Well Pointed Endofunctors

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22 September 2026

This is a preliminary version that is just intended to provide mathematical substance for what I presented in an online seminar on 19 June 2025. I have more to say on this subject, so please do not circulate copies of this paper, but download a new one from my website. *There are some comments in italics where I would appreciate either checking or other ideas from colleagues.*

1 Introduction

This work is prompted by two papers,
(a) *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on* by Max Kelly [Kel80], and
(b) *Une Méthode d'élimination des Nombres Transfinis des Raisonnements Mathématiques* by Kazimierz Kuratowski [Kur22],

whose titles demand the application of one to the other.

Kelly showed how various categorical recursion problems can be translated into one another. In so doing, he identified the notion of a ***well pointed endofunctor*** $S : \mathcal{X} \to \mathcal{X}$ on a category, that is, one with a natural transformation $\sigma : \mathsf{id} \to S$ satisfying $S\sigma = \sigma S$.

Kelly was not the first to consider well pointed endofunctors. I need to investigate the other literature.

Any idempotent monad (M, η, μ) on $\mathcal{X}$ is a well pointed endofunctor (M, η).
(a) (M, η, μ) may be regarded as the *big step* that produces a free algebra *directly* from its generators, whereas
(b) a general well pointed endofunctor (S, σ) is a *small step* towards that construction.

Kelly saw the problem of obtaining an idempotent monad from a well pointed endofunctor as the *simplest* of the questions on his menu. In fact we will reduce it further, to a Galois connection, but that recursive construction is the central subject matter of this paper.

Even though Kelly was the grandfather of Australian 2-category theory, regrettably he did not develop his notion of well pointed endofunctor in that style. That is why we are doing so now. He would have found that this notion goes a long way towards solving the problems that he considered.

Instead he resorted to transfinite recursion, even though it merely amounts to using directed colimits to give (unnecessary) names to the iterates of a functor. Kuratowski showed how to eliminate these labels from the analogous situation for funct*ions*, reducing it to the impredicative second order logic of Ernst Zermelo's axioms for set theory.

We will show that pointed endofunctors *already* form a directed category, but in a "proof relevant" way: instead of the mere existence of some bound for any two elements, they are provided by an algebraic structure.

However, we cannot deduce the existence of fixed points or adjoints in categories as easily as we can in the order case. We only consider these issues superficially, at the end of this paper, adopting the same naïve foundations as in Kelly's paper.

Remark 1.1 There are various grades of logical complexity governing the additional assumptions:
(a) The situation may be simple enough for the adjoint to be constructed "with bare hands",

requiring no further assumption.

(b) It may be ***finitary***, where the endofunctors preserve filtered colimits and it is enough to use one over an $\mathbb{N}$-indexed chain.

(c) The functor(s) may have ***rank*** or be ***accessible***, where the functor preserves colimits of diagrams that must obey stronger versions of the filtered property, but the fixed point is still found *uniformly* for all starting objects from the underlying category.

(d) The iteration may go on "forever" *in general* but terminate at different stages, so long as the starting object is "well founded".

The termination of any of these recursive methods may require an additional foundational assumption beyond the logic of an elementary topos, namely the axiom-scheme of replacement. That is considered in [Tay26] (which focuses on problem of type (d)) but not in this paper (where we have type (c) in mind). Instead we suppose here (without detailed discussion) that the categories are *internal* to some topos and that they have appropriate colimits indexed by objects of that enclosing topos.

2 Order-theoretic fixed points

We will see that iterating well pointed endofunctors is a *mild categorical generalisation* of the ordered case, which we review briefly in this section.

In both cases, there was *for a very long time* an intellectual blind spot, that for

- inflationary monotone endofunctions $s : X \to X$ of posets, defined by

$$\forall x.\ x \leq sx \quad \text{and} \quad \forall xy.\, x \leq y \Longrightarrow sx \leq sy,$$

- and well pointed endofunctors of categories,

one could *compose* them and thereby obtain a *directed system*.

$$\begin{tikzcd}
r(sx) \arrow[d, dash] \arrow[drr, dash] & & s(rx) \arrow[d, dash] \arrow[dll, dash] \\
rx \arrow[dr, dash] & & sx \arrow[dl, dash] \\
& x &
\end{tikzcd}$$

It apparently never occurred to Kuratowski or to many many others, almost throughout the twentieth century before Dito Pataraia in 1997, to do this for functions. A related observation (Lemma 6.11) about well pointed endofunctors is similarly absent from Kelly's account.

In the order-theoretic case we may take the join over this directed system and thereby obtain fixed points. In this section we give the proof of that, following exactly the same plan as the later categorical construction.

Notation 2.1 Let X be a dcpo (partial order with joins of all directed subsets) and $\mathsf{id} \leq S \subset [X \to X]$ a family of inflationary monotone endofunctions of X. Then write

$$\mathbf{Alg}\, S \ \equiv\ \{x \in X \mid sx \leq x\},$$

although, since $\mathsf{id} \leq s$, in fact $sx = x$.

Theorem 2.2 There is a closure operator

$$\mathsf{id}\ \leq\ m\ =\ m \cdot m\ :\ X \longrightarrow X$$

such that $\mathbf{Alg}\, m = \mathbf{Alg}\, S$.

Proof

(a) For any inflationary monotone endofunction $s : X \to X$ and element $a \in X$, we write

$$s \perp a \quad \text{for} \quad sa = a$$

and extend this to subsets by

$$S^\perp \;\equiv\; \mathbf{Alg}\, S \;\equiv\; \{a \in x \mid \forall s \in S.\, s \perp a\}$$

and
$$^\perp\! A \;\equiv\; \{s : X \to X \mid \mathsf{id} \le s \quad \& \quad \forall a \in A.\, s \perp a\}.$$

Hence these two operators define a contravariant adjunction or Galois connection,

$$A \subset S^\perp \equiv \mathbf{Alg}\, S \iff S \perp A \iff S \subset {}^\perp\! A\,.$$

(b) Then $\mathsf{id} \in {}^\perp\! A$ trivially, whilst if $r \in {}^\perp\! A$ and $s \in {}^\perp\! A$ then $r \cdot s \in {}^\perp\! A$, since $(r \cdot s)a \equiv r(sa) = sa = a$, and these satisfy $\mathsf{id} \le r, s \le r \cdot s$.

(c) Hence ${}^\perp\! A$ is a directed set of inflationary monotone endofunctions of a dcpo. It therefore has a pointwise join m, which is the greatest element of ${}^\perp\! A$.

(d) By the observation in (b), $m \cdot m \in {}^\perp a$, so $m \cdot m \le m$ since m is the greatest element. But $\mathsf{id} \le m$ since we're only dealing with inflationary functions and then $m \le m \cdot m$. So they're equal and m is a closure operator.

