Equideductive Logic
and CCCs with Subspaces

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www.PaulTaylor.EU/ASD

Abstract Stone Duality

- Lattice part: $\top$, $\bot$, $\wedge$, $\vee$ for open sets, $=$ for discrete spaces, $\neq$ for Hausdorff, $\forall$ for compact and $\exists$ for overt ones.
- Categorical part: λ-calculus for $\Sigma^{(-)}$, and the adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$ is monadic: gives definition by description, Dedekind completeness and Heine–Borel.

The categorical part only handles locally compact spaces.

It needs to be generalised.

We will get a CCC, but that's not important, because the exponential Y^X is tested by incoming maps, but its topology by outgoing ones.

We certainly need products, $\Sigma^{(-)}$ and equalisers.

CCCs with all finite limits

$$
\begin{tikzcd}
 & E \arrow[dr, tail, "i"] & & \\
\Gamma \arrow[ur, dotted, "b"] \arrow[rr, "a"] & & X \arrow[r, shift left, "\tilde{\alpha}"] \arrow[r, shift right, "\tilde{\beta}"'] & \Sigma^Y \\
 & E \times Y \arrow[uu] \arrow[dr, "i \times \mathrm{id}_Y"] \arrow[ddr] & & \\
\Gamma \times Y \arrow[uu] \arrow[ur, dotted] \arrow[rr, "a \times \mathrm{id}_Y"] \arrow[drr, "\alpha(a, y) = \beta(a, y)"'] & & X \times Y \arrow[uu] \arrow[r, shift left, "\tilde{\alpha} \times Y"] \arrow[r, shift right, "\tilde{\beta} \times Y"'] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & \Sigma^Y \times Y \arrow[uu] \arrow[dl, "\mathrm{ev}"] \\
 & & \Sigma &
\end{tikzcd}
$$

Want to write $E = \{x \mid \forall y.\ \alpha x y = \beta x y\}$.

Equideductive logic

$$\vdash \quad \top \qquad x : \mathbf{0} \quad \vdash \quad \mathfrak{p}$$

$$\mathfrak{p},\ \mathfrak{q}\ \vdash\ \mathfrak{p}\&\mathfrak{q} \qquad \mathfrak{p}\&\mathfrak{q}\ \vdash\ \mathfrak{p} \qquad \mathfrak{p}\&\mathfrak{q}\ \vdash\ \mathfrak{q}$$

$$\frac{\Gamma,\ x : A,\ \mathfrak{p}(x)\ \vdash\ \alpha x\ =\ \beta x}{\Gamma\ \vdash\ \forall x : A.\ \mathfrak{p}(x)\ \Rrightarrow\ \alpha x\ =\ \beta x}\ \forall I$$

$$\frac{\Gamma\ \vdash\ a : A,\ \mathfrak{p}(a) \qquad \Gamma\ \vdash\ \forall x : A.\ \mathfrak{p}(x)\ \Rrightarrow\ \alpha x\ =\ \beta x}{\Gamma\ \vdash\ \alpha a\ =\ \beta a}\ \forall E$$

All the variables on the left of $\Rrightarrow$ must be bound by $\forall$.
Maybe add some dependent types later.
Must have subsitution (cut) for free variable x.

Interpretation of equideductive logic

- The obvious set-theoretic one — the construction to follow will give Dana Scott's equilogical spaces.
- In locales — but I'm not sure whether this works (Does $(-) \times X$ preserve epis? I have both a proof and a counterexample!)
- In Formal Topology, if this works.
- Proof-theoretic, taking the rules just as they are (as we shall do for most of this lecture).
- In another type theory such as Coquand's Calculus of Constructions or Coq.
- With additional axioms of our choosing.

Interaction with the lattice structure

The implication $\Rightarrow$ in equideductive logic depends on the categorical structure (equalisers and $\Sigma^{(-)}$).

If Σ also has lattice structure, with induced order $\Rightarrow$, then these interact very nicely.

