] & \mathcal{Y}[x]^*(SX) \arrow[r, "\underline{Sx}"'] & SX \end{tikzcd}$$

the right-hand square is a pullback because the top and bottom are horizontal maps, whilst the rectangle is since ϕ is cartesian, so the left-hand square is also. Hence

$$\begin{tikzcd} \mathcal{Y}[X']/SX' \arrow[r, shift right, "|Sx|_!"'] & \mathcal{Y}[X']/\mathcal{Y}[x]^*(SX) \arrow[l, shift right, "|Sx|^*"'] \arrow[r, shift right, "\mathcal{Y}[x]_!"'] & \mathcal{Y}[X]/SX \arrow[l, shift right, "\mathcal{Y}[x]^*"'] \end{tikzcd}$$

$$\begin{tikzcd} SX' & \mathcal{Y}[x]^*(SX) \arrow[l, mapsto] & SX \arrow[l, mapsto] \end{tikzcd}$$

so $X \mapsto \mathcal{Y}[X]/SX$ defines a functor $\mathcal{X} \to \mathcal{RC}$ and by the same argument as Proposition 2.6.4 $\mathsf{Sect}(P)/S$ is the limit of the diagram of rigid homomorphisms. □

Exercise 2.36 Any stable functor $\mathcal{Z} : \mathcal{X} \to \mathcal{RC}$ such that $\mathcal{Z}[X]$ has a terminal object may occur in this way. $\square$

There are several ways of finding *filtered* colimits in the dependent product. There's brute force. We could also find a "dependent trace" and reproduce the method of section 3.2. The simplest method, however, is to observe that since $\ulcorner \mathsf{id} \urcorner$ and postcomposition with P are continuous, the equaliser above creates filtered colimits, which we know to exist and be calculated pointwise in $[\mathcal{X} \to \mathbf{Y}]$.

Proposition 2.37 $\mathsf{Sect}(P) = \Pi X : \mathcal{X}.\mathcal{Y}[X]$ has filtered colimits. $\square$

Calculating polycolimits in dependent products (or function-spaces) is an entirely different matter. We shall find later that the existence and uniqueness (connectedness) of polycolimit candidates depends not only on polycolimits in the codomain (as with slices) and those in the domain, but also on the slices of the codomain. Their automorphisms, however, can be bounded by those of the codomain.

Proposition 2.38 Suppose that each $\mathcal{Y}[X]$ has filtered colimits which are preserved by $\mathcal{Y}[x]^*$ and polycolimits of type $\mathcal{I}$ and of type $\mathcal{X}\backslash X$. Then $\Pi X.\mathcal{X} : \mathcal{Y}[X]$ has polycolimits of type $\mathcal{I}$.

Proof Let $S_i \to S$ be a cocone in $\Pi X.\mathcal{X} : \mathcal{Y}[X]$. For $X \in \mathcal{X}$ let $S_i X \to F_0 X \to SX$ be the polycolimit candidate in $\mathcal{Y}[X]$; note that F_0 is not necessarily stable (unless pullbacks preserve colimits of type $\mathcal{I}$) and the natural transformations $S_i \to F_0 \to S$ are not cartesian. However for any $X_0 \in \mathcal{X}$, we may define a stable functor on $\mathcal{X}/X_0$ by

$$(x : X \to X_0) \mapsto \mathcal{Y}[x]^*(F_0 X_0 \to S X_0)$$

and it is easy to show that this is the initial stable functor between S_i and S *on this slice*. (Imagine this functor like a cable-car coming down a mountain.) This is fine if X_0 is terminal, but otherwise we have to patch together lots of functors like this. Thus we define

$$\big(F_{n+1}X \to SX\big) = \underset{x:X\to X'}{\mathsf{colim}}\, \mathcal{Y}[x]^*(F_n X' \to SX')$$

which gives an increasing sequence of functors between S_i and S. (This is where polycolimits of type $\mathcal{X}\backslash X$ come in.) Now we have

$$S_i X \to F_n X \to F_\omega X = \underset{n}{\mathsf{colim}}^{\uparrow}\, X \to SX$$

and I claim that F_ω is stable and the natural transformation $F_\omega \to S$ is cartesian. This depends solely on preservation of filtered colimits by pullbacks: indeed so long as pullbacks have "rank" in some sense we may perform the same argument by transfinite iteration.

Let $x : X \to X'$; then there is a commutative square with $SX \to SX'$ on the top and $F_n X \to F_n X'$ on the bottom, so there is a mediator to the pullback, $F_n X \to \mathcal{Y}[x]^* F_n X'$. On the other hand, the latter is a component of the colimit making $F_{n+1}X$, and so we have a "sandwich"

$$\cdots \longrightarrow F_n X \longrightarrow \mathcal{Y}[x]^* F_n X' \longrightarrow F_{n+1} X \longrightarrow \cdots$$

Since $\mathcal{Y}[x]^*$ preserves filtered colimits, we deduce that $\mathcal{Y}[x]^* F_\omega X' \cong F_\omega X$ as required, *i.e.* $F_\omega \to S$ is cartesian and hence F_ω is stable. It is easy to check that this is the required polycolimit candidate. $\square$

Corollary 2.39 Polycolimits (where they exist) are computed pointwise iff pullback functors preserve colimits in slices. $\square$

The explicit description of polycolimits enables us to calculate their automorphism groups.

Proposition 2.40 The automorphism group of a polycolimit candidate in $\Pi X : \mathcal{X}.\mathcal{Y}[X]$ is obtained from automorphism groups of polycolimits in $\mathcal{Y}$ by a process of forming products, subgroups and extensions.

