Review of the
Well Founded Coalgebras Programme
(1991–9 and 2019–26)

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My Objectives in the early 1990s

(1) To provide a constructive and categorical account of transfinite recursion for my book *Practical Foundations of Mathematics* (CUP 1999),

building on

- Gerhard Osius, *Categorical set theory: a characterisation of the category of sets* (JPAA 1974) and
- Christian Mikkelsen, *Lattice-theoretic and logical aspects of elementary topoi* (thesis 1976, TAC Reprints 2022).

(2) To find a Categorical Replacement for Replacement, *i.e.* in the native language of category theory (adjunctions), not by transliterating set theory.

This lecture is a sketchy history of this programme.

The purpose of this lecture

This is a discursive overview of a long personal research programme, for which the papers and slides are at

`www.paultaylor.eu/ordinals/`

The central paper is called Well founded coalgebras and recursion.

It has been with a journal editor since November 2021.

This has obliged me (relucantly) to study the period 1904–23 in the history of set theory, the short window when it contributed foundations as most mathematicians use them.

More or less everything in that topic is mis-attributed or simply wrong in the textbooks.

So I have translated Bourbaki, Hartogs and Mirimanoff at

`www.paultaylor.eu/trans/`

where you will also find my historical paper on

Old and new proofs of the order-theoretic fixed point theorem.

Classical transfinite recursion

We often see

- $f(0)$ = some base construction
- $f(\alpha^+) = s\big(f(\alpha)\big)$ step construction
- $f(\lambda) = \bigcup_{\alpha\in\lambda} f(\alpha)$ unions,

giving a fixed point., purportedly — but this is not a proof!

One reason is that ordinals provide induction for predicates, but we require recursion for functions.

Johann (John) von Neumann, *Zur Einführung der transfiniten Zahlen* (1923) and *Über die Definition durch transfinite Induktion und...* (1928) showed how to derive recursion from induction.

The other is that we need a "big enough" ordinal.

Friedrich Hartogs, *Über das Problem der Wohlordnung* (1915) provided this.

However, there is no accurate presentation of Hartogs' construction in any set theory textbooks (I've checked a few).

There is no coherent account of the whole fixed point proof.

In Memoriam Friedrich Hartogs (1874–1943)

His main work was in Complex Analysis in Munich.
The 1915 paper was his only one on set theory.
It was the first substantial use of Zermelo's 1908 axioms
and one of very few citations before 1919.
It carefully cited the axioms and Cantor's 1897 work.
It built the set of all well-orderings of subsets of a given set
— a very complicated construction for its day.
It introduced the quotient of an abstract set by an equivalence relation as the set of equivalence classes.
It proved an isomorphism between structures of different $\in$-depth.

However, even his student and friend Abraham Fraenkel failed to give an accurate account in his set theory textbook.

As a Jew, he took his own life (as Felix Hausdorff did) to avoid being sent to a concentration camp.

There is no biography of him longer than a few pages.

Kazimierz Kuratowski 1922

As a 26-year old post-doc he wrote
Une méthode d'élimination des nombres transfinis des raisonnements mathématiques,
which clearly included what were later mis-named

- (Max) Zorn's Lemma (1935);
- (Alfred) Tarski's fixed point theorem (1955);
- the (Nicolas) Bourbaki– (Ernst) Witt theorem (1949–51).

This built on Ernst Zermelo's second proof (using Choice) that every set can be well ordered.

The key lemma is that the subset K generated by $\bot$, the successor operation s and unions satisfies

$$\forall xy \in K. \quad sx \leq y \ \vee\ x = y \ \vee\ sy \leq x$$

Zermelo's proof only worked for his application, but Kuratowski gave the first abstract proof, although Bourbaki's proof is neater.

Well founded relations and coalgebras

Induction scheme for a well founded relation $\prec$:

$$\forall\phi.\quad \frac{\forall a{:}A.\ \ (\forall b{:}A.\ \ b \prec a \Rightarrow \phi b) \Longrightarrow \phi a}{\forall a{:}A.\ \ \phi a}$$

Represent $(A, \prec)$ by $\alpha : A \to \mathcal{P}(A)$ where $\alpha(a) \equiv \{b \mid b \prec a\}$.