(e) Since $\{m\} \subset {}^\perp\! A$ we have $A \subset m^\perp \equiv \mathbf{Alg}\, m$, whilst any $s \in {}^\perp\! A$ satisfies $s \le m$ and so $\mathbf{Alg}\, m \subset \mathbf{Alg}\, s$. Putting $A \equiv \mathbf{Alg}\, S$ in this,

$$\mathbf{Alg}\, S \;\subset\; \mathbf{Alg}\, m \;\subset\; \bigcap \{\mathbf{Alg}\, s \mid s \in S\} \;\equiv\; \mathbf{Alg}\, S. \qquad \Box$$

Corollary 2.3 For any $x \le a \in X$ with $a \in \mathbf{Alg}\, S$,

$$x \;\le\; \bigvee\!\!{}^{\uparrow} \{rx \mid r \in {}^\perp S^\perp\} \;=\; mx \;\le\; a.$$

That is, mx is the least common fixed point of S above x and is the directed join of the rx for $r \in {}^\perp S^\perp$. In particular, if X has a least element $\perp$ then $m\perp$ is the least common fixed point of S and it is the directed join of the $r\perp$. $\Box$

In the categorical version of this proof that we shall give, we cannot make a blanket assumption that all directed colimits exist. Besides this, the interesting category theory is actually all *finitary.* We will work over an *arbitrary* category X, that is, we initially make *no further assumption* about it. We will introduce some colimits later.

In order to show that the finitary discussion yields *exactly* the required colimit *diagram* we will prove a converse to the above result. In the ordered setting this is:

Proposition 2.4 If $A \subset X$ is the image $\mathbf{Alg}\, m$ of a closure operator $\mathsf{id} \le m = m \cdot m$ then m is the greatest element of ${}^\perp\! A$.

Proof Firstly, $A \subset m^\perp \equiv \mathbf{Alg}\, m$ means $m \in {}^\perp\! A$.

On the other hand, for each $s \in {}^\perp\! A$ we have $\mathbf{Alg}\, m \equiv A \subset \mathbf{Alg}\, s$, which means $\forall y.\, my = y \Rightarrow sy = y$. For any $x \in X$, put $y \equiv mx$ and then $my = m(mx) = mx = y$. Then $s \le m$ because

$$sx \;\le\; s(mx) \;\equiv\; sy \;=\; y \;\equiv\; mx.$$

Therefore m is the greatest element of ${}^\perp\! A$. $\Box$

3 Reflective subcategories

Definition 3.1 Recall that a subcategory $U : \mathcal{A} \subset \mathcal{X}$ is ***reflective*** if it is

(a) ***full***: for each pair $A, B \in \mathcal{A}$, every morphism $f : A \to B$ in $\mathcal{X}$ is also in $\mathcal{A}$;

(b) ***replete***: if $A \in \mathcal{A}$ and $f : A \cong X \in \mathcal{X}$ then $X \in \mathcal{A}$ too; and

(c) the inclusion has a ***left adjoint*** $F : \mathcal{X} \to \mathcal{A}$: each $X \in \mathcal{X}$ has $\eta : X \to UFX$ such that for each $A \in \mathcal{A}$ and $f : X \to UA$ in $\mathcal{X}$ there is a unique $g : FX \to A$ in $\mathcal{A}$ such that $f = \eta_X \,; Ug$.

Lemma 3.2 An adjunction $F \dashv U$ with $U : \mathcal{A} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{A}$ has U full and faithful iff the counit $\epsilon : F \cdot U \to \mathsf{id}$ is invertible [Mac71, Thm. IV 3.1]. $\square$

Definition 3.3 A ***monad*** $\mathsf{M} \equiv (M, \eta, \mu)$ on $\mathcal{X}$ is an endofunctor $M : \mathcal{X} \to \mathcal{X}$ with natural transformations $\eta : \mathsf{id} \to M$ and $\mu : M \cdot M \to M$ such that

$$\eta T \,;\, \mu \;=\; \mathsf{id} \;=\; T\eta \,;\, \mu \qquad \text{and} \qquad T\mu \,;\, \mu \;=\; \mu T \,;\, \mu.$$

Any adjunction $F \dashv U$ with unit η and counit ϵ defines a monad by $M \equiv U \cdot F$ and $\mu \equiv U\epsilon F$. If ϵ is invertible, so is μ:

Lemma 3.4 For a monad (M, η, μ), each μ_X is invertible iff $M\eta = \eta M$. Such a monad is called ***idempotent***.

Proof If μ is invertible then the monad laws say that both $M\eta$ and ηM serve as its inverse, so they are equal.

Conversely, naturality of μ with respect to η or *vice versa*,

$$\begin{tikzcd}
MM \arrow[r, "\mu"] \arrow[d, "M\eta M"', "MM\eta"] & M \arrow[d, "M\eta"] \\
MMM \arrow[r, "\mu M"] & MM
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
MM \arrow[r, "\mu"] \arrow[d, "M\eta M"', "\eta MM"] & M \arrow[d, "\eta M"] \\
MMM \arrow[r, "M\mu"] & MM
\end{tikzcd}$$

together with the monad laws give the converse:

$$\mu \,;\, \mathcal{M}\eta \;=\; MM\eta \,;\, M\mu \;=\; M\eta M \,;\, M\mu \;=\; \mathsf{id}\,. \qquad \square$$

Lemma 3.5 The following are equivalent for an idempotent monad and an object $X \in \mathcal{X}$:
(a) X is a *fixed point*: $\eta X : X \cong MX$;
(b) X carries a (unique) (M, η, μ)-*algebra* structure $\xi : MX \to X$ such that

$$\eta_X \,;\, \xi \;=\; \mathsf{id}_X \quad \text{and} \quad M\xi \,;\, \xi \;=\; \mu_X \,;\, \xi;$$

(c) X is an *image*: $X \cong MY$ for some Y. $\square$

Putting all of these structures together:

Proposition 3.6 The following are uniquely interdefinable:
(a) reflective subcategory;
(b) an adjunction with invertible counit;
(c) a monad for which μ is invertible; and
(d) a monad for which $M\eta = \eta M$. $\square$

The class of reflective subcategories of any category $\mathcal{X}$ therefore forms a large partial order (***poclass***) that has a greatest member. This can also be expressed in terms of monads under the correspondence that we have just demonstrated:

Lemma 3.7 Let (M, η, μ) and (N, θ, ν) be monads with μ and ν invertible. Then any natural transformation $\phi : M \to N$ that makes the triangle commute also makes the rectangle from $M \cdot M$ to N commute:

$$\begin{tikzcd}
& \mathsf{id}_{\mathcal{X}} \arrow[dl, "\eta"'] \arrow[dr, "\theta"] & \\
M \arrow[rr, "\phi"] & & N
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M \cdot M \arrow[r, "M\phi"] \arrow[d, "\mu"'] & M \cdot N \arrow[r, "\phi N"] & N \cdot N \arrow[d, "\nu"] \\
M \arrow[rr, "\phi"] \arrow[u, dotted, "M\eta"'] \arrow[ur, dotted, "M\theta"'] & & N \arrow[ul, dotted, "\eta N"] \arrow[u, dotted, "\theta N"]
\end{tikzcd}$$

along with the similar one with $N \cdot M$. Such ϕ is called a ***monad morphism***.

Moreover there is at most one such morphism betweem any two idempotent monads.

Proof Since μ and ν are invertible, it suffices to show that the rectangle from M to $N \cdot N$ commutes, but that consists of M applied to the triangle and naturality of ϕ with respect to θ. Also, $\eta N \,;\, \phi N \,;\, \nu = \theta N \,;\, \nu = \mathsf{id}$, so ηN is mono.

Suppose that $\psi : M \to N$ is another such morphism. Then

$$M\theta \;=\; M\eta\,; M\phi \;=\; \phi\,; \eta N \qquad \text{and likewise} \qquad M\theta \;=\; \psi\,; \eta N$$

by naturality of ϕ and ψ with respect to η. Therefore $\phi = \psi$ since ηN is mono. □

Corollary 3.8 Idempotent monads on $\mathcal{X}$ form a poclass that is a full subcategory of the category of well pointed endofunctors $\mathcal{X} \to \mathcal{X}$ and their morphisms. It is equivalent to the poclass of full reflective subcategories of $\mathcal{X}$. □

Remark 3.9 Does this poclass admit intersections?

