Local Compactness and Bases in various formulations of Topology

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Abstract

A basis for a locally compact space is a family of pairs of subspaces, one open and the other compact, where containment of the compact subspace indicates whether the open one contributes to the union expressing a general open subspace. This is captured abstractly by saying which unions sets of basic opens contain (or cover) a basic compact subspace. This "way-below" relation was previously axiomatised for systems that are closed under unions and intersections: in this paper we do so without this assumption, so that balls in a metric space provide an example. We show how to reconstruct a space from an abstract basis in Point–Set Topology, Locale Theory, Formal Topology and Abstract Stone Duality. These four constructions respectively rely on the different logical foundations in which these approaches are usually presented. We also characterise continuous functions by means of relations called matrices that generalise the way-below relation. Hence our category defined using relations is weakly equivalent to that of locally compact spaces in each of these four formulations of topology, according to its appropriate logical foundations. Subsequent work will develop abstract bases towards computation.

Note (to referees) regarding the length of this paper:

The principal objective is to establish definitively the axioms for an abstract basis so that future work can build on them. Everything up to Section 7 is needed to show that they are sound and complete in Point–Set Topology, since it turns out to be necessary to go *via* Locale Theory and Formal Topology. Sections 8–11 are about my own subject (ASD) and for me Section 11 contains the core result. For technical reasons, the next paper, which will show that bases and matrices provide a model of ASD, must restrict to overt spaces (Section 13) with bases using compact subspaces (Section 12). Finally, we sum up the complicated argument as equivalences of categories (Section 14).

Arguably, however, I tend to include too much detail in my proofs, so I am open to opinions that particular results are obvious.

Introduction

I find it extremely difficult to write introductions and it is likely that this one will be re-written several times yet. I would appreciate help with citations for the milestones in the history of the ideas that I am using in this paper.

When a mathematical notion has several different axiomatic formulations that are equivalent as a theorem, we may argue that this is a discovery of nature rather than a human invention. We feel that the textbook definition of a topological space is merely a human convention, whilst the notion of a *locally compact space* is part of nature. Reformulating general topology solely in terms of open subspaces rather than points (Locale Theory) has freed the subject from the ubiquitous reliance on the Axiom of Choice and Excluded Middle, allowing it to be interpreted in the logic of an elementary topos. However, the general definitions of topological space in these two settings do not exactly match, whereas (distributive) continuous lattices do provide exactly the localic account of locally compact spaces.

Dana Scott used continuous lattices to build on the analogy that had long been known between topology and recursion theory, thereby founding the disciplines of domain theory and the denotational semantics of programming languages. Subsequent work in this tradition has allowed topology to be developed on even weaker logical foundations, turning this analogy into a formal equivalence. Abstract Stone Duality is a computable axiomatisation of topology as a λ-calculus that gives yet another characterisation of local compactness.

Any presentation of pure mathematics in a computable form necessarily involves coding, so it is important to develop this in such a form that the manipulations that we want to make for mathematical reasons may be performed in a straightforward way *within* the chosen formalism, without going back to the semantic setting.

A locally compact space is one that has enough compact subspaces for them to determine which basic open subspaces contribute to the expression for a general open subspace. This is similar to the way that a dual basis for a vector space says how much each basic vector contributes towards the expression of a general one. For vector spaces the *number* of basis vectors is an invariant and completely characterises the space, but for other forms of algebra and topology we need more information about the relations amongst the generators.

In our case, this information is provided by saying which (finite collections of) basic open subspaces *cover* the basic compact subspaces.

Achim Jung and Philipp Sünderhauf [JS96] gave a complete axiomatisation of this cover relation on the assumption that finite unions and intersections of basic open subspaces are also basic. They exploited this lattice structure to illustrate Lawson duality between the open subspaces of one space and the compact ones of another. However, to use bases of this kind for real-valued computation would require the manipulation of *lists* of open intervals.

The innovation in the present work is to use "individual" basis elements, such as single intervals in the real case, so that the basis does not have these lattice operations. The outcome of this is that working with abstract bases for general locally compact spaces shows features that are similar to computation with real intervals. In particular, the notion of *roundedness* that was prominent in earlier work with continuous lattices bifurcates, the second form being called *locatedness*.

In this paper we axiomatise this cover relation without lattice structure, not just showing that it satisfies certain conditions but also recovering the locally compact space given only the abstract data.

When we set out to recover a traditional topological space by first defining its points, we find that we can only do so if the basis is countable. In order to overcome the obstacle we need first to construct the continuous lattice of open subspaces and then derive the points. In fact, we find that our abstract bases are most naturally related to Formal Topology, an approach that is founded on Martin-Löf Type Theory, and the continuous lattice or localic account is best obtained from that.