Generalise this to any functor T that preserves monos.

$\alpha : A \to TA$ is a well founded coalgebra if
in any pullback diagram of the form

$$\begin{tikzcd}
TU \arrow[rr, tail, "Ti"] & & TA \\
H \arrow[u] \arrow[ur, phantom, "\lrcorner", very near start] \arrow[r, tail, "j"] & U \arrow[r, tail, "i"] & A \arrow[u, "\alpha"']
\end{tikzcd}$$

the maps i and therefore j are necessarily isomorphisms.

Subsets as subcoalgebras

Dimitry Mirimanoff, *Remarques sur la théorie des ensembles et les antinomies Cantoriennes I* (1917)

- characterised the subset relation as a simulation (in the sense of modern process algebra);
- defined the rank of a set ($\in$-depth); and
- defined what was later called the non Neumann hierarchy.

Gerhard Osius (1974) characterised the subset relation
as a coalgebra homomorphism
(probably without knowing about Mirimanoff).

Christian Mikkelsen (1976) proved
a categorical version of the recursion theorem
(without knowing about von Neumann's proof — I asked him).

Osius became a statistician and Mikkelsen a schoolteacher.

The recursion theorem

From any well founded coalgebra (A, α) to any algebra (Θ, θ)
there is a unique homomorphism

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & T\Theta \arrow[d, "\theta"] \\
A \arrow[u, "\alpha"] \arrow[r, "f"] & \Theta
\end{tikzcd}$$

Proof (following von Neumann):

- consider all such homomorphisms from subcoalgebras $B \hookrightarrow A$ (attempts);
- there is a unique one $\emptyset \to \Theta$;
- there is a bijection between those $B \to \Theta$ and $sB \to \Theta$, where sB is the successor coalgebra; and
- colimits extend uniqueness.

Induction on subcoalgebras gives the unique total attempt.

The relative successor subcoalgebra

Let $i : (B, \beta) \hookrightarrow (A, \alpha)$ be a subcoalgebra.
Then its relative successor $k : (C, \gamma) \hookrightarrow (A, \alpha)$
is given by pullback of α and Ti.

$$\begin{tikzcd}
 & A \arrow[r, "\alpha"] & TA & \\
 & C \arrow[u, hook, "k"] \arrow[r, "c"] \arrow[ur, phantom, "\lrcorner", very near start] \arrow[drr, "\gamma"'] & TB \arrow[u, hook, "Ti"'] \arrow[dr, hook, "Tj"] & \\
B \arrow[uur, hook, "i"] \arrow[ur, "j"] \arrow[urr, "\beta"'] & & & TC \arrow[uul, hook, "Tk"']
\end{tikzcd}$$

The pullback mediator $j : B \to C$ makes
$(B, \beta) \hookrightarrow (C, \gamma) \hookrightarrow (A, \alpha)$ as subcoalgebras
when we define $\gamma \equiv c \,;\, Tj$.

Where I left this in 1999

The recursion theorem is in my book (Section 6.3)
for functors $T : \mathcal{S} \to \mathcal{S}$ where $\mathcal{S}$ is an elementary topos
assuming T preserves inverse images (pullbacks of monos).

The proof took the union of all attempts,
which form a complete lattice.

I learned of Pataraia's simple argument (next slide)
shortly before the book was finished.

This wrecked my self-confidence.

I had also got interested in Abstract Stone Duality.

I returned in 2019 to answer the challenge
that T need only preserve monos.

The attempts would no longer have all joins.
I would have to use Pataraia's result.

Pataraia's lemma (1997)

A subset $I \subset X$ of a poset is directed if

$$\exists x{:}X.\ x \in I \qquad \text{and} \qquad \forall xy \in I.\ \exists z \in I.\ x \leq z \geq y;$$

A dcpo (directed-complete poset) has joins of all directed subsets, written $\bigvee^{\uparrow}$.