Let's call objects of our reflective subcategory *nice*.

Suppose we have a second reflective subcategory, whose objects are *pretty*.

Do *nice pretty* objects form a reflective subcategory too?

Not obviously. (Idempotent) monads *don't compose*: We have to make the given object *nice*, then *pretty*, then *nice* again, then *pretty* again, ... and when we've done this infinitely often, we're still not finished, ...

But, instead of doing something infinitely often and then some more, let's do some category theory!

4 Well pointed endofunctors

Since the invertible multiplication μ is redundant for an idempotent monad, forget it! The significant equation is $M\eta = \eta M$.

Definition 4.1 A ***pointed endofunctor*** of any category $\mathcal{X}$ is an endofunctor $S : \mathcal{X} \to \mathcal{X}$ together with a natural transformation $\sigma : \mathsf{id}_X \to S$. It is ***well pointed*** if $S\sigma = \sigma S$.

An ***algebra*** for a (well) pointed endofunctor is $X \in \mathcal{X}$ with $\xi : SX \to X$ such that $\sigma_X ; \xi = \mathsf{id}_X$.

A ***morphism*** of (well) pointed endofunctors is a natural transformation that makes this triangle commute:

$$\begin{tikzcd}
& \mathsf{id}_{\mathcal{X}} \arrow[dl, "\rho"'] \arrow[dr, "\sigma"] & \\
R \arrow[rr, "\phi"] & & S
\end{tikzcd}$$

In the absence of μ, the natural transformations $S\sigma$ and σS need no longer be invertible, but this equality alone still gives them the essence of the relationship with full subcategories [Kel80, Prop. 5.2].

Lemma 4.2 The following are equivalent for any well pointed endofunctor $\sigma : \mathsf{id} \to S$ and any object $X \in \mathcal{X}$:

(a) X has some ***algebra*** structure $\xi : SX \to X$ with $\sigma_X \,;\, \xi = \mathsf{id}_X$;

(b) σ_X is an isomorphism;

(c) X is ***orthogonal*** to σ_Y for all $Y \in \mathcal{X}$, in the sense that, for any map $f : Y \to X$ in $\mathcal{X}$ there is a unique map $g : SY \to X$ with $f = \sigma_Y \,;\, g$.

$$\begin{tikzcd}
Y \arrow[rr, "\sigma_Y"] \arrow[dr, "f"'] & & SY \arrow[dl, dotted, "\exists! g"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
SX \arrow[r, shift left, "S\sigma X"] \arrow[r, shift right, "\sigma SX"'] \arrow[d, "\xi"'] & SSX \arrow[d, "S\xi"] \\
X \arrow[r, "\sigma X"] & SX
\end{tikzcd}$$

Proof [a⇒b] Suppose that $X \xrightarrow{\sigma X} SX \xrightarrow{\xi} X$ is id_X. Then

$$\xi\,;\sigma X \;=\; \sigma SX\,; S\xi \;=\; S\sigma X\,; S\xi \;=\; S\mathsf{id}_X$$

by naturality of σ with respect to ξ. Hence ξ and σX are inverse and ξ is unique.

[b⇒c] $g \equiv (\sigma_X)^{-1}\,; f$.

[c⇒a] If $f \equiv \mathsf{id} : X \to X$ then $\xi \equiv g$ defines an algebra.

[b⇒a] In particular, $\xi \equiv (\sigma X)^{-1}$ provides the unique algebra structure on X. □

Lemma 4.3 A morphism $\phi : (R,\rho) \to (S,\sigma)$ of pointed endofunctors induces a faithful functor $\mathbf{Alg}\,(S,\sigma) \to \mathbf{Alg}\,(R,\rho)$ by $\xi \mapsto \phi_X\,;\xi$. If they are well pointed endofunctors, this is also full.

$$\begin{tikzcd}
X \arrow[r, "\xi"] \arrow[d, "f"'] & RX \arrow[r, "\phi_X"] \arrow[d, "Rf"'] & SX \arrow[r, "\sigma_X"] \arrow[d, "Sf"] & X \arrow[d, "f"] \\
Y \arrow[r, "\upsilon"] & RY \arrow[r, "\phi_Y"] & SY \arrow[r, "\sigma_Y"] & Y
\end{tikzcd}$$

Proof If $\sigma_X\,;\xi = \mathsf{id}_X$ then $\rho_X\,;\phi_X\,;\xi = \sigma_X\,;\xi = \mathsf{id}_X$ and homomorphisms $f : X \to Y$ stay the same.

In the well pointed case, ρ_X, ρ_Y, σ_X and σ_Y are all invertible, so $\phi_X = \rho_X\,;\sigma_X^{-1}$ and then f is an R-homomorphism iff it is an R-homomorphism. □

Any well pointed endofunctor defines a full replete subcategory of objects that it fixes and our principal goal is to find the left adjoint (reflection) onto that subcategory, thereby defining a monad. So this process reflects well pointed endofunctors onto the idempotent monads that they *generate.*

5 How do they arise?

I intend to add more constructions from Kelly's paper and elsewhere to this section.

Remark 5.1 Kelly gives some (frankly rather contrived) constructions that rely on finite colimits:

(a) Given a plain endofunctor H, there is a pointed one $T \equiv \mathsf{id} + H$.

(b) Given $\tau : \mathsf{id} \to T$, the coequaliser of $T\tau, \tau T : T \rightrightarrows TT$ is well pointed, but not universal as such.

(c) Combining these gives $S \equiv \mathsf{id} + H(\mathsf{id} + H)$, where τ takes H to the inner id, but this duplicates H. □

However, there is a more natural construction that requires nothing of the underlying category:

Lemma 5.2 Let $T : \mathcal{X} \to \mathcal{X}$ be a plain endofunctor, *i.e.* without any given natural transformation $\mathsf{id} \to T$, and let $\mathcal{Y} \equiv \mathbf{CoAlg}_{\mathcal{X}}T$ be the category of coalgebras for T over $\mathcal{X}$. Then for $Y \equiv (\xi : X \to TX) \in \mathcal{Y}$, the diagram

$$\begin{tikzcd}
Y \arrow[r, "\sigma_Y"] & SY \arrow[r, "\sigma_{SY} \equiv S\sigma_Y"] & SSY
\end{tikzcd}$$

$$\begin{tikzcd}
TX \arrow[r, "T\xi"] & TTX \arrow[r, "TT\xi"] & TTX \\
X \arrow[r, "\xi"] \arrow[u, "\xi"] & TX \arrow[r, "T\xi"] \arrow[u, "T\xi"'] & TX \arrow[u, "TT\xi"']
\end{tikzcd}$$

illustrates the definition of a well pointed endofunctor $\sigma : \mathsf{id} \to S : \mathcal{Y} \to \mathcal{Y}$ such that $T = \mathsf{cod} \cdot S$. Moreover $Y \equiv (\xi : X \to TX)$, considered as an object of $\mathcal{Y}$, is a fixed point of the well pointed endofunctor (S,σ) on $\mathcal{Y}$ iff ξ makes X a fixed point of the plain endofunctor T on $\mathcal{X}$. □

Remark 5.3 The motivation of this construction is that we intend to iterate the plain endofunctor T, starting at the initial object $\emptyset$ and then form a colimit.