On the other hand, there is still some debate about the most appropriate way in which to define local compactness in Formal Topology. We argue, with reference to what has been said in this debate, that our notion of abstract basis should be adopted as the definition in this discipline.

Meanwhile, in Abstract Stone Duality, the technology for defining particular spaces has undergone several stages in its evolution from the initial categorical idea and it has hitherto been quite laborious to construct individual objects in it. We argue here too that the abstract bases of this paper should be taken as the practical definition.

There are yet other settings in which one might define local compactness, but our thesis is that abstract bases as we define them here provide a common definition that is applicable across all foundational systems and therefore serve as a way of translating data from one to another.

Necessity and sufficiency does not, however, entirely determine how a system of axioms is best formulated, especially when we subsequently intend to work with the axioms alone, instead of with their motivating examples.

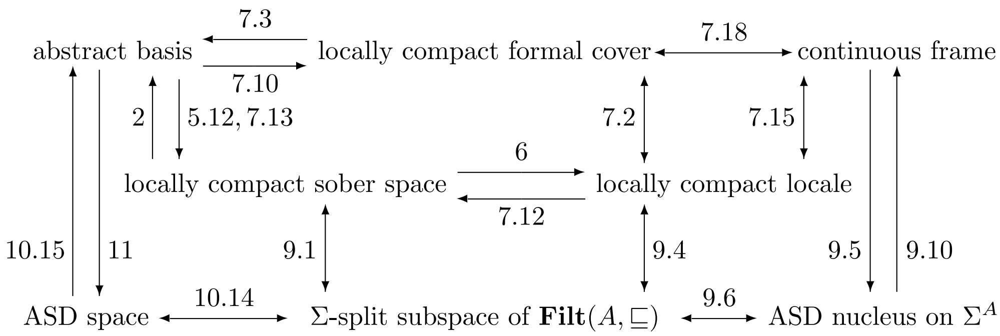
It is usual when introducing the axioms for some mathematical notion to state them in the form that is most natural and convenient for the subsequent *development and applications* of the theory. Sometimes, however, one of the axioms is derivable from the others, such as one of the distributive laws for a ring or lattice. In other cases, there may be some more parsimonious scheme that is less convenient for applications but for which it is easier to *build* models or prove the *fundamental* result of the subject, whilst models of the standard system may be obtained in a straightforward way. (Such a method is called *bootstrapping* in software development.)

Nevertheless, in these cases, the richer system of axioms (in our case consisting of both the primary and secondary ones) is the one that we *export* from the introductory account, the simpler (primary) system being solely for *internal* use. We hope that the system that we export will turn out to be the *definitive* one for abstract bases for locally compact spaces across many foundational settings.

Such a distinction arises in this investigation because (contrary to what may be suggested by real intervals), the identification of which open subspaces are to be treated as "individual" (rather than unions) need not be determined by an intrinsic property such as connectedness, but is a matter for our choice. Beyond the initial goal of justifying *some* complete axiomatisation (which we call the *primary axioms*), we would also like to design one that facilitates computation with the abstract basis alone and without reverting to the space. We find that any given concrete or abstract basis can be modified to yield another that also satisfies convenient *secondary axioms.*

As well as justifying primary and secondary axioms for bases for spaces, we also have to consider continuous functions between them. Continuing the loose analogy with linear algebra, we call the corresponding structures *matrices.* In this case we only satisfy the primary goal, leaving the consideration of more computationally convenient formulations of the axioms for matrices to later work.

The numerous equivalences amongst formulations of local compactness that we consider in this paper are summed up by the following diagram:



From the point of view of the *information content* of these equivalences, it will be convenient in this paper to regard a *locally compact space* as being one that is equipped with a *specified* concrete basis (the family (K_a) in Definition 1.3 or its equivalents). On the other hand, the notion of *continuous lattice* (Proposition 7.15) depends only on the lattice of all open subspaces, not a choice of compact ones, so it contains less information. This is like the distinction between a vector space on its own and one that is equipped with a particular basis.

The next section summarises the primary and secondary axioms for concrete and abstract bases. Section 2 shows that these are satisfied in Point–Set Topology and introduces a weaker notion of concrete basis that corresponds more closely to those that are used in the three constructive disciplines. Section 3 shows how bases obeying the primary axioms may be "improved" to satisfy the secondary ones too. Section 4 characterises continuous functions between spaces with given bases using relations that we call matrices.