Proof Let $\phi : F_\omega \to F_\omega$ be a natural automorphism which commutes with the cocone $S_i \to F_\omega$, and let $X \in \mathcal{X}$. Since $S_i X \to F_0 X \to F_\omega X$ is a polycolimit candidate, there is a unique map $\phi_0 X : F_0 X \to F_0 X$ which commutes with the diagram involving ϕX, and ϕ_0 is a natural automorphism of F_0. Now we construct $\phi_n X$ by mimicking the construction of $F_n X$: for $x : X \to X'$ we have a cocone $\mathcal{Y}[x]^*\phi_n X' : \mathcal{Y}[x]^* F_n X' \to \mathcal{Y}[x]^* F_n X'$ and hence a unique mediator $\phi_{n+1} X : F_{n+1} X \to F_{n+1} X$ below ϕX. Then also Φ_{n+1} is a natural automorphism of F_{n+1}.

Now the construction $\phi \mapsto \phi_0 X$ is a group homomorphism from the required automorphism group (of $S_i \to F_\omega$) to that of the pointwise colimit $S_i X \to F_0 X$. Automorphisms of the *functor* F_0 must be natural, which we may express as a subgroup of the product of the automorphism groups at each component. For the higher stages a similar argument applies: with ϕ_n fixed, $\phi_{n+1} X$ belongs to the automorphism group of the polycolimit of type $\mathcal{X}\backslash X$. Allowing the lower candidates to vary, we obtain an extension group with the component for $\phi_{n+1} X$ at the bottom and that for ϕ_0 at the top; again naturality of ϕ_{n+1} may be expressed as a subgroup of a product.

Finally ϕ is determined by the sequence (ϕ_n) since we have assumed the existence of filtered colimits (not "poly"), and so ϕ belongs to the cofiltered limit of the groups for ϕ_n (if the filtered colimit were poly, there would simply be yet another factor below the limit). □

Corollary 2.41 If each $\mathcal{Y}[X]$ has equalisers then so does $\Pi X : \mathcal{X}.\mathcal{Y}[X]$. □

Lamarche has observed that we cannot make stable functors *preserve* equalisers if we want cartesian closure, but his semigranular categories and aggregates *have* them. This result explains why this is possible, although our proof, unlike his (implicitly), does not depend on distributivity.

2.6 Categories of Stable Categories

Hitherto we have set out the theory of stable categories in the most general form possible, whereas other authors have studied more restricted versions. The remainder of this paper will be devoted to showing how their results can be recovered by working "top-down" and to a semi-systematic survey of the possible subcategories of stable categories which admit dependent sums and products (including function-spaces). We find that although the restrictions which previous authors have placed on their stable domains are not necessary for cartesian closure (and in some cases proved a substantial burden to them) they are of interest in themselves, and so we shall study their *axioms* one by one, and hence recover their results by taking suitable combinations of axioms.

As we have seen, the characteristic feature of stable domain theory is really *pullbacks*. Write **Pbk** for the 2-category of categories with pullbacks, functors which preserve them and cartesian transformations.

Theorem 2.42 The abstract adjunctions in **Pbk** are exactly rigid adjunctions, and the category $\mathbf{Pbk}^{\mathrm{rc}}$ of categories with pullbacks and rigid comparisons between them has pullbacks. **Pbk** is cartesian closed and admits dependent sums and products.

Proof Left adjoints are isotomies by Lemma 2.5.1. **Pbk** has binary products from Exercise 1.8.2. We went a long way round to find stable function-spaces, but it is an easy **exercise** to show that we may calculate pullbacks in function-spaces and $\mathcal{RC}$ pointwise. For dependent sums and products, Propositions 3.3.2 and 3.5.3 still apply. □

Everything else is built up from here. Write $\mathbf{Pbk}^\kappa_\lambda$ for the 2-category of locally small categories which

(i) have wide pullbacks of size λ

(ii) and filtered colimits of size κ (which must be preserved by pullbacks),

functors preserving them and cartesian transformations. Similarly write $\mathbf{SCat}^\kappa$ for the 2-category of locally small categories which

(i′) have wide pullbacks, a set of coseparators and are well-powered

(ii) and have filtered colimits of size κ (which must be preserved by pullbacks), functors preserving them and cartesian transformations.

Theorem 2.43 $\mathbf{Pbk}^{\kappa}_{\lambda}$ and $\mathbf{SCat}^{\kappa}$ admit dependent sums and products, which are created by the forgetful functors to **Pbk**. Moreover the categories of domains and rigid comparisons have appropriate wide pullbacks and filtered colimits. □

This is the first aspect of *flavour* of stable categories, which we call (algebraic) *structure*. Clearly there's nothing very exciting to be said about this, and we shall concentrate on the most restrictive case: continuous stability. This classification really applies to stable *functors*, since we may use spin (see below) to say that equalisers *exist*, whereas we may not require them to be *preserved*.

The second aspect of flavour is *size*: we may seek to restrict our stable categories to be such that there is, for instance, a countable set of objects from which others may be obtained by countable filtered colimits. Restrictions like this are popular in Computer Science for reasons of effectivity, but again nothing very much needs to be said about them, *except* that they may be inconsistent with the behaviour of spread, essentially because of a result adapted from Smyth's theorem about maximal cartesian closed categories of countably-based algebraic posets and continuous functions. Thus if we want to restrict *size* or spread by large cardinals we have to use $< \beth_\omega$ rather than $< \aleph_1$.

Exercise 2.44 Let $\mathcal{X}$ be an algebraic stable poset with some pair of finitely presentable objects which have $\kappa \geq 2$ multijoins. Then $[\mathcal{X} \to \mathcal{X}]$ has a pair of finitely presentable objects with at least 2^κ multijoins.

Besides the ubiquity of pullbacks in the theory we have forever been discussing *slices*, and we have shown that traces, function-spaces, $\mathcal{RC}$, dependent sums and dependent products have slices of the same kind as the given domains.

Definition 2.45 A *class of slices* is a class of complete categories closed under equivalence and the following operations:

(i) Singleton, cartesian product and slices.

(ii) Arrow categories $\mathcal{Y} \downarrow T$, where $T : \mathcal{X} \to \mathcal{Y}$ is a homomorphism.

(iii) Limits of diagrams whose arrows are pullback functors $X^* : \mathcal{X} \to \mathcal{X}/X$.

