Equideductive Logic and CCCs with Subspaces

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Abstract Stone Duality

ASD's axiomatisation of general topology consists of

- a lattice part: $\top$, $\bot$, $\wedge$, $\vee$ for open sets, $=$ for discrete spaces, $\neq$ for Hausdorff, $\Cup$ for compact and $\exists$ for overt ones (we'll see the reason for the new symbol $\Cup$ in place of $\forall$);
- a categorical part: λ-calculus for $\Sigma^{(-)}$, and the adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$ is monadic: gives definition by description, Dedekind completeness and Heine–Borel.

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We will get a CCC, but that's not important, because

- the exponential Y^X is tested by incoming maps,
- but its topology by outgoing ones.

We certainly need products, $\Sigma^{(-)}$ and equalisers.

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Even though this is much weaker than the correct definition, these two ideas are surprisingly powerful.

Don't worry — this is not a category theory talk!
Besides constructive topologists,
it's aimed at (some particular) type theorists.

CCCs with all finite limits

Working with nested equalisers and exponentials is clumsy.
Want to write $E = \{x \mid \forall y.\, \alpha x y = \beta x y\}$.

$$\begin{tikzcd}
 & E \arrow[dr, tail, "i"] & & \\
\Gamma \arrow[ur, dotted, "b"] \arrow[rr, "a"] & & X \arrow[r, shift left, "\tilde{\alpha}"] \arrow[r, shift right, "\tilde{\beta}"'] & \Sigma^Y \\
 & E \times Y \arrow[uu] \arrow[dr, "i \times \mathsf{id}_Y"] \arrow[ddr] & & \\
\Gamma \times Y \arrow[uu] \arrow[ur, dotted] \arrow[rr, "a \times \mathsf{id}_Y"] \arrow[drr, "\alpha(a,y) = \beta(a,y)"'] & & X \times Y \arrow[uu] \arrow[r, shift left, "\tilde{\alpha} \times Y"] \arrow[r, shift right, "\tilde{\beta} \times Y"'] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & \Sigma^Y \times Y \arrow[uu] \arrow[dl, "\mathsf{ev}"] \\
 & & \Sigma &
\end{tikzcd}$$

This can be stated without mentioning Σ^Y
as a universal property called a partial product.

Equideductive logic

The symbolic rules for $\forall \Rightarrow$ are as you would expect:

$$\frac{\Gamma,\ x : A,\ \mathfrak{p}(x)\ \vdash\ \alpha x\ =\ \beta x}{\Gamma\ \vdash\ \forall x : A.\ \mathfrak{p}(x)\ \Rightarrow\ \alpha x\ =\ \beta x}\ \forall I$$

$$\frac{\Gamma\ \vdash\ a : A,\ \mathfrak{p}(a) \qquad \Gamma\ \vdash\ \forall x : A.\ \mathfrak{p}(x)\ \Rightarrow\ \alpha x\ =\ \beta x}{\Gamma\ \vdash\ \alpha a\ =\ \beta a}\ \forall E$$

Of course, we need substitution (cut) for the free variable x.
It is given by a small change to the partial product diagram.

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This logic also has conjunction, with

$$\vdash\ \top \qquad \mathfrak{p},\ \mathfrak{q}\ \vdash\ \mathfrak{p}\&\mathfrak{q} \qquad \mathfrak{p}\&\mathfrak{q}\ \vdash\ \mathfrak{p} \qquad \mathfrak{p}\&\mathfrak{q}\ \vdash\ \mathfrak{q},$$

given by equalisers targeted at products. So, although $\forall \Rightarrow$ fundamentally has an equation on the right, we may define

$$\forall y.\ \big(\mathfrak{p}(y)\ \Rightarrow\ \forall z.\ (\mathfrak{q}(z) \Rightarrow \alpha xyz = \beta xyz)\big)$$

as $$\forall yz.\ (\mathfrak{p}(y)\ \&\ \mathfrak{q}(z)\ \Rightarrow\ \alpha xyz = \beta xyz).$$

The variable-binding rule

In the expression $\forall \vec{y}.\ \mathfrak{p}(\vec{y}) \implies \alpha\vec{x}\vec{y} = \beta\vec{x}\vec{y}$,
all of the variables on the left of $\Rightarrow$ must be bound by $\forall$.