The diagram for this colimit needs to have some edges, since otherwise it would just be a big coproduct, with no meaning for the notion of "fixed point". The coalgebra construction takes the data for these general edges from *just the first step* (ξ), instead of using the later values of some natural transformation $\tau : \mathsf{id} \to T$. This means that it is not clear how to adapt this construction when the given functor is "already" pointed or well pointed, *i.e.* when such a τ *has* been given.

Unfortunately, *general* T-algebras in $\mathcal{X}$, *i.e.* those that are not necessarily fixed points, seem not to fit into this picture. At first sight, this should not matter, because if T has an *initial* algebra then that *is* a fixed point, by a famous lemma due to Joachim Lambek.

Lemma 5.4 Suppose that $\mathcal{X}$ has an initial object $\emptyset$. Then
(a) the unique map $\emptyset \to T\emptyset$ provides the initial coalgebra (object of $\mathcal{Y}$); and
(b) $\alpha : TA \to A$ in $\mathcal{X}$ is the initial T-algebra in $\mathcal{X}$ iff α has an inverse $\xi : X \to TA$ that is the initial (S, σ)-algebra in $\mathcal{Y}$. $\square$

On the other hand, the covariant powerset functor $\mathcal{P}$ has plenty of algebras, such as any complete semilattice, but it has no fixed point, by Cantor's theorem. This is an example of Remark 1.1(d).

6 Composition as a tensor

In order to combine two (well) pointed endofunctors, Kelly invoked their pushout,

$$\begin{tikzcd} \mathsf{id} \arrow[r, "\sigma"] \arrow[d, "\rho"'] & S \arrow[d, "\nu_1"] \\ R \arrow[r, "\nu_0"'] & P, \end{tikzcd}$$

and showed that $\mathbf{Alg}\, P = \mathbf{Alg}\, R \cap \mathbf{Alg}\, S$.

We will instead follow the Pataraia's example for the order-theoretic case in Section 2 and just use the *composites*. We will not assume the existence of any finitary (limits or) colimits such as pushouts or coequalisers, although if the pushout P exists then it has mediators $P \to R \cdot S$ and $P \to S \cdot R$.

The endofunctors of any category form another category whose morphisms are the natural transformations between them. But the composition of successive functors also defines a "tensor product" or monoidal structure on this category. In this section we show that pointed and well pointed endofunctors inherit this structure. The "points" provide further structure that is called *co-affine*.

This analogy between composition and tensor product surely pre-dates category theory, but it was treated abstractly by Jean Bénabou [Bén63, Bén67] and Saunders Mac Lane [Lan63].

Definition 6.1 We will use two related notions:
(a) a category is ***directed*** if it has some object T and any two objects R and S have morphisms $R \to T \leftarrow S$ to some third object T; but
(b) it is ***filtered*** if, further, any pair of morphisms $R \rightrightarrows S$ has a further morphism $S \to T$ that has equal composites.

$$T \qquad \begin{tikzcd}[row sep=small] R \arrow[dr, dotted] & \\ & T \\ S \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & S \arrow[r, dotted, "\theta"] & T \end{tikzcd}$$

Both of these are related to finiteness, although directedness is more usually applied to posets than categories. However, our situation is more like type or proof theory, in which we are replacing the *fact* of directedness with the *(two) ways of proving it.*

General pointed endofunctors

Lemma 6.2 Given any two objects (R, ρ) and (S, σ), *i.e.* pointed endofunctors of $\mathcal{X}$, and form the naturality square

$$\begin{tikzcd}
\mathsf{id}_{\mathcal{X}} \arrow[r, "\rho"] \arrow[d, "\sigma"'] \arrow[dr, "\kappa \equiv \rho \cdot \sigma"] & R \arrow[d, "R\sigma"] \\
S \arrow[r, "\rho_S"] & Q \equiv R \cdot S
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
X \arrow[r, "\rho X"] \arrow[d, "\sigma_X"'] \arrow[dr, "\kappa_X"] & RX \arrow[d, "R\sigma_X"] \\
SX \arrow[r, "\rho_{SX}"] & QX \equiv R(SX)
\end{tikzcd}$$

Then the common diagonal

$$\kappa \equiv \rho \cdot \sigma \ \equiv \ \rho \,;R\sigma = \sigma \,; \rho_S \ : \ \mathsf{id}_{\mathcal{X}} \longrightarrow Q \equiv R \cdot S$$

defines another pointed endofunctor and there are morphisms (natural transformations)

$$(R, \rho) \xrightarrow{R\sigma} (Q, \kappa) \xleftarrow{\rho_S} (S, \sigma). \qquad \square$$

Remark 6.3 Since R and S need not commute, there is also $\lambda : \mathsf{id} \to S \cdot R$ defined by

$$\begin{tikzcd}
\mathsf{id}_{\mathcal{X}} \arrow[r, "\rho"] \arrow[d, "\sigma"'] \arrow[dr, "\lambda \equiv \sigma \cdot \rho"] & R \arrow[d, "\sigma R"] \\
S \arrow[r, "S\rho"] & T \equiv S \cdot R
\end{tikzcd}$$

and the two need not be related by morphisms.

Lemma 6.4 This composition is functorial in each argument with respect to morphisms of pointed endofunctors.

$$\begin{tikzcd}
\mathsf{id} \arrow[r, "\rho"] \arrow[drr, "\rho'"'] \arrow[dd, "\sigma"'] & R \arrow[dr, "\phi"] \arrow[dd, "R\sigma"' near end] & \\
& & R' \arrow[dd, "R'\sigma"] \\
S \arrow[r, "\rho_S"] \arrow[drr, "\rho'_S"'] & R \cdot S \arrow[dr, "\phi_S"] & \\
& & R' \cdot S
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{id} \arrow[rr, "\rho"] \arrow[d, "\sigma"'] \arrow[ddr, "\sigma'" near start] & & R \arrow[d, "R\sigma"'] \arrow[ddr, "R\sigma'"] & \\
S \arrow[rr, "\rho_S"] \arrow[dr, "\psi"'] & & R \cdot S \arrow[dr, "R\psi"'] & \\
& S' \arrow[rr, "\rho_{S'}"] & & R \cdot S'
\end{tikzcd}$$

Proof On the left this is naturality of ϕ with respect to σ and on the right it is that of ρ with respect to ψ This respects identity and composition for ϕ and ψ. $\square$

Lemma 6.5 Given $\phi : (R, \rho) \to (R', \rho')$ and $\psi : (S, \sigma) \to (S', \sigma')$, the two forms of composition obey the interchange law and define $\phi \cdot \psi$ as the diagonal of the naturality square

$$\begin{tikzcd}
R \cdot S \arrow[r, "R\psi"] \arrow[d, "\phi S"'] & R \cdot S' \arrow[d, "\phi S'"] \\
R' \cdot S \arrow[r, "R'\psi"] & R' \cdot S'
\end{tikzcd}$$

This is the same in the 2-category of categories, functors and natural transformations. $\square$

Lemma 6.6 Composition of pointed endofunctors is strictly associative, as is composition of their morphisms.

$$\begin{tikzcd}
& R \arrow[rr, "R\sigma"] \arrow[dd, "R\tau"'] \arrow[ddrr, dotted] & & R\cdot S \arrow[dd, "R(S\tau)"] \\
\mathsf{id} \arrow[ur, "\rho"] \arrow[rr, "\sigma"] \arrow[dd, "\tau"'] \arrow[urrr, dashed] \arrow[ddrr, dotted] & & S \arrow[ur, "\rho_S"'] \arrow[dd, "S\tau"] & \\
& R\cdot T \arrow[rr, "R\sigma_T"] & & R\cdot S\cdot T \\
T \arrow[ur, "\rho_T"] \arrow[rr, "\sigma_T"'] \arrow[urrr, dashed] & & S\cdot T \arrow[ur, "\rho_{S\cdot T}"'] &
\end{tikzcd}$$

Proof Composition of functors is strictly associative because it is defined by application to objects and morphisms and then λ-abstraction. So we just need to consider the natural transformations.