We shall split the problem of the complexity of polycolimits into two parts: the number of components (*spread*) and their automorphism groups (*spin*). The second problem is more tractable than the first.

Definition 2.46 A *class of spins* is a class of groups closed under isomorphism and the following operations:

(i) Singleton (trivial group).

(ii) Extensions.

(iii) Products and subgroups (alternatively arbitrary diagrams of group homomorphisms).

Theorem 2.47 Let $\mathcal{S}$ be a class of slices and $\mathcal{G}$ be a class of spins. Write $\mathbf{SCat}[\mathcal{S}, \mathcal{G}]$ for the full sub-2-category of stable categories whose slices belong to $\mathcal{S}$ and the automorphism groups of whose polycolimit candidates belong to $\mathcal{G}$. Then $\mathbf{SCat}[\mathcal{S}, \mathcal{G}]$ is cartesian closed, and indeed admits dependent sums, and dependent products indexed by *arbitrary* stable categories.

Proof As regards slices, condition (i) provides finite products and (ii) gives dependent sums; we need (iii) for cofiltered diagrams to construct filtered colimits in function-spaces and for general diagrams to construct dependent products. For spins, we used limits and extensions. □

We shall consider separately each of the restrictions which other authors have placed on their stable domains, showing that they may be expressed in terms of slices, spin and spread and why they arise naturally. Cartesian closure of their categories of domains considered will follow from the above results and such others as we can prove about preservation of spread by function-spaces and dependent products.

Exercise 2.48 Let $\mathcal{X}$ be a stable category and $\mathcal{I}$ a diagram-type (directed graph or category). Write $\mathcal{I}+1$ for the type obtained by freely adjoining a terminal object, and

$$\Delta : \mathcal{X}^{\mathcal{I}+1} \to \mathcal{X}^{\mathcal{I}}$$

for the forgetful functor between the categories of diagrams of these types in $\mathcal{X}$. Then the diagonal universals for Δ are exactly the polycolimit candidates and

$$\mathsf{Spec}_{\Delta}(X_i) \in \mathbf{Gpd}$$

is the groupoid of polycolimits of the diagram (X_i). □

Exercise 2.49 Let $S : \mathcal{X} \to \mathcal{Y}$ be stable with trace $\mathcal{T}$ and (T_i) be a diagram of type $\mathcal{I}$ in $\mathcal{T}$. Show that polycolimit candidates T for this diagram are given by pairs (X, Y, u) where X and Y are polycolimit candidates in $\mathcal{X}$ and in $\mathcal{Y}$ which are compatible in the sense that there is a (unique) morphism u making the diagram commute. By Lemma 1.5.8 this is diagonally universal. Show also that an automorphism of the candidate T is a pair $\langle x, y\rangle$ of compatible automorphisms, and hence that $\mathsf{Aut}_{\mathcal{T}}(T_i \to T)$ is the intersection of $\mathsf{Aut}_{\mathcal{T}}(T)$ and $\mathsf{Aut}_{\mathcal{X}}(X_i \to X) \times \mathsf{Aut}_{Y}(Y_i \to Y)$ in $\mathsf{Aut}_{\mathcal{X}}(X) \times \mathsf{Aut}_{Y}(Y)$. □

The following two results relate binary products and equalisers to spread (connectedness of polycolimits) and spin (discreteness). It is immediate from the adjoint functor theorem that a stable category has a terminal object iff it has colimits.

Exercise 2.50 A stable category has equalisers iff all polycolimits are discrete (multicolimits in Diers' terminology). This happens if (but not only if) all polycoequalisers are non-empty, or if the category has (pullbacks and) binary products. □

Exercise 2.51 A stable category $\mathcal{X}$ has binary products iff every diagram in $\mathcal{X}$ which has a cocone has a colimit (not poly). In this case $- \times - : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a stable functor, continuous iff $\mathcal{X}$ is pullback-continuous, and $\pi_0 : - \times X \to \mathsf{id}$ is cartesian.

Exercise 2.52 If all multijoins in a stable poset are non-empty then it has a top element. Find an example of a stable category in which all polycoproducts are non-empty but some of them are disconnected. [Hint: fields.]

What about the spread of $\mathcal{RC}$? We mentioned something analogous to disjoint coproducts (disjoint polypushouts) in Exercise 3.7.12. The problem with $\mathcal{RC}$ is that its spread tends to far exceed that of its objects:

Example 2.53 Consider $\mathcal{X} = \mathcal{Y} = 2 \in \mathcal{RC}$. A polycoproduct candidate for $\mathcal{X}$ and $\mathcal{Y}$ is a domain $\mathcal{Z}$ consisting of three points in a "V", bounded by an arbitrary groupoid. Hence the polycoproduct is a large groupoid. □

2.7 Posets and Monomorphisms

In every account of stable domain theory (except [G85], which in any case did not define a cartesian closed category) the domains have satisfied the following:

Exercise 2.54 The following are equivalent for a category $\mathcal{X}$:

(α) Every morphism $x : X' \to X$ of $\mathcal{X}$ is mono.

(β) Every slice $\mathcal{X}/X$ is a preorder.

(γ) If the parallel pair $f, g : X' \rightrightarrows X$ has an non-empty polycoequaliser then $f = g$. □

Proposition 2.55 Complete lattices form a class of slices, and hence stable categories in which all maps are mono form a relatively cartesian closed category. □

This condition is found in the mathematical examples of Diers as well: the categories of fields and linear orders have all maps mono, although they are not preorders. This phenomenon is partly explained by Exercise 1.3.5 but mainly by the

Exercise 2.56 Let $\mathbf{T}$ be a first-order theory. Every homomorphism of models of $\mathbf{T}$ is mono iff there is a *positive* formula $\phi(x, y)$ [*i.e.* one involving $\wedge$, $\vee$, $\exists$ and $\forall$ but not $\neg$ or $\Rightarrow$] such that $\mathbf{T} \vdash \forall x, y.\phi(x, y) \iff x \neq y$. □

Examples 2.57

(a) For fields, $x \neq y \iff \exists z.z(x - y) = 1$.