That is, if we assume the Phoa principle.
In the Gentzen style, this is

$$\frac{x : X,\ \alpha x = \top \ \vdash\ \beta x = \top}{x : X \ \vdash\ \alpha x \ \Rightarrow\ \beta x} \qquad \text{and} \qquad \frac{x : X,\ \beta x = \bot \ \vdash\ \alpha x = \bot}{x : X \ \vdash\ \alpha x \ \Rightarrow\ \beta x}$$

which we rewrite as

$$(\forall x. \alpha x = \top \Rightarrow \beta x = \top) \quad \Longleftrightarrow \quad (\forall x. \alpha x \Rightarrow \beta x)$$

$$\Longleftrightarrow \quad (\forall x. \beta x = \bot \Rightarrow \alpha x = \bot)$$

This is also the definition of $\alpha \leqslant \beta$.

Interaction with topological structure

Similarly, equality $=_N$ in a discrete space N is a special case of general equality of terms:

$$n = m \iff (n =_N m) = \top, \quad \text{whilst} \quad h = k \iff (h \neq_H k) = \bot$$

in a Hausdorff space H.

The universal quantifier $\Cup$ in a compact space is related to $\forall$:

$$(\forall x.\, \phi x = \top) \iff (\Cup x.\, \phi x) = \top$$

Similarly

$$(\forall x.\, \phi x = \bot) \iff (\exists x.\, \phi x) = \bot$$

in an overt space.

See *Foundations for Computable Topology*, §12, for more discussion: `www.PaulTaylor.EU/ASD/foufct`

Equideductive spaces

Urtypes: generated from $\mathbf{0}$, $\mathbf{1}$ and $\mathbb{N}$ by $+$, $\times$ and $((-) \to \Sigma)$.
Combinators, including

$$\mathbb{I} : (A \to \Sigma) \to A \to \Sigma, \qquad \mathbb{K} : (A \to \Sigma) \to B \to A \to \Sigma,$$

$$\mathbb{C} : \big((B \to \Sigma) \to (C \to \Sigma)\big) \to \big((A \to \Sigma) \to (B \to \Sigma)\big) \to (A \to \Sigma) \to C \to \Sigma$$

$$\mathbb{T} : \mathbf{1}, \qquad \nu_0 : A \to (A + B), \qquad \nu_1 : B \to (A + B),$$

$$\pi_0 : \big((A + B) \to \Sigma\big) \to A \to \Sigma, \qquad \pi_1 : \big((A + B) \to \Sigma\big) \to B \to \Sigma,$$

$$\langle\rangle : \big((C \to \Sigma) \to A \to \Sigma\big) \to \big((C \to \Sigma) \to B \to \Sigma\big) \to (C \to \Sigma) \to (A{+}B) \to \Sigma.$$

$$\mathbb{A} : \big(((A \to \Sigma) + A) \to \Sigma\big) \to \mathbf{1} \to \Sigma,$$

$$\mathbb{L} : \big(((A + B) \to \Sigma) \to \mathbf{1} \to \Sigma\big) \to (A \to \Sigma) \to (B \to \Sigma) \to \Sigma.$$

with appropriate equational axioms, such as
$\forall M N \phi c.\ \mathbb{C} N M \phi c \ = \ N(M\phi)c$, without $\Rightarrow$.

Equideductive spaces

An equideductive space X is $(A, \mathfrak{p}, \mathfrak{q})$ where A is an urtype, $\mathfrak{p}$ is an urstatement on Σ^A and $\mathfrak{q}$ one on A, for which

$$\phi, \psi : \Sigma^A, \quad \mathfrak{p}(\phi), \quad \forall a : A.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a \quad \vdash \quad \mathfrak{p}(\psi).$$

This rule is important in the construction.
Later, we tighten it to ensure that all spaces are definable using exponentials and equalisers.

LHS is a partial equivalence relation.