This is because the target of the equaliser was Σ^Y,
not a dependent type.

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Maybe we can add some dependent types later, but
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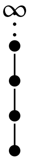
Write ϖ for the ascending natural number domain,



$$
\begin{tikzcd}
\mathbf{1} \arrow[r, "\infty"] & \varpi \\
\mathbf{0} \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near start] & \mathbb{N} \arrow[u]
\end{tikzcd}
$$

Then $\mathbb{N} \to \varpi$ is epi but not surjective,
since ∞ has no inverse image, *i.e.* its pullback is the initial object.

Therefore, a category of "sober" spaces and Scott-continuous
functions cannot be locally cartesian closed.

Equideductive translation of rules

An algebraic theory may be presented using judgements

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Then a rule

$$\frac{x : X,\ y : Y,\ \ldots,\ a = b,\ c = d,\ \ldots \quad \vdash \quad e = f}{u : U,\ v : V,\ \ldots,\ g = h,\ k = \ell,\ \ldots \quad \vdash \quad m = n}$$

is re-written as

$$\begin{aligned} &\big(\forall x : X.\forall y : Y.\ldots\ a = b \mathbin{\&} c = d \mathbin{\&} \cdots \Rightarrow e = f\big) \\ \Longrightarrow &\big(\forall u : U.\forall v : V.\ldots\ g = h \mathbin{\&} k = \ell \mathbin{\&} \cdots \Rightarrow m = n\big). \end{aligned}$$

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$$\Longrightarrow \big(\forall u : U.\forall v : V.\ldots\ g = h \mathbin{\&} k = \ell \mathbin{\&} \cdots \Rightarrow m = n\big).$$

But $\Rightarrow$ can be nested arbitrarily deeply, so we write induction as

$$\forall n.\ \ \mathfrak{p}(0) \mathbin{\&} \big(\forall m.\, \mathfrak{p}(m) \ \Rightarrow\ \mathfrak{p}(m+1)\big) \ \Longrightarrow\ \mathfrak{p}(n).$$

A "double negation" property

If $\mathfrak{p}(a)$ is $\top$, $\mathfrak{q}(a)\&\mathfrak{r}(a)$ or $\forall y.\ \mathfrak{q}(y) \Rightarrow \alpha a y = \beta a y$ then

$$\mathfrak{p}(a) \quad \dashv\vdash \quad \forall \phi \psi.\ (\forall a'.\ \mathfrak{p}(a') \Rightarrow \phi a' = \psi a') \implies \phi a = \psi a$$

where $a : A$ and $\phi, \psi : \Sigma^A$.

Disjunction and existential quantification

Using

$$\mathfrak{p}(a) \quad \dashv\vdash \quad \forall \phi\psi.\ (\forall a'.\, \mathfrak{p}(a') \Rightarrow \phi a' = \psi a') \implies \phi a = \psi a$$

we may also define $(\mathfrak{p} \vee \mathfrak{q})(a)$ as

$$\begin{aligned} \forall \phi\psi. \qquad & (\forall a'.\, \mathfrak{p}(a') \Rightarrow \phi a' = \psi a')\ \& \\ & (\forall a''.\, \mathfrak{q}(a'') \Rightarrow \phi a'' = \psi a'') \Rightarrow \phi a = \psi a \end{aligned}$$

and $(\exists x.\, \mathfrak{p})(a)$ as

$$\forall \phi\psi. (\forall a' x.\, \mathfrak{p}(x, a') \Rightarrow \phi a' = \psi a') \Rightarrow \phi a = \psi a$$

satisfying the distributive and Frobenius laws (???).