(b) For linear orders, $x \neq y \iff x < y \vee y < x$.

(c) For (tokens of) coherence spaces, $x \neq y \iff x \smile y \vee x \frown y$, where $\smile$ and $\frown$ denote *strict* (in)coherence. In this case the model homomorphisms are rigid embeddings. □

We are going to give a number of examples to show that the various axioms restricting stable domains may be expressed in terms of slices and spread. Some of them may appear to be a long-winded way of saying something very simple! Notice that we can deduce the existence of equalisers (and hence that polycolimits are discrete) from (ii) alone, and the existence of binary products (meets) from (iii) alone.

Example 2.58 A stable category $\mathcal{X}$ is a preorder iff

(i) every slice is a preorder and

(ii) every parallel pair has at least (exactly) one polycoequaliser candidate.

It is boundedly complete iff additionally

(iii) every pair of objects has at most one polycoproduct candidate. □

Proposition 2.59 Stable posets form a relatively cartesian closed category. □

Conjecture 2.60 Suppose that the polycoequaliser of any parallel pair in $[\mathcal{X} \to \mathcal{X}]$ is nonempty; then $\mathcal{X}$ is a preorder. Hence stable posets are the *largest* cartesian closed category of stable categories with coequalisers.

Example 2.61 A boundedly complete poset is

(i) a qualitative domain iff every slice is a complete atomic Boolean algebra and

(ii) a coherence space iff additionally the polycoproduct of a set of objects is the wide pullback of the polycoproducts of the subsets containing at most two objects. □

The last part shows that spread may or may not be determined by simple diagrams (polyinitial families, polycoequalisers and binary polycoproducts). The study of this phenomenon suggests the use of descent or homology. We do not quite have the tools to assert that qualitative and coherence domains form a cartesian closed category, but

Exercise 2.62 Complete atomic Boolean algebras form a class of slices, and hence Lamarche's *semigranular categories* (which are stable categories with Boolean slices) form a relatively cartesian closed category. □

2.8 Algebraic Stable Categories

The theme of Diers' work is the generalisation of features of the study of algebraic theories expressed in terms of functors with left adjoints to disjunctive theories using stable functors. The most famous result about the use of category theory to describe algebraic theories is the *Gabriel-Ulmer duality* between **LFP**, whose objects are categories of algebras, and **Lex**, whose objects are algebraic theories. More precisely, **LFP** is the 2-category of locally finitely presentable categories, homomorphisms and natural transformations, and **Lex** consists of categories with finite limits and functors preserving them. Diers' result extended the class of *objects* of these 2-categories by admitting (discretely) stable categories with **LFP**-slices on the one hand and categories with *multi*pullbacks on the other. However he did not generalise the morphisms, and he also use natural rather than cartesian transformations.

Definition 2.63 A category is *locally finitely poly-presentable* if it has wide pullbacks, filtered colimits and a set of (isomorphism classes of) finitely presentable objects, and every object may be expressed as a filtered colimit of finitely presentable objects. The category is *locally finitely multi-presentable* if it also has equalisers. Write **LFPP** (**LFMP**) for the 2-category of locally finitely poly- (multi-)presentable categories, continuous stable functors and cartesian transformations.

Definition 2.64

(a) A *plex-category* is a small category with finite poly-limits (its opposite has finite polycolimits); it is called a *mlex-category* if the poly-limits are discrete.

(b) A *homomorphism* of plex-categories $C : \mathbb{Y} \to \mathbb{X}$ is a functor which preserves poly-limits.

(c) A *trace* from a plex-category $\mathbb{Y}$ to another $\mathbb{X}$ is a pair of functors $C : \mathbb{T} \to \mathbb{X}$ and $F : \mathbb{T} \to \mathbb{Y}$ such that C preserves poly-limits and F is an equivalence on each *coslice*. The trace is *discrete* if F preserves coequalisers.

(d) A *diagonal* between traces $\Phi : \mathbb{T}' \to \mathbb{T}$ is an isomorphism class of functors which preserve poly-limits and make the C- and F-triangles commute.

Write **PLex** (**MLex**) for the 2-category of plex- (mlex-)categories, traces and diagonals.

Theorem 2.65 **LFPP** and **PLex** are dual in the sense that there is a 2-equivalence which is contravariant on 1-cells and covariant on 2-cells. **LFMP** and **MLex** are also dual, and both dualities restrict to discrete traces and to homomorphisms.

Proof If $\mathcal{X}$ is a locally finitely poly-presentable category, put $\mathbb{X} = (\mathcal{X}_{\text{fp}})^{\text{op}}$; conversely put $\mathcal{X} = \mathsf{Ind}(\mathbb{X}^{\text{op}})$. Lemma 1.5.8 and Exercise 1.9.9 show that this translation also applies to the trace, whilst Propositions 1.7.3 and 2.1.4 relate diagonals and cartesian transformations. The results easily restrict. $\square$

Our interest is in relatively cartesian closed categories:

Proposition 2.66 **LFP** is a class of slices.

Proof

(i) Singleton, cartesian product and slices: easy.

(ii) Arrow categories $\mathcal{Y}{\downarrow}T$, where T is continuous, not necessarily a homomorphism. The typical finitely presentable object is $y : Y \to TX$ where $X \in \mathcal{X}_{\text{fp}}$ and $Y \in \mathcal{Y}_{\text{fp}}$. It is easy to show that these approximate.

($\text{iii}_{\downarrow}$) Cofiltered limits of (not necessarily rigid) homomorphisms, or equivalently filtered colimits of comparisons. Comparisons preserve finite presentability, and so the typical finitely presentable object of $\mathsf{bilim}\,\mathcal{X}_i$ is $\Phi_i(X_i)$ where $X_i \in \mathcal{X}_{i\text{fp}}$. These approximate images, whilst images approximate all objects.