The dashed lines show $\kappa \equiv \rho\cdot\sigma : \mathsf{id} \to R\cdot S$ and the dotted ones $\lambda \equiv \sigma\cdot\tau : \mathsf{id} \to S\cdot T$. The *â priori* two ways for calculating that for the triple composite are given by the body diagonal. However, all of the faces of the cube commute by naturality of ρ, σ, τ, κ and λ, and so these two ways are the same.

Composition of morphisms of pointed endofunctors is also associative, because it is inherited from that of natural transformations. $\square$

Lemma 6.7 The object $\mathsf{id} \xrightarrow{\mathsf{id}} \mathsf{id}$ is both the initial object and the strict tensor unit. $\square$

A monoidal category where this happens is called ***co-affine*** and the tensor product makes it ***directed***.

Theorem 6.8 Pointed endofunctors, their morphisms and the respective compositions form a strictly monoidal structure that is co-affine and therefore directed but not symmetric. $\square$

Remark 6.9 In the subcategory generated by the points ρ, σ, τ, ... (rather than any other possible morphisms) of a collection of pointed endofunctors, the morphisms all go from shorter to longer strings of functors.

We can describe them more precisely as *embeddings* of the shorter string in the longer one. They names are strings of functors and points, where the functors are the same as the source (domain) but these are interspersed with the points belonging to the functors in the target (codomain) that are missing from the image of the embedding.

Beware, however, that whilst the points may occur in any order, the functors must be in the same order as in the domain.

For example, the morphims from SRT to $SRRSTSRT$ include $SR\rho\sigma T\sigma\rho\tau$, $SR\rho\sigma\tau\sigma\rho T$, $S\rho R\sigma\tau\sigma\rho T$, $S\rho\rho\sigma\tau\sigma RT$, $\sigma\rho\rho S\tau\sigma RT$ and $\sigma\rho\rho\sigma\tau SRT$. $\square$

The well pointed case

Kelly and others seem to have missed the following lemma. It not only shows that the monoidal structure restricts to well pointed endofunctors but that the result is drammatically simpler.

The equation survives the insertion of another pointed endofunctor:

Lemma 6.10 Let $\rho : \mathsf{id} \to R$ and $\sigma : \mathsf{id} \to S$ be pointed endofunctors, where S is well pointed. Then $S(\rho\cdot\sigma) = (\sigma\cdot\rho)S : S \to S\cdot R\cdot S$.

$$\begin{tikzcd}
& S \arrow[dl, "S\rho"'] \arrow[d, "S\sigma\,|\,\sigma S"] \arrow[dr, "\rho S"] & \\
S\cdot R \arrow[dr, "SR\sigma"'] & S\cdot S \arrow[d, "S\rho S"] & R\cdot S \arrow[dl, "\sigma RS"] \\
& S\cdot R\cdot S &
\end{tikzcd}$$

Proof The left triangle is naturality of $S\rho$ with respect to σ and defines $S(\rho\cdot\sigma)$, whilst the right triangle is naturality of σ with respect to ρS and defines $(\sigma\cdot\rho)S$. □

Lemma 6.11 Composition preserves well-pointedness.

Proof Let $\sigma : \mathsf{id} \to S$ and $\rho : \mathsf{id} \to R$ be well pointed endofunctors. Then Lemma 6.2 defines

$$\kappa \equiv \rho\cdot\sigma : \mathsf{id} \to R\cdot S \quad\text{and}\quad \lambda \equiv \sigma\cdot\rho : \mathsf{id} \to S\cdot R,$$

and Lemma 6.10 showed that they satisfy

$$S\kappa = \lambda S \qquad\text{and}\qquad R\lambda = \kappa R.$$

Therefore $\kappa : \mathsf{id} \to R\cdot S$ and $\lambda : \mathsf{id} \to S\cdot R$ are well pointed because

$$RS\kappa = R\lambda S = \kappa RS \qquad\text{and}\qquad SR\lambda = S\kappa R = \lambda SR.$$ □

Lemma 6.12 The subcategory generated by the points of any set of well pointed endofunctors is a preorder.

Proof Remark 6.9 described the pattern of the morphisms between general pointed endofunctors.

In the well pointed case, Lemma 6.10 showed that any occurrence of a point (such as σ) may be exchanged in this pattern with any occurrence of its functor (S). Thus

$$\alpha\cdot S\cdot\beta\cdot\sigma\cdot\gamma = \alpha\cdot\sigma\cdot\beta\cdot S\cdot\gamma,$$

where α, β and γ are any strings of the functors and their points. For example, the successive members of the list in Remark 6.9 are related by swaps of this kind.

Since pair-swaps generate the full permutation group, all of these embeddings result in the same morphism. □

Other morphisms between well pointed endofunctors may also be brought into this simplification:

Lemma 6.13 Let $\phi : (S,\sigma) \to (R,\rho)$ be a morphism of pointed endofunctors where (S,σ) is well pointed. Then

$$\rho S = \phi\,;R\sigma \quad\text{and}\quad S\rho = \phi\,;\sigma R.$$

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\sigma S\,|\,S\sigma"'] \arrow[dr, "\rho S"] & R \arrow[d, "R\sigma"] \\
SS \arrow[r, "\phi S"] & RS
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\sigma S\,|\,S\sigma"'] \arrow[dr, "S\rho"] & R \arrow[d, "\sigma R"] \\
SS \arrow[r, "S\phi"] & SR
\end{tikzcd}$$

Proof Naturality of ϕ with respect to σ and *vice versa*. □

Corollary 6.14 If $\phi, \psi : S \rightrightarrows R$ are parallel morphisms of well pointed endofunctors then there is another morphism $\theta : R \to T$ of them such that

$$\phi ; \theta \;=\; \psi ; \theta,$$

namely either $R\sigma : R \to RS$ or $\sigma R : R \to SR$. $\square$

It could be that any such ϕ and ψ are already equal, as they are for idempotent monads (Lemma 3.7), but I can't prove that at the moment.

Theorem 6.15 Well pointed endofunctors and their points generate a subcategory of that in Theorem 6.8 that is closed under the monoidal structure and is in fact a directed preorder. Moreover, the full subcategory that they generate (including the other morphisms) is filtered. $\square$

7 Subcategories of algebras

Let's return to Remark 3.9 about intersections of reflective subcategories, making use of the monoidal category of (well) pointed endofunctors. The following repeats Lemma 4.3.

Lemma 7.1 Any algebra $q : R(SA) \to A$ gives rise to algebras $r : RA \to A$ and $s : SA \to A$ and *vice versa* by

$$r \;\equiv\; \sigma RA ; q, \quad s \;\equiv\; S\rho A ; q \quad \text{and} \quad q \;\equiv\; Sr ; s.$$

These formulae recover r and s from q but not necessarily conversely.