Constructive topology

Remember that, so far, we have just been working in a category with products, equalisers and a kind of partial product.

Not necessarily even a cartesian closed category.
(The CCC motivated the partial product and so $\forall \Rightarrow$,
but we then looked at a subcategory.)

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Remember that, so far, we have just been working in a category with products, equalisers and a kind of partial product.

Not necessarily even a cartesian closed category.
(The CCC motivated the partial product and so $\forall \Rightarrow$,
but we then looked at a subcategory.)

So far, Σ has needed no special properties.

So what does all of this have to do with constructive topology?

Equilogical spaces

Dana Scott introduced equilogical spaces.
They are given by partial equivalence relations
on algebraic lattices.

They provide a cartesian closed extension
of the textbook category of topological spaces.

There are many variations, including
Martin Hyland's filter spaces and Alex Simpson's QCB.

Giuseppe Rosolini related these categories to presheaves on,
and exact completions of, the textbook category.

However, they include many objects
that owe more to set theory than to topology.

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that are restrictions of congruences.

So we replace one, two-argument partial equivalence relation
with two one-argument predicates ($\mathfrak{p}$ and q).

Also, instead of set theory, we use equideductive logic,
possibly with some other interpretation.

What other interpretation?
That's a question for you — at the end of this lecture!

Equideductive spaces

Urtypes: generated from $\mathbf{0}$, $\mathbf{1}$ and $\mathbb{N}$ by $+$, $\times$ and $((-) \to \Sigma)$.
Combinators, including

$$\mathbb{I} : (A \to \Sigma) \to A \to \Sigma, \qquad \mathbb{K} : (A \to \Sigma) \to B \to A \to \Sigma,$$

$$\mathbb{C} : \big((B \to \Sigma) \to (C \to \Sigma)\big) \to \big((A \to \Sigma) \to (B \to \Sigma)\big) \to (A \to \Sigma) \to C \to \Sigma$$

$$\mathbb{T} : \mathbf{1}, \qquad \nu_0 : A \to (A + B), \qquad \nu_1 : B \to (A + B),$$

$$\pi_0 : \big((A + B) \to \Sigma\big) \to A \to \Sigma, \qquad \pi_1 : \big((A + B) \to \Sigma\big) \to B \to \Sigma,$$

$$\langle\rangle : \big((C \to \Sigma) \to A \to \Sigma\big) \to \big((C \to \Sigma) \to B \to \Sigma\big) \to (C \to \Sigma) \to (A{+}B) \to \Sigma.$$

$$\mathbb{A} : \big(((A \to \Sigma) + A) \to \Sigma\big) \to \mathbf{1} \to \Sigma,$$

$$\mathbb{L} : \big(((A + B) \to \Sigma) \to \mathbf{1} \to \Sigma\big) \to (A \to \Sigma) \to (B \to \Sigma) \to \Sigma.$$

with appropriate equational axioms, such as
$\forall MN\phi c.\, \mathbb{C}NM\phi c \;=\; N(M\phi)c$, without $\Rightarrow$.

Equideductive spaces

An equideductive space X is $(A, \mathfrak{p}, \mathfrak{q})$ where A is an urtype, $\mathfrak{p}$ is a predicate on Σ^A and $\mathfrak{q}$ one on A, for which

$$\phi, \psi : \Sigma^A, \quad \mathfrak{p}(\phi), \quad \forall a : A.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a \quad \vdash \quad \mathfrak{p}(\psi).$$

This rule is important in the construction.
It can be tightened to ensure that all spaces are definable using exponentials and equalisers.

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LHS is a partial equivalence relation.