($\mathrm{iii}_{\mathrm{fin}}$) The limiting cone over a diagram of homomorphisms between *complete* LFP-categories consists of homomorphisms. Typical finitely presentable objects of the limit are finite colimits of images of finitely presentables. □

Corollary 2.67 The following full sub-2-categories of **SCat** are relatively cartesian closed:

(a) locally finitely poly-presentable categories,

(b) locally finitely multi-presentable categories,

(c) categories with filtered colimits whose slices are algebraic lattices and

(d) the same, but also having equalisers.

(e) algebraic stable posets. □

Exercise 2.68 A stable category is *profinite* (*i.e.* it can be expressed as a limit of a diagram of finite stable categories and stable functors) iff its slices are algebraic lattices. □

Exercise 2.69 Continuous categories in the sense of [Johnstone & Joyal 1982] form a class of slices, and hence the continuous analogue of Corollary 3.8.5 holds (continuous stable posets were studied in [T88]). □

Bifinite stable categories (those which can be expressed as cofiltered limits of finite domains and *rigid homomorphisms*) are much more special, because *rigid* comparisons, being isotomies, preserve more than just finite presentability.

Definition 2.70 $X \in \mathcal{X}$ is called *strongly finite* if $\mathcal{X}/X$ is a finite category. $\mathcal{X}$ is *strongly presentable* if every object can be expressed as a filtered colimit of strongly finite objects.

Exercise 2.71

(a) Show that rigid comparisons create strong finiteness, and strongly finite objects of pullback-continuous categories are finitely presentable.

(b) Strongly presentable categories form a class of slices contained in the class of algebraic lattices.

(c) Suppose that the polycoproduct of any pair of strongly finite objects is finite; then any finite set of strongly finite objects is contained in a rigidly embedded finite subcategory.

(d) Hence show that a stable category is bifinite iff it is strongly presentable and the polycoproduct of any pair of strongly finite objects is finite. □

Exercise 2.72 Show that the following are bifinite stable categories:

(a) algebraic number fields (*i.e.* algebraic field extensions of $\mathbb{Q}$) and field homomorphisms;

(b) **LOrd** (linear orders and strict monotone functions);

(c) dI-domains and rigid embeddings. □

Jung [1988] has shown how to extend the polycoproduct condition to arbitrary objects by imposing the Scott or Lawson topologies on the domain, so that *finite* becomes *compact*, but this is outside the scope of this paper. Although we cannot use spread to prove cartesian closure, the following result follows easily from Theorem 3.1.6, Exercise 3.4.6 and Proposition 3.5.3 (Coquand mentions bifinite stable posets).

Proposition 2.73 The 2-category of bifinite stable categories is relatively cartesian closed. □

Conjecture 2.74 The category of bifinite stable posets admits dependent products indexed by its category of rigid embeddings, and hence models System F. The interest of such a model is that $\Pi\alpha.\alpha \to \alpha \to \alpha$ would have only the points $\bot$, $\mathbf{t}$ and $\mathbf{f}$, and not "intersection" as in the coherence space model.

Question 2.75 What about finitely presentable categories as slices?

The most interesting classes of slices are the subclasses of the class of locally cartesian closed categories, which we shall discuss after we have shown why this condition has a serious effect on spread. Let us just conclude this subsection with the trivial case.

Exercise 2.76 Every slice of $\mathcal{X}/X$ is trivial (equivalent to the singleton category) iff every morphism of $\mathcal{X}$ is invertible. Trivial categories form a class of slices, and hence **Gpd** is relatively cartesian closed. Trivial groups also form a class of spins, and hence **Set** is relatively cartesian closed. In **Set** and **Gpd** every morphism is a display, and so these categories are *locally* cartesian closed. □

2.9 Distributivity

By far the most interesting restriction we may place on stable categories is distributivity. All accounts of stable domain theory except [T88] have assumed this because it makes the computation of function-spaces substantially easier. However the importance of this condition lies in the possibility of *topological* representations, of which there is some hint in [CGW], and in the application to *linear logic*.

We cannot define a "class of spreads" and prove a simple analogue of Theorem 3.6.5. In particular we cannot make the spread trivial (*i.e.* all colimits exist) without making the category of domains trivial too. The problem is that whereas slices and spin are essentially independent of each other and of spread, the spread of function-spaces is critically affected by the slices and spin of the domains.

Proposition 2.77 Suppose $\mathcal{X}$ and $[\mathcal{X} \to \mathcal{X}]$ are boundedly complete stable posets. Then meets distribute over joins (where they exist) in $\mathcal{X}$.

Proof It is well-known that any non-distributive lattice contains one or other of the following as sub-lattices:

$$
\begin{tikzcd}[column sep=small]
 & a \vee b & \\
A \arrow[ur] & & \\
 & & b \arrow[uul] \\
a \arrow[uu] & & \\
 & a \wedge b \arrow[ul] \arrow[uur] & \\
 & \perp \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \vee b & \\
a \arrow[ur] & b \arrow[u] & c \arrow[ul] \\
 & a \wedge b \arrow[ul] \arrow[u] \arrow[ur] & \\
 & \perp \arrow[u] &
\end{tikzcd}
$$

(In this context, a "lattice" has bottom but not top.) Let $j : \mathcal{Y} \subset \mathcal{X}$ be such a sublattice, so that $p : \mathcal{X} \to \mathcal{Y}$ with p and j stable, and $j \dashv p$. Define the three endo-functions of $\mathcal{Y}$:

$$\hat{a} : y \mapsto a \wedge y \qquad \hat{b} : y \mapsto b \wedge y \qquad h : y \mapsto (a \wedge y) \vee (b \wedge y)$$

and verify case-wise that they are stable and h is a multijoin of $\hat{a}$ and $\hat{b}$ in $[\mathcal{Y} \to \mathcal{Y}]$. Then $p\,;h\,;j$ is a multijoin of $p\,;\hat{a}\,;j$ and $p\,;\hat{b}\,;j$ in $[\mathcal{X} \to \mathcal{X}]$, and $p\,;j$ also lies above them (in the Berry order). By the hypothesis that $[\mathcal{X} \to \mathcal{X}]$ is boundedly complete, the *multijoin* should be a *join*, but if we test cartesianness of $p\,;h\,;j \le p\,;j$ at $y \le a \vee b$ with $y = A$ or $y = c$ we obtain the pullback

$$
\begin{tikzcd}
(a \wedge (a \vee b)) \vee (b \wedge (a \vee b)) \arrow[r, equal] & a \vee b \\
(a \wedge y) \vee (b \wedge y) \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner" very near start] & (a \vee b) \wedge y \arrow[u]
\end{tikzcd}
$$

from which distributivity follows. (This argument appears to be a proof by contradiction, but with slight re-wording it can be made constructive.) $\square$