For pointed endofunctors we therefore have a retraction

$$\mathbf{Alg}\, R \;\cap\; \mathbf{Alg}\, S \;\triangleleft\; \mathbf{Alg}\,(SR),$$

but this becomes an equality in the *well* pointed case:

Lemma 7.2 If (R, ρ) and (S, σ) are well pointed endofunctors then the previous result becomes a bijection

$$\mathbf{Alg}\,\big((R,\rho)\cdot(S,\sigma)\big) \;=\; \mathbf{Alg}\,\big((S,\sigma)\cdot(R,\rho)\big) \;=\; \mathbf{Alg}\,(R,\rho) \cap \mathbf{Alg}\,(S,\sigma).$$

Proof By Lemma 4.2, when the functors are *well* pointed, an object carries a (unique) algebra structure iff the relevant component of the natural transformation is invertible. This ensures that $q \mapsto (r, s) \mapsto q$ in Lemma 7.1 is the identity. $\square$

Corollary 7.3 $\mathbf{Alg}\,(S^n) = \mathbf{Alg}\,(S)$.

Proof By S^n we mean the n-fold iterate of the functor, together with the point $\mathsf{id} \to S^n$ given by the relevant composites of S and σ. The induction step uses the Lemma with $R \equiv S$. $\square$

Proposition 7.4 The operation **Alg** that forms the full subcategory of fixed points of a well pointed endofunctor takes

(a) the initial object (identity functor and tensor unit) to the whole category;

(b) morphisms to reverse inclusions (Lemma 4.3);

(c) composition (tensor product) to intersection (Lemma 7.2) and

(d) colimits to intersections too (Lemma 9.6). $\square$

This construction suggests that there may be a (contravariant) adjunction or Galois connection between well pointed endofunctors and their full subcategories of fixed point.

8 Galois connection with algebras

We now prove the finitary part of the categorical analogue of Theorem 2.2, starting with the Galois connection in 2.2(a).

Notation 8.1 For any well-pointed endofunctor (S, σ) on $\mathcal{X}$ and object $X \in \mathcal{X}$, write

$$(S, \sigma) \perp X \quad \equiv \quad X \text{ fixed by } (S, \sigma) \quad \equiv \quad \sigma X : X \cong SX,$$

where the inverse provides the unique (S, σ) algebra structure on X, by Lemma 4.2. More generally, we may consider a *family* $\mathcal{S}$ of these functors and write: $\mathcal{S}^{\perp} \equiv \mathbf{Alg}\,\mathcal{S} \subset \mathcal{X}$ for the full subcategory

$$\bigcap \{\mathbf{Alg}\,(S, \sigma) \mid (S, \sigma) \in \mathcal{S}\} \;=\; \{X \in \mathcal{X} \mid \forall (S, \sigma) \in \mathcal{S}.\, \sigma X \text{ invertible}\}.$$

Hence we may use this to form the *intersection* of a family of reflective subcategories: First forget the μ-parts of the corresponding idempotent monads and just regard them as well pointed endofunctors. Then the monad and reflector for the intersection are the ones that are obtained below.

Now, *any* binary relation between two sets gives rise to a Galois connection or contravariant adjunction between their lattices of subsets. Thus, for any collection or full subcategory $\mathcal{S}$ of well pointed endofunctors or $\mathcal{A} \subset \mathcal{X}$ of objects, we define
(a) $\mathcal{S}^{\perp}$ is just another notation for $\mathbf{Alg}\,\mathcal{S}$ and more generally $\mathcal{S} \perp \mathcal{A} \iff \mathcal{A} \subset \mathbf{Alg}\,\mathcal{S}$.
(b) ${}^{\perp}\mathcal{A} \equiv \{X \in \mathcal{X} \mid \forall (S, \sigma) \in \mathcal{S}.\, \sigma X \text{ invertible}\}$, so $(S, \sigma) \in {}^{\perp}\mathcal{A} \iff \mathcal{A} \subset \mathbf{Alg}\,(S, \sigma)$.

The new form of 2.2(e) differs only by replacing the inequality $S \leq M$ with a morphism $\phi(S, \sigma) \to (M, \eta)$.

Lemma 8.2 If ${}^{\perp}\mathcal{A}$ has a terminal object (M, η) then
(a) $\mathcal{A} \subset \mathbf{Alg}\,(M, \eta)$ and
(b) if $(S, \sigma) \in {}^{\perp}\mathcal{A}$ then $\mathbf{Alg}\,(M, \eta) \subset \mathbf{Alg}\,(S, \sigma)$.
In particular,
(c) if ${}^{\perp}(\mathbf{Alg}\,\mathcal{S}) \equiv {}^{\perp}(\mathcal{S}^{\perp})$ has a terminal object (M, η) then $\mathbf{Alg}\,(M, \eta) = \mathbf{Alg}\,\mathcal{S}$.

Proof $\mathcal{A} \subset \mathbf{Alg}\,(M, \eta)$ because $(M, \eta) \in {}^{\perp}\mathcal{A}$ by Notation 8.1.

On the other hand, if $(S, \sigma) \in {}^{\perp}\mathcal{A}$ then, since (M, η) is terminal there, there is a morphism $(S, \sigma) \to (M, \eta)$ and so $\mathbf{Alg}\,(M, \eta) \subset \mathbf{Alg}\,(S, \sigma)$.

Applying these with $\mathcal{A} \equiv \mathbf{Alg}\,\mathcal{S}$ and each $(S, \sigma) \in \mathcal{S}$ gives

$$\mathbf{Alg}\,\mathcal{S} \overset{(a)}{\rightarrowtail} \mathbf{Alg}\,(M, \eta) \overset{(b)}{\rightarrowtail} \bigcap \{\mathbf{Alg}\,(S, \sigma) \mid (S, \sigma) \in \mathcal{S}\} \equiv \mathbf{Alg}\,\mathcal{S}. \qquad \square$$

The next step explains why we have called the terminal object (M, η). It is the analogue of 2.2(b) and 2.2(d), although we needed the previous two sections to set up the category theory for this.

Lemma 8.3 If $(S, \sigma) \perp \mathcal{A}$ and $(R, \rho) \perp \mathcal{A}$ then $(S \cdot R, \sigma \cdot \rho) \perp \mathcal{A}$, so ${}^{\perp}\mathcal{A}$ is directed. In fact it's filtered by Theorem 6.15.

Proof Lemma 7.1 says this for a single object $\mathcal{A} \equiv \{X\}$ and it easily extends to sets of them. $\square$

Lemma 8.4 If ${}^{\perp}\mathcal{A}$ has a terminal object (M, η) then there is a unique morphism $\mu : M \cdot M \to M$ such that (M, η, μ) is an idempotent monad.

