Transfinite Iteration of Functors as an Extensional Reflection, being Part II of A Categorical Replacement for Replacement

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Aesthetic completeness of a theory

The first genuinely mathematical application of the Axiom-Scheme of Replacement was by Johann von Neumann in 1923, when he showed that any well-ordering is equivalent to one in which the order relation is membership.

Contemporary set theorists (including Ernst Zermelo, who was opposed to the re-formulation in First Order Logic) accepted the new axiom because they considered that their theory was incomplete without it.

It does have applications outside set theory, in particular to transfinite iteration of functors, of which the Cumulative ("von Neumann") Hierarchy is the leading example.

To me, Set Theory is a Religion and I am an Atheist.

This talk will be strictly about Category Theory.

The new principle that I will demonstrate is similarly one that "ought" to be part of a "complete" theory of our subject.

Induction and recursion as magic

You know from experience of proofs by induction that the cognitively difficult part is formulating a sufficiently general predicate.

Then it may be laborious to show that each step preserves the predicate.

After that, magically you can deduce that the predicate is always true, in particular at the end (so long as it was true at the beginning).

It all works because we assert Peano (or some other) induction as an axiom.

Categorical Replacement as magic

Similarly, when using "Categorical Replacement", the labour will be in constructing various categories and characterising the relevant properties in them.

These laborious properties will be:

- pullbacks,
- well-foundedness,
- factorisation systems and
- extensionality.

The magical result will be transfinite recursion.

The Proposed new principle was shown to be valid in Intuitionistic Zermelo–Fraenkel Set Theory in my ItaCa lecture on 18 November 2025 ("Part I"). See www.paultaylor.eu/ordinals/ for the slides.

Well founded coalgebras

A coalgebra $\alpha : A \to TA$ is well founded if:
for any subobject $i : U \rightarrowtail A$
such that the pullback H factors through U,

$$\begin{tikzcd} TU \arrow[rr, tail, "Ti"] & & TA \\ H \arrow[u] \arrow[r, tail] \arrow[ur, phantom, very near start, "\urcorner"] & U \arrow[r, tail, "i"] & A \arrow[u, "\alpha"'] \end{tikzcd}$$

in fact $i : U \cong A$. (I call this a broken pullback.)

In the case of $T \equiv \mathcal{P}$ in a topos, this happens (for all U) iff
the relation $(\prec)$ corresponding to α obeys the induction scheme

$$\frac{\forall y. \quad (\forall z.\, z \prec y \Rightarrow \phi z) \Longrightarrow \phi y}{\forall x.\, \phi x}$$

for all predicates ϕ on A.

Extensionality

The less-than-innocent first axiom of Zermelo set theory:

A binary relation $(\prec)$ is extensional if it satisfies

$$\frac{\forall z. \quad z \prec x \iff z \prec y}{x = y}$$

for all $x, y \in A$.

This is equivalent to $\alpha : A \rightarrowtail \mathcal{P}A$ being 1–1.

We generalise this to any functor $T : \mathcal{X} \to \mathcal{X}$ on any category
by saying that α is mono.

The initial T-algebra (if there is one) is
a (the terminal) extensional well founded coalgebra.

The key innovation is that we further generalise this
by replacing "plain" monos with
the $\mathcal{M}$ class of a factorisation system.

Extensional well founded coalgebras are like sets

Even with general functors and factorisation systems,

- there is at most one homomorphism
  between any two extensional well founded coalgebras;
- this homomorphism is in $\mathcal{M}$;
- there are directed unions;
- there are binary intersections, just like for sets;
- there are overlapping unions, just like for sets,
  so long as the base category has well behaved pushouts.

So these properties of set-theoretic inclusion *etc.*,
which are really weird from a categorical point of view,
naturally arise from the general situation that we are
considering.

This theory is developed in my paper
Well founded coalgebras and recursion.

The key properties for our application again

Because of their similarity to set theory,
extensional well founded coalgebras will be called ensembles.

The category **Ens** of ensembles and coalgebra homomorphisms
is actually a (large) preorder.

Applying the functor remains within this preorder.

Forming colimits also remains within this preorder,
but this requires some care.

Therefore whatever "transfinite" means,
iteration of the functor also remains within the preorder.

This matters because then the colimits are just joins.

Constructive ordinals are non-trivial posets,
not just successors or chains,
so they do not define "categorical" diagrams.

Factorisation systems

Recall that this has two classes $\mathcal{E}$ and $\mathcal{M}$
that each contain all isomorphisms,
such that every map factorises as $e\,;m$
and the classes are orthogonal in this sense:

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Q \arrow[d, "g"] \arrow[dl, dotted, "h"'] \\ U \arrow[r, tail, "m"] & Y \end{tikzcd}$$

There is a unique map h making both triangles commute.