Conjecture 2.78 If $[\mathcal{X} \to \mathcal{X}]$ has binary products then in $\mathcal{X}$ pullbacks preserve colimits.

We have already required pullbacks to preserve *filtered* colimits as a necessary part of continuity of stable functors, and now we are asking that they also preserve *finite* colimits. It is because previous authors have defined stable categories in terms of polycolimits that they have needed distributivity to prove cartesian closure.

Definition 2.79 A category $\mathcal{X}$ is *locally cartesian closed* if pullback functors x^* exist and have right adjoints (*cf.* Lemma 2.2.4). *Notice that we have not required there to be a terminal object.* We say $\mathcal{X}$ is an lccc.

Proposition 2.80 The following are equivalent for $\mathcal{X}$:

(α) pullback functors have right adjoints;

(β) every slice is cartesian closed;

(γ) $\mathcal{X}$ is relatively cartesian closed with respect to *all* maps;

and if $\mathcal{X}$ has wide pullbacks and a set of coseparators (...),

(δ) pullbacks preserve colimits;

(ϵ) pullbacks preserve the initial object (it is strict), coproducts (the pullback of $X \to X+Y \leftarrow Y$ against $Z \to X+Y$ is $X' \to X'+Y' = Z \leftarrow Y'$), coequalisers (similar) and filtered colimits (pullback-continuous).

Proof Standard exercise. □

Lemma 2.81 In an LCCC, pullbacks preserve exponentials.

Proof This is the Beck condition for Π; it states the commutativity of a square whose maps are the right adjoints of those in the Beck condition for Σ (exercise). □

Lemma 2.82 Let $S_1, S_2 : \mathcal{X} \rightrightarrows \mathcal{Y}$ be stable functors between locally cartesian closed categories. Suppose that the (pointwise) coproduct $S_1X + S_2X$ exists in $\mathcal{Y}$ for each $X \in \mathcal{X}$. Then this defines a stable functor $S_1 + S_2 : \mathcal{X} \to \mathcal{Y}$.

Proof Let $\nu_i X : S_i X \to S_1 X + S_2 X$ be the inclusion; *this is not a cartesian transformation without additional hypotheses* (which we shall investigate in the next subsection). Given $w : Y \to S_1X + S_2X$ in $\mathcal{Y}$, form its pullback $(\nu_i X)^* w : Y_i \to SX_i$ and factorise this as $(\nu_i X)^* w = u_i \,;\, S_i f_i$ with $u_i : Y_i \to S_i X_i$ diagonally universal and $f_i : X_i \to X$. Now, using preservation of coproducts by pullbacks, it is an **exercise** to show that $X_1 + X_2$ is the required intermediate object. □

Theorem 2.83 LCCCs with 1 form a class of slices. Hence

(a) LCCCs,

(b) LCCCs with equalisers and

(c) LCCCs with products

form relatively cartesian closed categories.

Proof Either exponentials or pullbacks and colimits may be constructed componentwise; hence [a] and [b] follow immediately from Theorem 3.6.5. For [c], having binary products is equivalent to having at most one polycolimit for any diagram. By Corollary 3.5.8, any polycolimit candidate for a diagram in the function-space is computed pointwise and so is unique. □

The relatively cartesian closed categories of stable domains which we have as a result of this are too numerous to mention individually, so we shall just list the ones which others have considered.

Corollary 2.84 The following classes of stable domains form relatively cartesian closed categories:

(a) boundedly complete, strongly presentable and distributive: *dI-domains*, Berry [1978] and Coquand, Gunter & Winskel [1986];

(b) boundedly complete and Boolean: *qualitative domains*, Girard [1984];

(c) boundedly complete and Boolean, where any pairwise bounded set has a join; *coherence spaces*, Girard [1986] and Girard, Lafont & Taylor [1988];

(d) mono, algebraic and distributive: *stable categories of embeddings*, Coquand [1988];

(e) mono, equalisers, strongly presentable and distributive: *aggregates*, Lamarche [1988];

(f) Boolean and equalisers: *semigranular categories*, Lamarche [1988].

In cases (a–c) we may understand *non-empty* if we please.

Proof Obviously Boolean algebras are strongly presentable distributive posets. That the binary condition (c) is reproduced in function-spaces follows from the form of condition 3.7.6(ii). □

[Any distributive domain has a reflection into any class of slices.]

Our reason for considering distributivity was that it was necessary for bounded completeness, whilst the fact that the domain may be taken to be a subset of a *frame* suggests *topological representations* of which [CGW] gives some hint. It also appears (although I do not know why) that distributivity is needed to interpret *linear logic*. Domains of this kind appear to be worthy of specify study, and Girard's qualitative/quantitative terminology seems too good to waste:

Definition 2.85 A *qualitative domain* is a boundedly complete poset with directed joins and bottom whose slices are frames.

Theorem 2.86 Qualitative domains form a relatively cartesian closed category.

Proof Special case of Theorem 3.9.7 where all maps are mono. □

Theorem 2.87 Qualitative domains admit second-order dependent types, *i.e.* they model Girard-Reynolds polymorphism.