An endofunction $s : X \to X$ is monotone if

$$\forall x, y{:}X. \quad x \leq y \quad \Longrightarrow \quad sx \leq sy,$$

and inflationary if $\quad \forall x{:}X. \quad x \leq sx.$

The poset of inflationary monotone endofunctions of any poset is directed, because the composites $r \,;\, s$ and $s \,;\, r$ both lie above both r and s in the pointwise order:

$$\forall x. \quad x \quad \leq \quad rx,\ sx \quad \leq \quad r(sx),\ s(rx).$$

Since directed completeness is inherited by the function space,
any dcpo has a greatest inflationary monotone endofunction,
which is idempotent.

Dito Pataraia (1963–2010)

The components of the two classical fixed point theorems were in place by 1923. The fixed points were even expressed parametrically in the base case, *i.e.* in function notation.

When directedness was first introduced for convergence in analysis and point–set topology by Eliakim Hastings Moore and Llewellyn Schmidt in *A general theory of limits* (1922) they called it the composition property.

Why did nobody before 1997 think to compose the functions of which they were taking fixed points?

Because people (including me) had Cantor's idée fixe of doing something infinitely often and then some more.

There was also a distracting trivial generalisation:
monotonicity was often omitted
because it can be deduced (on the relevant subset)
from the inflationary property (and also *vice versa*).

Well founded elements and maximal fixed points

In a dcpo X with $\bot, \wedge$ and an endofunction $s : X \to X$, define $x \in X$ to be well founded element if

$$x \leq sx \qquad \text{and} \qquad \forall y \in X.\, (sy \wedge x \leq y) \implies x \leq y.$$

Then in the subset $W \subset X$ of well founded elements, fixed points are maximal,

$$\forall x \in W.\, \forall y \in X. \quad sy = y \leq x \implies x = y.$$

(This is my contribution to the Pataraia theorem.)

- W has a greatest element, which we call $\top$;
- this is the unique fixed point in W and the least in X; and
- induction: any predicate ϕ on X for which
  - $\phi(\bot)$ holds,
  - $\phi(x) \Rightarrow \phi(sx)$ and
  - $\big(\forall x \in I.\, \phi(x)\big) \Rightarrow \phi(\bigvee^{\uparrow} I)$ for all directed $I \subset X$

  also satisfies $\phi(\top)$.

Extensionality

A relation $\prec$ (such as $\in$) is extensional if

$$\forall xy. \quad (\forall z.\, z \prec x \iff z \prec y) \implies x = y.$$

That is, $\forall xy. \quad \alpha(x) = \alpha(y) \implies x = y.$

In other words,
the structure map $\alpha : A \hookrightarrow TA$ is a monomorphism.

This definition is enough to re-construct many of the weird properties of sets, so we call an extensional well founded coalgebra an ensemble:

- between any two ensembles, there is at most one coalgebra homomorphism, and then it is mono;
- this preorder has meets: any two two ensembles have a largest common subensemble (intersection);
- if the underlying category has well-behaved pushouts, they also have an "overlapping" union.

"Mostowski collapse"

Von Neumann (1923) showed that every well-ordering is equivalent to an ordinal (in his sense), *i.e.* where $\prec$ is $\in$.
This was the first use of Replacement and his (class) recursion theorem.

Mirimanoff (1917) had had the same idea, but without the axiomatisation or the proof.

Andrzej Mostowski (*An undecidable arithmetical statement*, 1949) showed in the same way that every extensional well founded relation is equivalent to a unique set.

The same method can turn any well founded relation
into an extensional one (Mostowski collapse) and then
into a well-ordering, called the rank (Mirimanoff again).

Mostowski without Replacement

If we use explicit structure instead of $\in$, the two constructions

well founded $\longrightarrow$ extensional $\longrightarrow$ ordinal

can be considered as universal properties

and (in the classical cases)

can be carried out without Replacement,
as the quotient by an equivalence relation.

The equivalence relation is defined recursively,
so the quotient is defined co-recursively.