A morphism $M : X \equiv (A, \mathfrak{p}, \mathfrak{q}) \to Y \equiv (B, \mathfrak{r}, \mathfrak{s})$ is a realiser $M : (A \to \Sigma) \to B \to \Sigma$ such that

$$\phi : \Sigma^A, \quad \mathfrak{p}(\phi) \quad \vdash \quad \mathfrak{r}(M\phi)$$

$$\phi, \psi : \Sigma^A, \quad \mathfrak{p}(\phi), \quad \forall a.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a \quad \vdash \quad \forall b.\ \mathfrak{s}(b) \Rightarrow M\phi b = M\psi b,$$

where $M_1 = M_2$ if

$$\phi : \Sigma^A, \quad \mathfrak{p}(\phi) \quad \vdash \quad \forall b : B.\ \mathfrak{s}(b) \Rightarrow M_1\phi b = M_2\phi b.$$

The type structure

$\mathbf{1} \equiv (\mathbf{0}, \top, \top), \quad \Sigma \equiv (\mathbf{1}, \top, \top).$

The product is $(A, \mathfrak{p}, \mathfrak{q}) \times (B, \mathfrak{r}, \mathfrak{s}) \;\equiv\; \big(A + B, (\mathfrak{p} \cdot \pi_0 \& \mathfrak{r} \cdot \pi_1), [\mathfrak{q}, \mathfrak{s}]\big).$

The equaliser is

$$E \equiv (A, \mathfrak{t}, \mathfrak{q}) \rightarrowtail^{I} (A, \mathfrak{p}, \mathfrak{q}) \underset{N}{\overset{M}{\rightrightarrows}} (B, \mathfrak{r}, \mathfrak{s})$$

$$\mathfrak{t}(\phi) \quad \equiv \quad \mathfrak{p}(\phi) \quad \& \quad \forall b : B.\ \mathfrak{s}(b) \Rightarrow M\phi b = N\phi b,$$

The exponential of $X \equiv (A, \mathfrak{p}, \mathfrak{q})$ is $\Sigma^X \equiv (\Sigma^A, \mathfrak{q}^{\mathfrak{p}}, \mathfrak{p})$, where

$$\mathfrak{q}^{\mathfrak{p}}(F) \;\equiv\; \forall \phi, \psi : \Sigma^A.\ \mathfrak{p}(\phi) \,\&\, (\forall a : A.\ \mathfrak{q}(a) \Rightarrow \phi a = \psi a) \;\Rightarrow\; F\phi = F\psi,$$

cf. the "double negation" property earlier.
(The modulation $\mathfrak{p}(\phi)\& \cdots$ is the source of many difficulties.)

There's still nothing special about the object Σ

cf. the two-level structure of Abstract Stone Duality for locally compact spaces:

we have replaced the underlying categorical strucure with a new one,

although it's *not* actually a *generalisation*

(this is a problem that we shall try to solve later).

The structure on Σ

At least, a distributive lattice: $(\Sigma, \top, \bot, \wedge, \vee)$.

Classifying open subsets

We want Σ to be a dominance (Giuseppe Rosolini again):

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, "\top"] \\ X \arrow[r, "\chi_U"] & \Sigma \end{tikzcd}$$

- If $U \cong V$ then $\chi_U = \chi_V$ (*pace* Per Martin-Löf);
- id_X is a pullback of $\top : \top \to \Sigma$ (along $\lambda x.\, \top$);
- If $U \hookrightarrow V$ and $V \hookrightarrow W$ are pullbacks of $\top : \top \to \Sigma$ then so is their composite $U \hookrightarrow W$;
- i is Σ-split: there is $\exists_i : \Sigma^U \to \Sigma^X$ with $\Sigma^i \cdot \exists_i = \mathsf{id}_{\Sigma^U}$ and $\exists_i \cdot \Sigma^i = (-) \wedge \chi_U \leqslant \mathsf{id}_{\Sigma^X}$, so $\exists_i \dashv \Sigma^i$.

When is Σ a dominance?

Recall that the implication $\Rightarrow$ in equideductive logic depends on the categorical structure (equalisers and $\Sigma^{(-)}$).

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If Σ also has lattice structure, we write $\Rightarrow$ for the induced order.