A morphism $M : X \equiv (A, \mathfrak{p}, \mathfrak{q}) \to Y \equiv (B, \mathfrak{r}, \mathfrak{s})$ is an realiser $M : (A \to \Sigma) \to B \to \Sigma$ such that

$$\phi : \Sigma^A, \quad \mathfrak{p}(\phi) \quad \vdash \quad \mathfrak{r}(M\phi)$$

$$\phi, \psi : \Sigma^A, \quad \mathfrak{p}(\phi), \quad \forall a.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a \quad \vdash \quad \forall b.\ \mathfrak{s}(b) \Rightarrow M\phi b = M\psi b,$$

where $M_1 = M_2$ if

$$\phi : \Sigma^A, \quad \mathfrak{p}(\phi) \quad \vdash \quad \forall b : B.\ \mathfrak{s}(b) \Rightarrow M_1\phi b = M_2\phi b.$$

Categorical structure

$\mathbf{1} \equiv (\mathbf{0}, \top, \top)$, $\Sigma \equiv (\mathbf{1}, \top, \top)$.

The product is $(A, \mathfrak{p}, \mathfrak{q}) \times (B, \mathfrak{r}, \mathfrak{s}) \equiv \big(A + B, (\mathfrak{p} \cdot \pi_0 \& \mathfrak{r} \cdot \pi_1), [\mathfrak{q}, \mathfrak{s}]\big)$.

The equaliser is

$$
\begin{tikzcd}
E \equiv (A, \mathfrak{t}, \mathfrak{q}) \arrow[r, tail, "I"] & (A, \mathfrak{p}, \mathfrak{q}) \arrow[r, shift left, "M"] \arrow[r, shift right, "N"'] & (B, \mathfrak{r}, \mathfrak{s})
\end{tikzcd}
$$

$$\mathfrak{t}(\phi) \quad \equiv \quad \mathfrak{p}(\phi) \quad \& \quad \forall b : B.\ \mathfrak{s}(b) \Rightarrow M\phi b = N\phi b,$$

The exponential of $X \equiv (A, \mathfrak{p}, \mathfrak{q})$ is $\Sigma^X \equiv (\Sigma^A, \mathfrak{q}^{\mathfrak{p}}, \mathfrak{p})$, where

$$\mathfrak{q}^{\mathfrak{p}}(F) \equiv \forall \phi, \psi : \Sigma^A.\ \mathfrak{p}(\phi) \,\&\, (\forall a : A.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a) \Rightarrow F\phi = F\psi.$$

(The modulation $\mathfrak{p}(\phi) \& \cdots$ is the source of many difficulties.)

All objects are definable

If $\mathfrak{q}$ is defined using $\top$, equations, & and $\forall \Rightarrow$ then
$\mathfrak{q}(a) \dashv\vdash \mathfrak{q}^{\top}(\lambda\phi.\,\phi a)$.
$(A, \mathfrak{p}, \top) \cong (\Sigma^{\Sigma^A}, \mathfrak{p}^{\top} \,\&\, \mathsf{prime}, \top)$
$(A, \top, \mathfrak{q}) \cong (\Sigma^{\Sigma^A}, \top, \mathfrak{q}^{\top} \,\&\, \mathsf{prime}) \cong \Sigma^{(\Sigma^A, \mathfrak{q}^{\top} \& \mathsf{prime}, \top)}$.