Can we do this in a purely categorical way?

Mostowski as a categorical reflection

The Mostowski collapse of a well founded relation to an extensional one is a coalgebra homomorphism like this:

$$\begin{tikzcd}
TA \arrow[r, "Tq"] & TE \\
A \arrow[u, "\alpha"] \arrow[r, two heads, "q"] & E \arrow[u, hook, "\epsilon"']
\end{tikzcd}$$

where ϵ is mono (E is extensional)
and q is epi in the underlying category (**Set**).

It is the universal such diagram:
a reflection into the subcategory.

(So long as T preserves monos,
it's the only such diagram, up to isomorphism.)

Factorisation systems

In more interesting categories than **Set**,
we can ring the changes on what "epis" and "monos" are.

This uses a factorisation system $(\mathcal{E}, \mathcal{M})$
(Peter Freyd and Max Kelly,
Categories of continuous functors, I (JPAA 1972)).

There are two situations to explore:

- $\mathcal{E}$ is "under control" (co-well-powered); or
- $\mathcal{E}$ is "out of control" but $\mathcal{M}$ is under control (well-powered).

The co-well-powered case

There is only a "set" of $\mathcal{E}$-maps out of each object.

We also assume a "cancellation property"
that makes the out-going epis into a preorder.

(This is probably automatic classically, but with a categorical version of Pataraia's theorem it is not actually needed.)

In fact, $\mathcal{E}$-maps out of any coalgebra form
an ipo with a "successor" operation.

So long as T preserves $\mathcal{M}$ maps,
maximality of fixed points holds,

so there is a "longest" $\mathcal{E}$-map out of any coalgebra.

Even if T doesn't preserve monos,
there is still a least fixed point.

Either way,
there is a universal $\mathcal{E}$-map to an $\mathcal{M}$-extensional coalgebra.

Successor quotient

The successor quotient (C,γ) of any coalgebra (B,β) is given by factorising β like this:

$$\begin{tikzcd}
TTB & & \\
& TC \arrow[ul, "Ti"'] \arrow[dr, dotted, "Tg"] & \\
TB \arrow[uu, "T\beta"] \arrow[ur, "Te"] \arrow[rr, "Tf"'] & & TE \\
& C \arrow[ul, hook, "i"'] \arrow[uu, "\gamma"'] \arrow[dr, dotted, "g"] & \\
B \arrow[uu, "\beta"] \arrow[ur, two heads, "e"] \arrow[rr, "f"] & & E \arrow[uu, hook, "\epsilon"']
\end{tikzcd}$$

B is fixed $(e : B \cong C)$ iff it is extensional $(\beta \in \mathcal{M})$.

Co-well-powered examples: **Set**

For $\mathcal{P}$ on **Set** (a topos) with
its usual factorisation into onto and 1–1 maps,

this is the "Mostowski collapse"
of well founded relations to extensional ones.

Co-well-powered examples: **Pos**

The analogue for **Pos** (posets) is the "down-sets" functor.

Pos has at least three interesting factorisation systems.

One of them consists of epis and regular monos:

- $\mathcal{M}$-maps are subset inclusions with the restricted order; and
- $\mathcal{E}$-maps are monotone functions that are surjective on points.

Then extensional well founded coalgebras satisfy

$$\forall w,x,y. \quad (\forall z.\, z \prec y \Rightarrow z \prec x) \land (x \prec w) \Longrightarrow (y \prec w),$$

which is classically the same as well-orderings, but not intuitionistically.

The theory works to give a rank construction.

But I cannot see how to formulate the universal property
of transfinite recursion with successors and limits.

Co-well-powered examples: **Pos**

That was not the usual form of transitivity.

The commonest notion of "ordinal" is
an extensional well founded relation
with the usual transitivity property,

$$\forall w,x,y. \quad (y \prec x) \land (x \prec w) \Longrightarrow (y \prec w),$$

This does have a rank function with a universal property.

It also has transfinite recursion, being universal for
an inflationary but not monotone successor.