$i : U \hookrightarrow X$ is Σ-split iff $\Rrightarrow$ and $\Rightarrow$ are related by the Euclidean principle in the form

$$\sigma = \top \;\Rrightarrow\; \alpha = \beta \quad \vdash \quad \sigma \wedge \alpha = \sigma \wedge \beta$$

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This is the translation of the Gentzen-style rule

$$\frac{\sigma = \top \;\vdash\; \alpha = \beta}{\vdash\; \sigma \wedge \alpha = \sigma \wedge \beta}$$

Interaction of $\Rightarrowtriangle$ with $\Rightarrow$ and & with $\wedge$

Another way of writing the Euclidean principle is

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So it is natural to read

$$\begin{array}{lll} \sigma : \Sigma & \text{as} & \sigma = \top \\ \phi : \Sigma^X & \text{as} & \forall x.\, \phi x = \top \end{array}$$

making $\Rightarrow$ a special case of $\Rightarrowtriangle$.

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Then we have, as observed by Matija Pretnar,

$$\alpha = \top \ \& \ \beta = \top \qquad \dashv\vdash \qquad \alpha \wedge \beta = \top$$

making $\wedge$ a special case of &.

The Phoa principle

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Monotonicity says that

$$\forall a.\, \phi a \Rightarrow \psi a \quad \vdash \quad F\phi \Rightarrow F\psi$$

for $\phi, \psi : \Sigma^A$ and $F : \Sigma^A \to \Sigma$.

The dual Euclidean principle is

$$\sigma = \bot \Rightarrow \alpha = \bot \quad \dashv\vdash \quad \sigma \Leftarrow \alpha,$$

cf. the contrapositive in classical logic.

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Then the lattice-theoretic $\vee$ and $\exists$
are special cases of those defined earlier using $\forall \Rightarrow$ from the
categorical structure (??).

Interaction with topological structure

Similarly, equality $=_N$ in a discrete space N is a special case of general equality of terms:

$$n = m \;\dashv\vdash\; (n =_N m) = \top, \quad \text{whilst} \quad h = k \;\dashv\vdash\; (h \neq_H k) = \bot$$

in a Hausdorff space H.

The universal quantifier $\forall\!\!\!\forall$ in a compact space is related to $\forall$:

$$(\forall x.\, \phi x = \top) \;\dashv\vdash\; (\forall\!\!\!\forall x.\, \phi x) = \top$$

(Existential quantifiers in an overt space too???)

A more complicated example

Recall from Andrej Bauer's lecture that
an overt subspace $I \subset X$ defined by $\Diamond$ is connected if

$$\Diamond\top \;\Leftrightarrow\; \top \quad \text{and} \quad \ldots, \phi, \psi : \Sigma^X,\ \phi \vee \psi = \top_I \vdash \Diamond\phi \wedge \Diamond\psi \;\Rightarrow\; \Diamond(\phi \wedge \psi).$$

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The second clause of connectedness is

$$\forall\phi, \psi.\Big(\forall x.\ (\forall\theta.\theta x \Rightarrow \Diamond\theta) \Rightarrow \phi x \vee \psi x\Big) \Longrightarrow \Big(\Diamond\phi \wedge \Diamond\psi \Rightarrow \Diamond(\phi \wedge \psi)\Big).$$

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The variable-binding rule does not allow parameters in $\Diamond$.
What does this mean?

A new language for topology

Since $\Rightarrow, \wedge, \vee$ (in Σ) and $=_N$, $\biguplus$, $\exists$ (discrete, compact, overt) are special cases of $\Rightarrow$, &, $\vee$, $=$, $\forall$, $\exists$ we can just use the traditional symbols.

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- The inner one provides the terms of type Σ, which are observable properties or open subspaces; computably continuous functions are derived from these.

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But they generate two different logics:

- The inner one provides the terms of type Σ, which are observable properties or open subspaces; computably continuous functions are derived from these.
- The outer one is the logic of provable properties and general subspaces.