Proof The category of qualitative domains and rigid comparisons has small spread, so we are permitted to form dependent products over it. The slices of such a dependent product are frames, so it only remains to verify that it is boundedly complete; but the argument of Theorem 3.9.7 still applies. □

These dependent products satisfy Moggi's uniformity property, but the values of polymorphic types are likely to be much the same as in coherence spaces and dI-domains.

2.10 Quantitative Domains

Qualitative domains exploit the best we can do in demanding existence of coequaliers and uniqueness of coproducts; as we shall see, we cannot have both, so we shall now look for the reverse. The following is a special case of Conjecture 3.9.2:

Lemma 2.88 If $[\mathcal{X} \to \mathcal{X}]$ is connected then $\mathcal{X}$ (is empty or) has a *strict* initial object.

Proof Clearly $\mathcal{X}$ is connected, so let I be the unique polyinitial candidate. By hypothesis $\mathsf{id}, \ulcorner I \urcorner : \mathcal{X} \to \mathcal{X}$ are connected by some zig-zag, which using pullbacks we may reduce to

$$\mathsf{id} \xleftarrow{\ \kappa\ } M \xrightarrow{\ \phi\ } \ulcorner I \urcorner$$

Put $Z = MI \in \mathcal{X}$ and $p = \kappa_I : Z \to I$. Since I is a polyinitial candidate, there is a (unique) $j : I \to Z$ with $j\,;p = \mathsf{id}$, so p is split epi. For any $X \in \mathcal{X}$ we have some $\xi : I \to X$ and then the left-hand square is a pullback, making $M\xi$ an isomorphism:

$$\begin{tikzcd}
Z \arrow[r, "\phi_I"] \arrow[d, "M\xi"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "\mathsf{id}"] \\
MX \arrow[r, "\phi_X"] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
MZ \arrow[r, "\kappa_Z"] \arrow[d, "\cong"', "Mp"] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "p"] \\
Z \arrow[r, "p = \kappa_I"] & I
\end{tikzcd}$$

It follows that M takes *every* morphism to an isomorphism, and in particular Mp is one. Then the right-hand square shows that p is mono and hence iso, so $\phi : M \cong \ulcorner I \urcorner$. The argument also shows that any given $p' : Z' \to I$ is iso; in particular so are $g, (\phi^{-1} ; \kappa)_I : I \to I$. But suppose we have $g : I \cong I$; then naturality of $\phi^{-1} ; \kappa$ with respect to g shows that $g = \mathsf{id}$, so I is strict initial. $\square$

The following property of slices is essentially the difference between Girard's qualitative and quantitative domains: in a poset, binary coproducts (joins) are *idempotent*, *i.e.* $X \vee X \cong X$, whereas in **Set** or any topos they are *disjoint*, *i.e.* the pullback of the inclusions is the (strict) initial object. This is the reason why $[2 \to 2]$ is not 3 but a V-shaped domain.

Lemma 2.89 Suppose that the polycoproduct of any pair of objects of $[\mathcal{X} \to \mathcal{X}]$ is nonempty. Then $\mathcal{X}$ has *disjoint* polycoproducts.

Proof Clearly $[\mathcal{X} \to \mathcal{X}]$ is connected, and so by Proposition 3.9.1 $\mathcal{X}$ has a strict initial object 0. Let $X, Y \in \mathcal{X}$ and consider $\mathsf{id}, \ulcorner X \urcorner : \mathcal{X} \to \mathcal{X}$. By hypothesis these have a polycoproduct candidate S, with $\phi : \ulcorner X \urcorner \to S$ and $\psi : \mathsf{id} \to S$, where (ϕ and) ψ are cartesian. Then the square is a pullback:

$$\begin{tikzcd}
0 \arrow[rr, "\psi_0 = ?_{S0}"] \arrow[dr, "?_X"] \arrow[dd, "?_Y"'] & & S0 \arrow[dd, "S?_Y"] \\
& X \arrow[ur, "\phi_0"] \arrow[dr, "\phi_Y"] \arrow[dd, "\nu_1"] & \\
Y \arrow[rr, "\psi_Y"] \arrow[dr, "\nu_2"'] & & SY \\
& X+Y \arrow[ur, "m"'] &
\end{tikzcd}$$

where $m : X + Y \to SY$ is the coproduct in $\mathcal{X}/SY$ with inclusions ν_1 and ν_2. It is easy to see that the parallelogram from 0 to $X + Y$ is also a pullback. $\square$

A categorical logician might imagine that the analogous result for polycoequalisers is that equivalence relations are effective. Unfortunately by conjecture 3.7.6 the result is more drastic.

Proposition 2.90 Suppose that $[\mathcal{X} \to \mathcal{X}]$ has a terminal object. Then $\mathcal{X}$ is degenerate, *i.e.* equivalent to either 0 or 1.

Proof Suppose that $\mathcal{X}$ is connected. The polycoproduct of any pair of objects of $[\mathcal{X} \to \mathcal{X}]$ is nonempty, and so $\mathcal{X}$ has disjoint polycoproducts and a strict initial object 0. $\mathcal{X}$ also has colimits (since diagrams of constant functors have colimits) and hence a terminal object 1, binary products and equalisers.

Consider the stable functor $SX = X^2$ and the object $2 = 1 + 1$; as well as the identity, these carry "switch" automorphisms $s : 2 \to 2$ and $t : S \to S$, giving a parallel pair

$$\langle S, 2\rangle \underset{\langle \mathsf{id}, \mathsf{id}\rangle}{\overset{\langle t, s\rangle}{\rightrightarrows}} \langle S, 2\rangle$$

in $[\mathcal{X} \to \mathcal{X}] \times \mathcal{X}$. The latter has a terminal object, whence any stable functor out of it preserves equalisers. Clearly applying ev to this pair makes them equal, and so the equaliser of the images is 2^2.