However, it seems that this cannot be expressed
as an example of the method using factorisation systems.

Having got stuck down that rabbit-hole,
I did not go on to explore other categories
and other notions of constructive ordinal.

The well-powered but not co-well-powered case

Still with **Pos** and down-sets, now let
$\mathcal{M}$ be lower inclusions and
$\mathcal{E}$ be the cofinal monotone functions.

Then the extensional coalgebras are plump ordinals.

These are the "ordinals" that categorists
would have invented if they hadn't had the *idée fixe*:
they form the (class) initial algebra for
arbitrary joins and a monotone successor.

However, to construct plump $\omega \cdot 2$
in the category of presheaves on a single arrow
requires the axiom of Replacement.

The well-powered but not co-well-powered case

Generalising the example of plump ordinals,

Using the axiom-scheme of Replacement and unbounded Separation (comprehension) as in the official formulation of ZFC (which I now finally understand how to use),

on a category that has a "forgetful" functor to a topos
that creates filtered colimits,

so long as $\mathcal{M}$ is well powered and T preserves it,

every well founded coalgebra has an extensional reflection.

This is my proposal for a categorical replacement for Replacement,

i.e. as a new axiom to add to those for an elementary topos.

See my November 2025 ItaCa slides for this.

Constructions with the Replacement for Replacement

Most of 20th century mathematics is done in
Zermelo Set Theory or the logic of an elementary topos.

Some things go outside that framework.

The examples that I know use infinite or
transfinite iteration of functors.

The paradigm of that is the (Mirimanoff-) non Neumann hierarchy

$$V_0 \equiv \emptyset \qquad V_{\alpha^+} \equiv \mathcal{P}(V_\alpha) \qquad V_\lambda \equiv \bigcup_{\alpha\in\lambda} V_\alpha.$$

I will show how to use my Categorical Replacement for
Replacement to do this in my online CT26 presentation in July.

Categorical Pataraia?

Pataraia's result used composition of
inflationary monotone endofunctions.

Their closest categorical analogue is well pointed endofunctors.

A pointed endofunctor has a natural transformation

$$\begin{tikzcd} \mathsf{id} \arrow[r, "\sigma"] & S \end{tikzcd}$$

and it is well pointed if $S\sigma = \sigma S$.

This notion was introduced by Kenneth Hardie, *Symmetric unads and free algebras* (JPAA 1977), but principally exploited by Max Kelly, *A unified treatment of transfinite constructions for free algebras, ...* (1980).

The idée fixe again!

Categorical Pataraia?

It’s a pity that Kelly failed to explore the monoidal category of well pointed endofunctors.

As in the dcpo case, he didn’t compose them.

This is a *bit* tricky, but they do compose!

Instead of well pointed elements, we can use a Galois connection to cut the monoidal category down to what we want.

Then the classical transfinite construction becomes a directed colimit diagram. (Not a filtered one.)

See my June 2025 slides for the details.

Are transfinite numbers good for anything at all?

Kuratowski (1922) showed that they are not needed for “transfinite” induction for subsets or predicates,

Nor are they needed for “transfinite” recursion for functions.

The study of well pointed endofunctions shows that they are not needed for “transfinite” iteration of functors.

Also, the intuitionistic theory of ordinals is a mess. (Having been infected by the idée fixe, I helped to create it.)

They were the *idée fixe* that stopped people from making the observation that Pataraia made.

Re-stating the *idée fixe* will not clear up the intrinsic mess.

Are they good for anything at all?

Ordinals in Proof Theory

Ordinal arithmetic is used to measure the complexity of (transformations of) proofs.

Composition of well pointed endofunctors behaves like ordinal addition: it preserves colimits in one argument but not the other.

By considering endofunctors of the category of endofunctors we obtain something like ordinal multiplication.

Maybe if Proof Theorists applied categorical ideas to their methods of transforming proofs, they might obtain a more general algebraic structure of which the way they use the classical ordinals would be an example. (*Cf.* generalising from $\mathbb{Z}$ and $\mathbb{R}$ to rings and fields.)