Proof By the previous lemma, $(M \cdot M, \eta \cdot \eta) \in {}^{\perp}\mathcal{A}$ too, with

$$M\eta, \eta M : (M, \eta) \longrightarrow (M \cdot M, \eta \cdot \eta), \qquad \text{and so} \qquad \eta\,;M\eta \;=\; \eta \cdot \eta \;=\; \eta\,;\eta M.$$

By hypothesis, (M, η) is terminal in ${}^{\perp}\mathcal{A}$, so there is a morphism

$$\mu : (M \cdot M, \eta \cdot \eta) \longrightarrow (M, \eta).$$

Using uniqueness of maps into a terminal object,

$$M\eta\,;\mu \;=\; \mathsf{id} \;=\; \eta M\,;\mu : (M,\eta) \longrightarrow (M,\eta)$$

and
$$M\mu\,;\mu \;=\; \mu M\,;\mu : (M\cdot M\cdot M, \eta\cdot\eta\cdot\eta) \longrightarrow (M,\eta),$$

whence (M,η,μ) is a monad. In fact it's idempotent since we were working with well pointed endofunctors anyway (Lemma 3.7). □

For the converse, analogous to Proposition 2.4, we need to know more about morphisms of well pointed endofunctors $(S,\sigma) \to (M,\eta)$ when (M,η) is an idempotent monad. Quite an elaborate theory can be developed about these, but the following is more than enough:

Lemma 8.5 The following are equivalent:
(a) there is a morphism $\phi : (S,\sigma) \to (M,\eta)$;
(b) $\sigma M : M \to SM$ is invertible;
(c) $M\sigma : M \to MS$ is invertible;
(d) for all $X \in \mathcal{X}$, MX is an S-algebra;
(e) $\mathbf{Alg}\,\mathsf{M} \subset \mathbf{Alg}\,(S,\sigma)$.
Moreover ϕ in part (a) is unique.

Proof
[a⇒b] Any morphism $\phi : (S,\sigma) \to (M,\eta)$ must satisfy $\sigma\,;\phi = \eta$ so

$$\sigma M\,;\phi M\,;\mu \;=\; \eta M\,;\mu \;=\; \mathsf{id}\,.$$

But since (S,σ) is well pointed we have $\phi\,;\sigma M = S\eta$ by Lemma 6.13, so

$$\phi M\,;\mu\,;\sigma M \;=\; \phi M\,;\sigma MM\,;S\mu \;=\; S\eta M\,;S\mu \;=\; \mathsf{id}$$

by naturality of σ with respect to μ and the monad unit laws. Hence σM has inverse $\phi M\,;\mu$.
[b⇒a] If σM is invertible, put $\phi \equiv S\eta\,;(\sigma M)^{-1}$, so

$$\sigma\,;\phi \;\equiv\; \sigma\,;S\eta\,;(\sigma M)^{-1} \;=\; \eta\,;\sigma M\,;(\sigma M)^{-1} \;=\; \eta$$

by naturality of σ with respect to η, so $\phi : S \to M$ is a morphism.
[a⇔c] By similar arguments, with $(M\sigma)^{-1} \equiv M\phi\,;\mu$ and $\phi \equiv \eta S\,;(M\sigma)^{-1}$.
[b⇔d] By Lemma 4.2, σM is invertible iff for all $X \in \mathcal{X}$, MX is an S-algebra.
[d⇔e] Since all algebras for an idempotent monad are of the form MX, this suffices to give $\mathbf{Alg}\,M \subset \mathbf{Alg}\,(S,\sigma)$.
[unq] The translations between ϕ and $(\sigma M)^{-1}$ in (a⇔b) define a bijection because

$$\phi' \;\equiv\; S\eta\,;(\sigma M)^{-1} \;=\; S\eta\,;\phi M\,;\mu \;=\; \phi\,;M\eta\,;\mu \;=\; \phi$$

using naturality of ϕ with respect to η, whilst inverses are unique. □

Lemma 8.6 If $\mathcal{A} = \mathbf{Alg}\,\mathsf{M}$ then (M,η) is the terminal object of ${}^{\perp}\mathcal{A}$.
Proof If $\mathcal{A} \subset \mathbf{Alg}\,\mathsf{M}$ then $(M,\eta) \in {}^{\perp}\mathcal{A}$ by Notation 8.1.
If $\mathbf{Alg}\,\mathsf{M} \subset \mathbf{Alg}\,(S,\sigma)$ then there is a unique morphism $(S,\sigma) \to (M,\eta)$ by Lemma 8.5.
Hence if $\mathcal{A} = \mathbf{Alg}\,\mathsf{M}$ then (M,η) belongs to ${}^{\perp}\mathcal{A}$ and for any $(S,\sigma) \in {}^{\perp}\mathcal{A}$ there is a unique morphism $(S,\sigma) \to (M,\eta)$. □

Theorem 8.7 The following are equivalent:
(a) ${}^{\perp}(\mathbf{Alg}\,\mathcal{S})$ has a terminal object (M,η);
(b) there is an idempotent monad M with $\mathbf{Alg}\,\mathsf{M} = \mathbf{Alg}\,\mathcal{S}$;
(c) the inclusion $\mathbf{Alg}\,\mathcal{S} \rightarrowtail \mathcal{X}$ has a left adjoint.

Proof [a⇒b] By Lemmas 8.2 and 8.4, [b⇒a] by Lemma 8.6 and [b⇔c] is well known. □

Remark 8.8 For $\mathcal{A} = \mathbf{Alg}\,\mathsf{M}$, the point ρ of any $(R,\rho) \in {}^{\perp}\mathcal{A}$ behaves like a universal property, in that any $f : X \to A \in \mathcal{A}$ factorises $f = \rho X \,;\, g$:

$$\begin{tikzcd}
RX \arrow[r, "\phi X"] \arrow[drr, "g"] & MX \arrow[r, "Mf"] & MA \\
X \arrow[u, "\rho X"] \arrow[rr, "f"] & & A \arrow[u, "\cong", "\eta A"']
\end{tikzcd}$$

Remark 8.9 Suppose that $\mathcal{S}$ is a single well pointed endofunctor $S : \mathcal{X} \to \mathcal{X}$ that is the lifting of a plain endofunctor $T : \mathcal{Y} \to \mathcal{Y}$ to its category of coalgebras, as in Lemma 5.2. Then any morphism $g : RU \to A$ in $\mathcal{X}$ is of course a T-coalgebra homomorphism. But its target is a fixed point of both S and T, so it has an invertible T-*co*algebra structure map, which we may instead regard as a T-*algebra.*

$$\begin{tikzcd}
TRU \arrow[r, "Tg"] & TA \arrow[d, "\cong"', "a"] \\
RU \arrow[u, "\xi"] \arrow[r, "g"] & A
\end{tikzcd}$$

Then $g : RU \to A$ in $\mathcal{X}$ is a ***T-coalgebra-to-algebra homomorphism*** in the prior category.

9 Limits and colimits

Kelly and other authors who have considered how to find free algebras for non-finitary constructors invoked finitary (pushouts and coequalisers) and infinitary (transfinite) colimits *from the outset* of their development. We have shown, in contrast to this, that there is much to be learned about the finitary abstract algebra of iterating endofunctors *without assuming anything at all* about the underlying category $\mathcal{X}$.

However, we have now come to the end of what we can do finitarily. We know that, at least in the familiar situations, free algebras are term algebras that are the colimits of iterative partial constructions.

In the previous section we identified a *specific* colimit diagram in a certain category that we had *constructed* that is needed to construct the free monad on a well pointed endofunctor. We now show that it is sufficient to have colimits over this diagram in the *base* category $\mathcal{X}$. We need to show how they are inherited through the constructions that we have made.

First we recall some textbook facts:

Lemma 9.1 A category has a terminal object iff it has colimits for any diagram using itself as the shape. $\square$

Now we recall the various steps that we have made in building one category from another, in order to show that directed colimits are inherited through the whole procedure.