$$\begin{tikzcd}
(\Sigma^A, \mathsf{prime}, \top) \arrow[r, two heads] & (\Sigma^A, \top, \top) \arrow[r, shift left, "F \mapsto \lambda\mathcal{F}.\,\mathcal{F}F"] \arrow[r, shift right, "F \mapsto \lambda\mathcal{F}.\,F\left(\lambda a.\,\mathcal{F}(\lambda\phi.\,\phi a)\right)"'] & (\Sigma^3 A, \top, \top)
\end{tikzcd}$$

$$\begin{tikzcd}
(\Sigma^A, \mathfrak{p}^{\top} \& \mathsf{prime}, \top) \arrow[r, two heads] & (\Sigma^A, \mathsf{prime}, \top) \arrow[r, shift left, "\Sigma^2 M"] \arrow[r, shift right, "\Sigma^2 N"'] & (B, \top, \mathfrak{r}) \cong \Sigma^{(\Sigma^B, \mathfrak{r}^{\top} \& \mathsf{prime}, \top)}
\end{tikzcd}$$

$$\begin{tikzcd}
\{A \mid p\} \arrow[r, tail] & \{A \mid \top\} \arrow[r, shift left, "\Sigma^2 M"] \arrow[r, shift right, dashed, "\Sigma^2 N"'] & \cong \Sigma^{\{B \mid \mathfrak{r}\}}
\end{tikzcd}$$

An exactness property

$$
\begin{tikzcd}
Z \equiv \{\Sigma^A \mid \mathfrak{p}\} \equiv (A, \mathfrak{p}, \top) \arrow[r, tail, "i"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma^A \equiv (A, \top, \top) \arrow[d, two heads, "\Sigma^j"] & A \\
X \equiv \{\Sigma^{\{A|\mathfrak{q}\}} \mid \mathfrak{p}\} \equiv (A, \mathfrak{p}, \mathfrak{q}) \arrow[r, tail] & \Sigma^Y \equiv \Sigma^{\{A|\mathfrak{q}\}} \equiv (A, \top, \mathfrak{q}) & Y \equiv \{A \mid \mathfrak{q}\} \arrow[u, tail, "j"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
W \equiv (A, \mathfrak{q}^{\mathfrak{p}}, \top) \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma^2 A \equiv (\Sigma^A, \top, \top) \arrow[d, two heads, "\Sigma^j"] & \Sigma^A \\
\Sigma^X \equiv (\Sigma^A, \mathfrak{q}^{\mathfrak{p}}, \mathfrak{p}) \arrow[r, tail] & \Sigma^Z \equiv (\Sigma^A, \top, \mathfrak{p}) & Z \equiv \{\Sigma^A \mid \mathfrak{p}\} \arrow[u, tail, "i"']
\end{tikzcd}
$$

Exactness property

Let $\mathcal{L}$ be the full subcategory of objects $(A, \mathfrak{p}, \top)$.

(In the case of equilogical spaces, $\mathcal{L}$ consists of sober Bourbaki (= textbook) spaces.)

$\mathcal{L}$ is closed under $\times$, regular monos and $\Sigma^{\Sigma^{(-)}}$.

Σ is injective wrt regular monos in $\mathcal{L}$.

Given regular mono $(A, \mathfrak{p}, \top) \rightarrowtail (A, \top, \top)$,
$\Sigma^{(-)}$ takes it to a regular epi,
the pullback of this along any regular mono is still regular epi.

Set obeys similar (but stronger) properties.

A Chu-like construction

We can represent any equideductive space $(A, \mathfrak{p}, \mathfrak{q})$ by two $\mathcal{L}$-objects $(A, \mathfrak{p}, \top)$ and $(\Sigma^A, \mathfrak{q}^{\mathfrak{p}}, \top)$.

Similarly any morphism $(A, \mathfrak{p}, \mathfrak{q}) \to (B, \mathfrak{r}, \mathfrak{s})$ is given by $(A, \mathfrak{p}, \top) \to (B, \mathfrak{r}, \top)$ and $(\Sigma^A, \mathfrak{q}^{\mathfrak{p}}, \top) \leftarrow (\Sigma^B, \mathfrak{s}^{\mathfrak{r}}, \top)$.

$(\Sigma^A, \mathfrak{q}^{\mathfrak{p}}, \top) \leftarrow (\Sigma^B, \mathfrak{s}^{\mathfrak{r}}, \top)$ is a homomorphism of Σ^2-algebras.