For our application, additionally

- $\mathcal{M}$-maps must have the cancellation property of monos;
- the functor T must preserve this class; and
- $\mathcal{M}$ must also be closed under directed unions.

The Proposal for Replacement

The new axiom that we propose is that
every well founded coalgebra have an extensional reflection.

It is an axiom-scheme like the set-theoretic version
because we assert this for all suitable
categories, functors and factorisation systems.

As with other notions of recursion,
the expressivity comes from constructing suitable
categories, functors and factorisation systems.

There is some labour required to characterise
the pullbacks, well-foundedness, factorisation and
extensionality.

Ordinals as coalgebras

We replace **Set** by **Pos**,
where pullbacks are constructed as in **Set**.

Instead of the *full* covariant powerset $\mathcal{P}$,
we use the lower subsets functor $\mathcal{D}$.

We have a choice of factorisation systems, with $\mathcal{M}$-class

- $\mathcal{I}$ plain monos (1–1 on points): not closed under $\mathcal{D}$;
- $\mathcal{R}$ regular monos (subsets with the restricted order): $\mathcal{D}$ preserves $\mathcal{R}$ but not its inverse images;
- $\mathcal{L}$ lower subsets: $\mathcal{D}$ preserves $\mathcal{L}$ and inverse images.

Their orthogonal $\mathcal{E}$ classes are

- $\mathcal{R}^\perp$ plain epis (surjective on points); and
- $\mathcal{L}^\perp$ cofinal functions — by no means surjective!

The idea is to use $\mathcal{E}$ classes like $\mathcal{L}^\perp$ to build skyscrapers.

$\mathcal{R}$- and $\mathcal{L}$-ordinals

A coalgebra $A \to \mathcal{D}A$ in **Pos** has $\leq$ and $\prec$ related by

$$c \prec b \leq a \Longrightarrow c \prec a \quad \text{and} \quad c \leq b \prec a \Longrightarrow c \prec a.$$

Well-foundedness of $\prec$ is the same as in **Set** (for both $\mathcal{R}$ and $\mathcal{L}$).

Then $(A, \leq, \prec)$ is $\mathcal{R}$-extensional iff it is meta-transitive:

$$(\forall d.\, d \prec c \Rightarrow d \prec b) \wedge (b \prec a) \Longrightarrow (c \prec a).$$

For example, $(\mathbb{N}, \leq, <)$ satisfies this.

Intuitionistically, this is stronger than ordinary transitivity,
which cannot be characterised using these ideas.

$\mathcal{L}$-extensional well founded $\mathcal{D}$-coalgebras are plump ordinals.

$\mathcal{R}$- and $\mathcal{L}$-extensional reflection (rank)

Since $\mathcal{R}^\perp$ is well co-powered,
the theory in *Well founded coalgebras and recursion*,
within an elementary topos,
gives the $\mathcal{R}$-extensional reflection of any well founded relation.

Ignoring the distinction between transitive and meta-transitive, this is the usual notion of ordinal rank.

On the other hand, $\mathcal{L}^\perp$ is not well co-powered.

Plump $\omega \cdot 2$ in $\mathbf{Set}^\rightarrow$ requires the axiom of replacement.

In general plump rank is an example of our proposal.

We may use either $\mathcal{R}$ or $\mathcal{L}$ to define "transfinite" in what follows, but $\mathcal{R}$ has the advantage that "combinatorial" ordinals like $(N, \leq, <)$ are $\mathcal{R}$-extensional.

Transfinite recursion

Suppose we have a complete join-semilattice $(E, \bigvee)$ together with a monotone endofunction $s : E \to E$.

Then transfinite recursion over an ordinal $(A, \leq, \prec)$ means a monotone function $f : A \to E$ such that

$$f(a) \;=\; \bigvee_E \{\, s\,(f\,b\,) \mid b \prec a \,\}.$$

We also call this transfinite iteration of the function s.

We will construct a category, factorisation system and endo-function such that the extensional reflection is this transfinite recursion.

After that we will replace the legitimate $\bigvee$-semilattice E with the illegitimate preorder **Ens**.