Lemma 9.2 The forgetful functor $\mathbf{CoAlg}\,T \to \mathcal{X}$ from the category of coalgebras for a plain endofunctor T to the underlying category (Lemma 5.2) creates colimits and also whatever limits the functor preserves. $\square$

Lemma 9.3 Any functor category $[\mathcal{Y} \to \mathcal{X}]$ inherits limits and colimits pointwise from $\mathcal{X}$. $\square$

The next step is less obvious but still well known. Pointed endofunctors form the co-slice category $\mathsf{id} \downarrow [\mathcal{X} \to \mathcal{X}]$. Co-affine monoidal diagrams are directed and therefore connected.

Lemma 9.4 The forgetful functor $\mathsf{cod} : U \downarrow \mathcal{Z} \to \mathcal{Z}$ from any co-slice category creates limits and connected colimits (*i.e.* coequalisers, pushouts, directed and filtered colimits).

Proof Let $Z_{(-)} : \mathcal{I} \to U \downarrow \mathcal{Z}$ be any diagram.

(a) Let $L \in \mathcal{Z}$ be the limit of the diagram $\mathsf{cod} \cdot Z_{(-)}$ in $\mathcal{Z}$. The diagram $Z_{(-)}$ in $U \downarrow \mathcal{Z}$ is a cone over this in $\mathcal{Z}$, so there is a mediator $U \to L$, which is the limit of the given diagram.

(b) Let $\nu_I : Z_I \to C$ be the colimiting cocone of the diagram $\mathsf{cod} \cdot Z_{(-)}$ in $\mathcal{Z}$. For any edge $e : I \to J$ in $\mathcal{I}$, the diagram

$$\begin{tikzcd}
& Z_I \arrow[dd, "Z_e"] \arrow[dr, "\nu_I"] & \\
U \arrow[ur, "z_I"] \arrow[dr, "z_J"'] & & C \\
& Z_J \arrow[ur, "\nu_J"'] &
\end{tikzcd}$$

commutes. Therefore, so long as $\mathcal{I}$ is connected (and in particular inhabited), the composite $U \to C$ is the same for all vertices. This is the colimit in $U \downarrow \mathcal{Z}$.

(c) The initial object of $U \downarrow \mathcal{Z}$ is id_U, but U itself need not be initial in $\mathcal{Z}$. Let $U \equiv \mathbf{1}$ in $\mathcal{Z} \equiv \mathbf{Set}_f$; then the coproduct of $\mathbf{1} \to \mathbf{2}$ with itself in $U \downarrow \mathcal{Z}$ has three elements, but in $\mathcal{Z}$ it has four. $\square$

Next is the less familiar question of what colimits the subcategory of *well* pointed endofunctors inherits from the larger category of *general* pointed endofunctors. We know that $\mathsf{id} \xrightarrow{\mathsf{id}} \mathsf{id}$ is the common initial object, whilst composition (Section 6) preserves well-pointedness, but that is not a pushout. The general result comes from [Kel80, Prop. 9.1].

Lemma 9.5 Let $(\mathsf{id} \xrightarrow{\sigma_i} S_i)$ be a family of well pointed endofunctors and $(\nu_i : S_i \to T)$ their colimiting cocone as pointed endofunctors, so $\tau = \sigma_i \mathbin{;} \nu_i$. Then $(\mathsf{id} \xrightarrow{\tau} T)$ is well pointed too. The family may be a coproduct or general colimit, because the maps in the diagram play no role in the proof.

$$\begin{tikzcd}
S_i \arrow[rr, "\nu_i"] \arrow[dr, "S_i\tau"] \arrow[dd, "S_i\sigma_i"'] & & T \arrow[dd, "T\tau"] \\
& S_iT \arrow[dr, "\nu_iT"] & \\
S_iS_i \arrow[ur, "S_i\nu_i"] \arrow[dr, "\nu_iS_i"] & & TT \\
& TS_i \arrow[ur, "T\nu_i"] & \\
S_i \arrow[uu, "\sigma_iS_i"] \arrow[ur, "\tau S_i"] \arrow[rr, "\nu_i"'] \arrow[uuuu, bend left=60, "\mathsf{id}"] & & T \arrow[uu, "\tau T"']
\end{tikzcd}$$

Proof The small triangles come from $\tau = \sigma_i \mathbin{;} \nu_i$ and the quadrilaterals from naturality of ν_i and τ with respect to each other. The whole figure says that $\nu_i \mathbin{;} T\tau = \nu_i \mathbin{;} \tau T$, but ν_i is a colimiting cocone, so $T\tau = \tau T$ by uniqueness of the mediator. $\square$

Lemma 9.6 $\mathbf{Alg}\,(T) \;=\; \bigcap_I \mathbf{Alg}\,(S_I)$.

Proof Given $t : TX \to X$ with $\tau_X \mathbin{;} t = \mathsf{id}$, we obtain $s_i \equiv \nu_i \mathbin{;} t : S_iX \to TX \to X$ with

$$\sigma_i \mathbin{;} s_i \;=\; \sigma_i \mathbin{;} s_i \mathbin{;} \nu_i \mathbin{;} t \;= \tau \mathbin{;} t \;=\; \mathsf{id}_X.$$

Conversely, any family $(s_i : S_iX \to X)$ with $\sigma_i \mathbin{;} s_i = \mathsf{id}_X$ is a cocone under the colimit diagram and therefore has a unique mediator $t : TX \to X$ with $\nu_i \mathbin{;} t = s_i$ and $\tau \mathbin{;} t = \mathsf{id}_X$. $\square$

Corollary 9.7 In the Galois connection of the previous section, $\mathcal{S}^\perp$ is closed under under intersections and ${}^\perp\mathcal{A}$ under whatever colimits exist in $\mathcal{X}$. $\square$

Corollary 9.8 $\mathbf{Alg}\,(S^\alpha) = \mathbf{Alg}\,(S)$.

Proof This extends Corollary 7.3 with whatever colimits exist for diagrams consisting of iterates of S. $\square$

By way of showing that we have now assembled all of the necessary parts of our construction, we make one more outrageous assumption:

Proposition 9.9 The object (M, η) in Theorem 8.7 exists iff the underlying category $\mathcal{X}$ has all (filtered) colimits of shape ${}^\perp\mathcal{S}^\perp$.

Proof If the monad exists then it is terminal in ${}^\perp\mathcal{S}^\perp$ and all such colimits exist trivially, because they are computed simply by evaluation at the terminal object.

Conversely, the category ${}^\perp\mathcal{S}^\perp$ is filtered by Theorem 6.15. If $\mathcal{X}$ has these colimits then so does ${}^\perp\mathcal{S}^\perp$ by the results of this section, whence it has a terminal object and that is the monad. $\square$

The assumption is outrageous because the diagram ${}^\perp\mathcal{S}^\perp$ is much "bigger" than the category $\mathcal{X}$ in which we require the colimits. Classically, a small category can only be complete if it is a lattice. There may be ways to achieve that or get around it, but in the final section we consider more modest assumptions. What we have done here is to identify the *specific* diagram shape for which we require the colimit.

10 Rank

I have in mind a concluding section surveying the ways in which the "outrageous colimit" may be handled, including ideas about the way that ordinals and their arithmetic *emerge* from this situation (*cf.* the so-called Bourbaki–Witt Theorem in Kuratowski's paper), instead of being *imposed* from the outset.

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