On the General Recursion Theorem

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Since Lawvere's early 1960s ambition "to do set theory without elements" enormous progress has been made in our understanding of the connectives of logic and type theory. We have beautiful and powerful modern technology of both symbolic and diagrammatic kinds for dealing with these aspects of foundations.

The treatment of induction and recursion in their full generality lags way behind, and in writing the chapter on induction in my book *Practical Foundations* I felt embarrassed how old fashioned the material was in comparison to that on type theory. The established general form — well founded relations — reeks of set theory, and the full treatment of the ordinals was, until recently, thoroughly reliant on excluded middle.

There are, however, idioms of induction which do not arise as instances of well founded relations. The fixed point theorem which Tarski proved is demonstrated using closure conditions, whilst the form of it popular in computer science has been adapted to an idiom known as Scott induction, by specialising to predicates which are *ā priori* closed under directed joins. My own work on *The Fixed Point Property in Synthetic Domain Theory* employed a corresponding notion of "finite ordinals" which, unlike Cantor's, stop at ω, and domains of this kind have been generalised to "FIX objects" in more complicated inductive structures by Crole and Pitts.

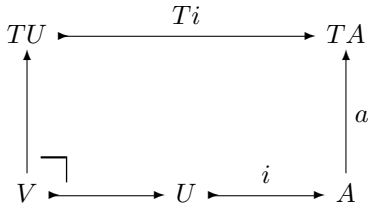
This work unifies many of these idioms of induction, with, for example, a single definition of ordinal which can be instantiated to three notions of intuitionistic ordinal and also the domain-theoretic kind. It also treats the construction and characterisation of free algebras for free theories and hence pattern matching and the unification algorithm.

The theorem at the centre of this work is Cantor's General Recursion Theorem (originally stated for ordinals), which extends inductive proof to recursive construction. The proof is by pasting together "attempts" (partial solutions). I shall show how this argument may be put in categorical form, thereby isolating those properties of the category of sets $\mathcal{C}$ (and the class of monos $\mathcal{M}$) on which it depends. The style of argument was first used by Osius.

Let $T : \mathcal{C} \to \mathcal{C}$ be a functor which preserves pullbacks of monos against arbitrary maps (and hence extends to an order-preserving endofunctor of the ordered category of partial maps). For example T may be $\mathcal{P}$, the covariant powerset functor (acting on morphisms as directed image), or of the form $\sum_r (-)^{\mathrm{ar}(r)}$ where r is a typical operation-symbol of arity $\mathrm{ar}(r)$ of a free algebraic theory. Notice that in both these cases T in fact preserves *arbitrary* intersections of monos.

The notion of T-algebra, $a : TA \to A$, is well known, and we ultimately find the free T algebra if it exists (which it does if T preserves κ-filtered colimits for some κ). However the approach is to characterise the "initial segments" of the free algebra, for example transitive sets (or, as I call them, ensembles) are the initial segments of the von Neumann hierarchy, which is the (inadmissible) free $\mathcal{P}$-algebra.

Here is the characterisation. We define a **well founded coalgebra** to be a structure $a : A \to TA$ (*sic*) such that in any diagram of the form



the mono $i : U \rightarrowtail A$ is already an isomorphism. In the case $T = \mathcal{P}$, this says that $a : A \to \Omega^A$ is (the exponential transpose of) a well founded relation $(\prec) \subset A \times A$: the diagram (with $i : U \equiv \{a : \phi[a]\} \rightarrowtail A$) expresses the premise

$$\forall a.(\forall b.b \prec a \Rightarrow \phi[b]) \Rightarrow \phi[a]$$

of the induction scheme, whose conclusion is that $i : U \cong A$. An initial segment is then a well founded coalgebra which is **extensional**, *i.e.* $a : A \rightarrowtail TA$ is mono.

The two most difficult steps in the categorical form of the proof of the general recursion theorem correspond to the most innocent observations in the classical treatment. One is used to paste together attempts and relies on the fact that unions of subsets are pushouts of sets: the corresponding result for epis has a counterexample involving a handful of elements. The other is the fact about well founded relations which everyone knows but most people don't know they know, that strictly monotone functions reflect well-foundedness (indeed the issue is the simplest case, an initial segment). A Heyting implication of monos is needed, but for epis this means at least that the lattice of equivalence relations is modular, which is true for Mal'cev theories but not sets. (This implication is already apparent in the intuitionistic proof of the "well known fact" but not in brute force classical versions, so here is another example where intuitionistic logic shows up the essential structure where classical logic obfuscates it.)

The applications which I am currently able to make replace sets with posets, transitive relations, semilattices and domains, giving the various notions of ordinal. What I wanted to do was "turn the arrows round" and hence explain co-induction, but by the preceding remarks this may not be possible.