Since ev preserves equalisers jointly, it must do so separately (although the converse is not true), and in particular equalisers of stable functors are calculated pointwise. The coproduct

$2 = 1 + 1$ is disjoint, so the equaliser of the above pair is $\langle \ulcorner 0 \urcorner, 0\rangle$, which becomes 0 on application of ev. Hence $2^2 \cong 0$ and $\mathcal{X}$ is degenerate. □

Corollary 2.91 In any non-trivial cartesian closed category of stable categories, there are objects for which some polycoproduct or some polycoequaliser is empty. □

Question 2.92 What feature, if any, of $[\mathcal{X} \to \mathcal{X}]$ would force $\mathcal{X}$ to have effective equivalence relations in slices? Is there a connection between this and when rigid adjunctions are monadic?

The following terminology arises from Giraud's *theorem* that a category is a *Grothendieck* topos iff it is of this form.

Definition 2.93 A *Giraud topos* is a locally small category which has set-indexed colimits and a set of generators and satisfies

(i) it is locally cartesian closed;

(ii) coproducts are disjoint;

(iii) equivalence relations are effective, *i.e.* if a subobject $R \subset X \times X$ is reflexive, symmetric and transitive then it is the kernel of its coequaliser;

(iv) epimorphisms are regular, *i.e.* the coequaliser of their kernels.

Lemma 2.94 In a topos, pullback functors preserve the subobject classifier Ω. □

Theorem 2.95 Giraud toposes form a class of slices, and hence stable domains whose slices are toposes form a relatively cartesian closed category.

Proof Either we may compute Ω componentwise, or else we compute pullbacks and colimits componentwise and verify (ii–iv) in the same way as (i). □

As with distributivity, our aim was to optimise spread, and again it seems appropriate to modify Girard's terminology. Here also we should look for topological representations, since any quantitative domain is a full subcategory of a topos.

Definition 2.96 A *quantitative domain* is a category with filtered colimits, a strict initial object, binary products and binary coproducts, whose slices are *toposes*.

Lemma 2.97 Let S_1 and S_2 be stable functors between quanitative domains. Then $\nu_1 X : S_1 X \to S_1 X + S_2 X$ is cartesian.

Proof Suppose $f : Y \to S_1 X' + S_2 X'$ and $g : Y \to S_1 X$ make the square commute. Let $Y = Y_1 + Y_2$, where $Y_i = f * (\nu_i X')$. Then $Y_2 \to 0$ since the coproduct $S_1 X + S_2 X$ is disjoint, and so $Y_2 \cong 0$ and $Y \cong Y_1 \to S_1 X'$. □

Theorem 2.98 Quantitative domains form a relatively cartesian closed category.

Proof (Sketch) The strict initial object of $[\mathcal{X} \to \mathcal{X}]$ is $\ulcorner 0 \urcorner$, and we have shown that coproducts are constructed pointwise. Coequalisers, where they exist, are also computed pointwise and hence are unique; it follows that any colimit diagram with a cocone has a colimit, which is equivalent to possessing binary products. The results are easily extended to dependent (sums and) products. □

Theorem 2.99 The category of quantitative domains and rigid comparisons is a quantitative domain, *i.e.* a *type of types*.

Proof The strict initial object is the singleton, and coproducts are given by products quantitative domains *quâ* categories. Binary product is a "tensor product" construction which for toposes is topological product. □

Conjecture 2.100 This is the largest category of stable domains with a type of types. $\square$

[The following result belongs in section 2.1; necessity was remarked by Lamarche.]

Lemma 2.101 Φ is full (as well as faithful) iff ϕ is mono.

Proof

[$\Rightarrow$] Suppose $w_1 ; \phi_X = w_2 ; \phi_X$, in which we may put $w_1 = u_1' ; S'x_1$ and $w_2 = u_2' ; S'x_2$. Then

$$u_1 ; Sx_1 = u_1' ; \phi_{X_1'} ; Sx_1 = u_1' ; S'x_1 ; \phi_X = u_2' ; S'x_2 ; \phi_X = u_2' ; \phi_{X_2'} ; Sx_2 = u_2 ; Sx_2$$

by naturality of ϕ, and so by diagonal universality of u_1 and u_2 there is a unique $h : X_1' \cong X_2'$ such that

$$u_2 = u_1 ; Sh \qquad \text{and} \qquad x_2 = h^{-1} ; x_1$$

The first equation makes $\langle h, \mathsf{id}\rangle : \Phi\langle X_1, Y, u_1'\rangle \to \Phi\langle X_2, Y, u_2'\rangle$ a morphism, and so by hypothesis $\langle h, \mathsf{id}\rangle : \langle X_1, Y, u_1'\rangle \to \langle X_2, Y, u_2'\rangle$ is also, *i.e.* $u_2' = u_1' ; S'h$. But then

$$w_2 = u_2' ; S'x_2 = u_1' ; S'h ; S'(h^{-1} ; x_1) = u_1' ; S'x_1 = w_1$$

as required.

[$\Leftarrow$] Let $\langle x, y\rangle : \Phi\langle X_1, Y, u_1'\rangle \to \Phi\langle X_2, Y, u_2'\rangle$ be a morphism. Then the square on the left commutes and that on the right is a pullback:

$$\begin{tikzcd} Y_1 \arrow[r, "u_1"] \arrow[d, "y ; u_2'"'] & SX_1 \arrow[d, "Sx"] \\ Y_2 \arrow[r, "\phi_{X_2}"] & SX_2 \end{tikzcd} \qquad \begin{tikzcd} S'X_1 \arrow[r, "\phi_{X_1}"] \arrow[d, "S'x"'] \arrow[dr, phantom, "\lrcorner", very near start] & SX_1 \arrow[d, "Sx"] \\ S'X_2 \arrow[r, "\phi_{X_2}"] & SX_2 \end{tikzcd}$$

Let $v : Y_1 \to S'X_1$ be the mediator. But then $v ; \phi_{X_1} = u_1 = u_1' ; \phi_{X_1}$ and ϕ_{X_1} is mono, so $v = u_1'$. Hence also $u_1' ; S'x = y ; u_2'$, *i.e.* $\langle x, y\rangle : \langle X_1, Y, u_1'\rangle \to \langle X_2, Y, u_2'\rangle$ is a morphism. $\square$