Like the real and imaginary parts of a complex number.

So equideductive spaces have a topological part and an algebraic one, *cf.* Stone duality.

However, $(A, \mathfrak{p}, \top)$ is not the reflection of $(A, \mathfrak{p}, \mathfrak{q})$ in $\mathcal{L}$, and indeed does not depend functorially on it.

What kind of theory

Should generalised topology be

- bipartite, with a topological ("real") part and an algebraic ("imaginary" one), or
- unitary, where the same (exactness) properties apply to all objects?

(In "free" equideductive logic, the exactness property only holds when the basic object is $(A, \top, \top)$, essentially a locally compact space.)

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An analogy from the history of Science:

- Aristotle had a bipartite theory, with rectilinear motion on Earth and circular motion for the planets.
- Galileo and Newton unified them.

Similarly, whilst $\mathbb{C}$ adds $\sqrt{-1}$ to $\mathbb{R}$,
it otherwise obeys the same laws of algebra.

A critical example

$B \equiv \mathbb{N}^{\mathbb{N}}$ is not locally compact,

so $i : B \equiv \mathbb{N}^{\mathbb{N}} \rightarrowtail R$ (where $R \equiv \Sigma^{\mathbb{N}\times\mathbb{N}}$ or $\mathbb{N}_{\perp}^{\mathbb{N}}$) is not Σ-split,

i.e. there is no $I : \Sigma^B \to \Sigma^R$ with $\Sigma^i \cdot I = \mathsf{id}$.

Hence there is no diagonal fill-in

$$\begin{tikzcd} B \times \Sigma^B \arrow[r, rightarrowtail, "i \times \mathsf{id}"] \arrow[d, "\mathsf{ev}"'] & R \times \Sigma^B \arrow[dl, dotted] \\ \Sigma & \end{tikzcd}$$

so $\Sigma^{i\times\mathsf{id}}$ is not surjective.

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Conjecture: $\Sigma^{i\times\mathsf{id}}$ could still be regular epi.

Question in recursion theory

Let $X \equiv \Sigma^R$ be the topology on the space R of binary relations (or partial functions if you prefer).

$B \equiv \mathbb{N}^{\mathbb{N}} \subset R$ induces an equivalence relation $\sim$ on X (this is definable in equideductive logic).

From this, define the notations

$$\begin{array}{rcl} (f \sim g) & \equiv & \forall x.\, fx \sim gx \\ ({\sim}f{=}) & \equiv & \forall xy.\, x \sim y \Rightarrow fx = fy \\ ({\sim}g{\sim}) & \equiv & \forall xy.\, x \sim y \Rightarrow gx \sim gy. \end{array}$$

Is the following extra rule consistent?

$$\frac{\forall fg.\, ({\sim}f{\sim})\ \&\ (f \sim g)\ \&\ ({\sim}g{\sim}) \;\Rightarrow\; \Phi f = \Phi g \qquad \forall f.\, ({\sim}f{=}) \;\Rightarrow\; \Phi f = \Psi f}{\forall g.\, ({\sim}g{\sim}) \;\Rightarrow\; \Phi g = \Psi g}$$

Need to analyse the proof of $\forall f.\, ({\sim}f{=}) \;\Rightarrow\; \Phi f = \Psi f$.

The goal for a new theory of topology

- All maps are automatically continuous and computable.
- They represent computationally observable properties.
- Subspaces represent provable properties.
- Define subspaces as mathematicians (not set theorists) use set theory, *e.g.* $K \equiv \{x : X \mid \forall \phi.\ \Box\phi \Rightarrow \phi x\}$.
- Generalised spaces have as many of the exactness properties of sets that they can have when all maps are continuous.

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- Generalised spaces have as many of the exactness properties of sets that they can have when all maps are continuous.

The new category of spaces would be highly non-pointed.

Potential applications? Measure, distribution or probability theory.