A new language for topology

Since $\Rightarrow$, $\wedge$, $\vee$ (in Σ) and $=_N$, $\biguplus$, $\exists$ (discrete, compact, overt) are special cases of $\Rrightarrow$, &, $\vee$, $=$, $\forall$, $\exists$ we can just use the traditional symbols.

But they generate two different logics:

- The inner one provides the terms of type Σ, which are observable properties or open subspaces; computably continuous functions are derived from these.
- The outer one is the logic of provable properties and general subspaces.

We may form $=$, $\neq$, $\forall$ or $\exists$ within the inner calculus so long as the relevant space is discrete, Hausdorff, compact or overt, as in the old calculus.
The other cases, including $\Rightarrow$, take us to the outer calculus.

The goal for a new theory of topology

- All maps are automatically continuous and computable.
- They represent computationally observable properties.
- Subspaces represent provable properties.
- Define subspaces as mathematicians (not set theorists) use set theory, *e.g.* $K \equiv \{x : X \mid \forall\phi.\ \Box\phi \Rightarrow \phi x\}$.
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Interpretation of equideductive logic

- The obvious set-theoretic one — the construction earlier gives Dana Scott's equilogical spaces.
- In locales — but I'm not sure whether this works (Does $(-) \times X$ preserve epis? I have both a proof and a counterexample!)
- In Formal Topology, if this works.
- Proof-theoretic, taking the rules just as they are (as we have done in this lecture).
- In another type theory such as Thierry Coquand's Calculus of Constructions or Coq.
- With additional axioms of our choosing.

A critical example

$B \equiv \mathbb{N}^{\mathbb{N}}$ is not locally compact,

so $i : B \equiv \mathbb{N}^{\mathbb{N}} \rightarrowtail R$ (where $R \equiv \Sigma^{\mathbb{N}\times\mathbb{N}}$ or $\mathbb{N}_\perp^{\mathbb{N}}$) is not Σ-split,

i.e. there is no $I : \Sigma^B \to \Sigma^R$ with $\Sigma^i \cdot I = \mathsf{id}$.

Hence there is no diagonal fill-in

$$\begin{tikzcd}
B \times \Sigma^B \arrow[rr, rightarrowtail, "i \times \mathsf{id}"] \arrow[d, "\mathsf{ev}"'] & & R \times \Sigma^B \arrow[dll, dotted] \\
\Sigma & &
\end{tikzcd}$$

so $\Sigma^{i\times\mathsf{id}}$ is not surjective.

($(-) \times \Sigma^B$ is crucial to this counterexample.)

Conjecture: $\Sigma^{i\times\mathsf{id}}$ could still be regular epi.

Question in recursion theory

Let $X \equiv \Sigma^R$ be the topology on the space R of binary relations (or partial functions if you prefer).

$B \equiv \mathbb{N}^{\mathbb{N}} \subset R$ induces an equivalence relation $\sim$ on X (this is definable in equideductive logic).

From this, define the notations

$$\begin{array}{rcl} (f \sim g) & \equiv & \forall x.\, fx \sim gx \\ (\sim f =) & \equiv & \forall xy.\, x \sim y \Rightarrow fx = fy \\ (\sim g \sim) & \equiv & \forall xy.\, x \sim y \Rightarrow gx \sim gy \end{array}$$

for $f, g : X \to X$.

Is the following extra rule consistent?

$$\frac{\forall fg.\, (\sim f \sim)\ \&\ (f \sim g)\ \&\ (\sim g \sim) \ \Rightarrow\ \Phi f = \Phi g \qquad \forall f.\, (\sim f =) \ \Rightarrow\ \Phi f = \Psi f}{\forall g.\, (\sim g \sim) \ \Rightarrow\ \Phi g = \Psi g}$$

where $\Phi, \Psi : \Sigma^{X^X}$.

Need to analyse the proof of $\forall f.\, (\sim f =) \ \Rightarrow\ \Phi f = \Psi f$.

Conjecture for a new interpretation

Equideductive logic $\longrightarrow$ Calculus of Constructions

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What is the extra logical axiom that this entails?