Transfinite recursion 2

We will work in the lax slice category $\mathbf{Pos}//E$ with

objects $\begin{tikzcd} A \arrow[d, "a"'] \\ E \end{tikzcd}$ and morphisms $\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "a"'] & \leq & B \arrow[dl, "b"] \\ & E & \end{tikzcd}$

that factorise as $f = e\,;m$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$ and

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[ddr, bend right, "a"'] & Q \arrow[r, tail, "m"] \arrow[d, tail, "m"] & B \arrow[ddl, bend left, "b"] \\
\leq & B \arrow[d, "b"] & = \\
& E &
\end{tikzcd}$$

Orthogonality of these classes $\mathcal{E}//E$ and $\mathcal{M}//E$ follows from that of $\mathcal{E}, \mathcal{M} \subset \mathbf{Pos}$, plus an easily provable inequality.

Transfinite recursion 3

Pullbacks and well-foundedness are the same as in **Pos** and **Set** but with suitable extra maps and inequalities:

$$\begin{tikzcd}
P \arrow[rr, "\pi_0"] \arrow[dd, "\pi_1"'] \arrow[dr, dotted, "p"] & \leq & A \arrow[dl, "a"'] \arrow[dd, "f"] \\
\geq & E & \geq \\
B \arrow[ur, "b"] \arrow[rr] & \leq \atop g & C \arrow[ul, "c"']
\end{tikzcd}$$

where $p \equiv \pi_0\,; a \wedge \pi_a\,; b$.

$$\begin{tikzcd}
T_0U \arrow[rrr, hook, "I"] \arrow[drr, "G"] & & = & T_0A \arrow[dl, "F"'] \\
\geq & & E & \geq \\
K \arrow[u, "h"] \arrow[urr, dotted, "k"] \arrow[r, dotted, "="] & U \arrow[ur, "g"] \arrow[rr, hook] & = \atop i & A \arrow[ul, "f"'] \arrow[uu, "\alpha"']
\end{tikzcd}$$

Transfinite recursion 4

The endofunctor $\mathsf{R} : \mathbf{Pos}//E \to \mathbf{Pos}//E$ on objects is

$$\mathsf{R}(A \xrightarrow{f} E) \;\equiv\; (\mathcal{D}A \xrightarrow{F} E),$$

where $FU \;\equiv\; \bigvee\{s(fa) \mid a \in U\}$ and on morphims it is

$$\mathsf{R}p \equiv \mathcal{D}p : U \mapsto \bigvee \{b \mid \exists a \in U.\, b \le pa\}.$$

$$\begin{tikzcd} A \arrow[rr, "p"] \arrow[dr, "f"'] & \le & B \arrow[dl, "g"] \\ & E & \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \mathcal{D}A \arrow[rr, "\mathcal{D}p"] \arrow[dr, "F"'] & \le? & \mathcal{D}B \arrow[dl, "G"] \\ & E & \end{tikzcd}$$

which is a $\mathbf{Pos}//E$-map because

$$FU \;\equiv\; \bigvee\{fa \mid a \in U\} \;\le\; \bigvee\{g(pa) \mid a \in U\} \;=\; G(\mathcal{D}pU).$$

This is equality if $f = p\,;g$, so R preserves $\mathcal{M}//E$-maps.

The transfinite recursion theorem

An R-coalgebra

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "f"'] & \le & \mathcal{D}A \arrow[dl, "F"] \\ & E & \end{tikzcd}$$

has $\qquad fa \;\le\; F(\alpha a) \;\equiv\; \bigvee\{s(fb) \mid b \prec a\}.$

It is extensional iff $\le$ is equality and $\alpha \in \mathcal{M} \subset \mathbf{Pos}$.

A solution of the transfinite recursion equation
is therefore exactly an $\mathcal{M}//E$-extensional R-coalgebra
whose underlying $\mathcal{M}$-extensional well founded $\mathcal{D}$-coalgebra
is the ordinal over which the recursion is to hold.

Our axiom says magically that this exists.

For example, the $\mathcal{L}//E$-extensional reflection of

$$\lambda n.\, \bot : (\mathbb{N}, \le, <) \to E \qquad \text{is} \qquad \lambda n.\, s^n \bot.$$

From functions to functors

The idea is to replace the internal $\bigvee$-semilattice E
with the external category **Ens** that is a preorder.

We don't use universes (big sets, internal toposes, *etc.*), because
they detract from what makes Replacement interesting.

The order relation in **Ens** corresponds to underlying functions.

(For simplicity, we consider iterating an endofunctor of **Set**.)

The lex slice category $\mathbf{Pos}//E$ becomes
the category **DFPos** of discrete opfibrations of posets.

But making the Grothendieck construction part of the argument
uses Replacement more or less as set theorists state it.

We have to build the opfibred analogue of **Ens**
so that the colimits become joins
in the functor that encodes recursion over an ordinal.

Discrete opfibrations

The object $X_{(-)} : A \to E$ of $\mathbf{Pos}//E$ becomes $\mathsf{X} \equiv \Sigma_a X_a \to A$.

Lax slice morphisms become commutative squares,
with a factorisation system like this:

$$\begin{tikzcd} \mathsf{X} \arrow[r] \arrow[d] & \mathsf{U} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathsf{Y} \arrow[d] \\ A \arrow[r, two heads, "e \in \mathcal{E}"] & U \arrow[r, tail, "m \in \mathcal{M}"] & B \end{tikzcd}$$

Pullbacks are constructed in the obvious way,
as commutative cubes in which two faces are pullbacks.

Opfibred coalgebras

The functor $T : \mathbf{Set} \to \mathbf{Set}$ has opfibred coalgebras like this:

$$
\begin{tikzcd}
\mathsf{X} \arrow[dr, "\xi"] \arrow[dd] & \\
& \mathsf{TX} \arrow[dl] \\
A &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_a \arrow[rr] \arrow[dr, "\xi_a"] \arrow[dd, mapsto] & & X_b \arrow[dr, "\xi_b"] \arrow[dd, mapsto] & \\
& TX_a \arrow[rr] \arrow[dl, mapsto] & & TX_b \arrow[dl, mapsto] \\
a \arrow[rr, phantom, "\leq"] & & b &
\end{tikzcd}
$$

At first we just take A to be a poset, not a $\mathcal{D}$-coalgebra or ordinal.

Well-foundedness and extensionality in the fibre over A are defined by a simple generalisation of the **Set** case or by treating this fibre as the presheaf topos $\mathbf{Set}^A$.

In this way we define the category (preorder) $\mathbf{Ens}^A$ of *opfibred* extensional well founded T-coalgebras over A.

Invoking the theory twice

We have just defined extensional well founded T-coalgebras in the fibre $\mathbf{Set}^A$.

Previously we constructed transfinite recursion using $\mathcal{R}$- or $\mathcal{L}$-extensional well founded $\mathcal{D}$-coalgebras in the base category **Pos**.

These are two separate invocations of the theory, with different functors and factorisation systems.

The simplifying assumption of just considering $T : \mathbf{Set} \to \mathbf{Set}$ means that the "monos" within the fibres are just plain ones.

Replacing **Set** with another category would just require an understanding of what its opfibred version is (over posets).

The final step is to combine these two parts.

Transfinite iteration of functors

Adapting the contruction in $\mathbf{Pos}//E$ we have

$$
\begin{tikzcd}
\mathsf{X} \arrow[rrr] \arrow[dr, "\xi"] \arrow[ddr] & & & \mathsf{RX} \arrow[dr, "\mathsf{R}\xi"] \arrow[ddr] & \\
& \mathsf{TX} \arrow[rrr] \arrow[d] & & & \mathsf{TRX} \arrow[d] \\
& A \arrow[rrr, "\alpha"] & & & \mathcal{D}A
\end{tikzcd}
$$

where $$(\mathsf{RX})_U \equiv \bigvee_{\mathbf{Ens}} \{TX_b \mid b \in U\},$$

so that the fibres look like

$$
\begin{tikzcd}
X_a \arrow[rrr] \arrow[dr, "\xi_a"] \arrow[ddr, mapsto] & & & (RX)_{\alpha a} = \bigvee \{TX_b \mid b \prec a\} \arrow[dr, "(\mathsf{R}\xi)_{\alpha a}"] \arrow[ddr, mapsto] & \\
& TX_a \arrow[rrr] \arrow[d, mapsto] & & & T(RX)_{\alpha a} \arrow[d, mapsto] \\
& a \arrow[rrr, mapsto, "\alpha"] & & & \alpha a \equiv \{b \mid b \prec a\}
\end{tikzcd}
$$

Transfinite iteration of functors

We need to characterise pullbacks, well-foundedness and extensionality.

A morphism of opfibred coalgebras is a pullback iff its "underlying objects" form a pullback. (This does not require T to preserve inverse images.)

It is well founded iff the base morphism $\alpha : A \to \mathcal{D}A$ is well founded in **Pos**.

It is extensional iff $\alpha \in \mathcal{M} \subset \mathbf{Pos}$ and the opfibred coalgebras form a pullback.

But that happens iff this encodes transfinite iteration:

$$X_a = (\mathsf{RX})_{\alpha a} \equiv \bigvee_{\mathbf{Ens}} \{TX_b \mid b \prec a\}.$$

This exists magically using our proposed axiom!