Section 5 begins the reconstruction of spaces from abstract bases with the classical (point–set) setting, but only manages this in the countable case. Section 6 introduces Locale Theory and begins the construction of the distributive continuous lattice of open subspaces from the abstract basis. These tasks are completed in Section 7 by introducing Formal Topology, where we recall the different ways in which local compactness has been defined and argue for the use of our abstract bases.

Section 8 shows that there are exponentials (function-spaces) of the form Σ^X, where Σ is the Sierpiński space. Using these, Section 9 shows how bases correspond to inclusions $i : X \hookrightarrow \Sigma^A$ for which there is a map $I : \Sigma^X \hookrightarrow \Sigma^{\Sigma^A}$ with $\Sigma^i \cdot I = \mathsf{id}$.

Section 10 introduces a symbolic calculus (Abstract Stone Duality) that exploits this intrinsic structure and Section 11 demonstrates the equivalence between abstract bases and the nuclei that were used in previous work on ASD.

Most of the formal work in this paper uses the weaker notion of basis, with Scott-open filters of open subspaces. However, in subsequent applications it will be much more convenient to adopt a secondary axiom that amounts to using compact subspaces. The justification of this, which is not as trivial as for the other secondary axioms, is given in Section 12.

It will also be very useful in further work to restrict attention to those spaces in which no basic compact subspace is covered by the empty collection of basic opens. Such spaces are called *overt.*

This property is a computationally natural one, whilst it holds vacuously in the classical setting. This is studied in Section 13.

In the concluding Section 14 we summarise how the results of this paper provide equivalences of categories, where that for each of the four formulations of topology relies on the corresponding logical foundation:

(a) traditional Point–Set Topology in set theory with the Axiom of Choice,

(b) Locale Theory in the logic of an elementary topos,

(c) Formal Topology in Martin-Löf Type Theory and

(d) abstract Stone duality over an arithmetic universe.

Abstract bases therefore provide a unifying framework across these four formulations of topology and we can say logically that a space or continuous function exists in each subject iff it is definable in the appropriate logic.

1 Concrete and abstract bases

We begin with a summary of the axioms and notation for bases that we shall consider in the rest of the paper.

Definition 1.1 In a (not necessarily locally compact) topological space X, a ***concrete basis using open subspaces*** indexed by a preorder $(A, \sqsubseteq)$ consists of

(a) for each element $a \in A$, an open subspace $U_a \subset X$; such that

(b) if $a \sqsubseteq b$ then $U_a \subset U_b$;

(c) if $x \in U_a$ and $x \in U_b$ then $x \in U_c$ for some $c \in A$ with $a \sqsupseteq c \sqsubseteq b$; and

(d) if $x \in U \subset X$ with U open then $x \in U_a \subset U$ for some $a \in A$.

The last part may alternatively be written as $U = \bigcup\{U_a \mid U_a \subset U\}$ and is called the ***basis expansion*** of U. We say "using open subspaces" in this paper to distinguish this widely used notion that is usually just called a basis from our main subject, so please do not use this phrase elsewhere without clear necessity and explanation.

Definition 1.2 A space X is ***locally compact*** if it has the ***interpolation property*** that, given $x \in V \subset X$ with V open, there are $x \in U \subset K \subset V \subset X$ with U open and K compact. This definition is suitable for non-Hausdorff (but sober) spaces and was given by Karl Hofmann and Michael Mislove [HM81]. The interpolation property may easily be extended, replacing the point x by a compact subspace L with $L \subset V$, obtaining $L \subset U \subset K \subset V \subset X$.

Definition 1.3 A ***concrete basis using compact subspaces*** for a locally compact space X is a family of pairs (U_a, K_a) of subspaces of X indexed by a preorder $(A, \sqsubseteq)$ such that

(a) each U_a is open and K_a is compact;

(b) if $a \sqsubseteq b$ then $U_a \subset U_b$, whilst $K_b \subset U \Longrightarrow K_a \subset U$ for any open $U \subset X$;

(c) if $x \in U_a$ and $x \in U_b$ then $\exists c.\, x \in U_c \ \wedge\ (a \sqsupseteq c \sqsubseteq b)$ and

(d) $x \in V \iff \exists a.\, x \in U_a \ \wedge\ K_a \subset V$, or $V = \bigcup\{U_a \mid K_a \subset V\}$.

Part (d) is the basis expansion, in which the compact subspaces K_a are playing a role like that of a ***dual basis*** in linear algebra: they specify *which* basic open subspaces U_a should contribute to the union in the last axiom above.

Proposition 5.14 shows why it is convenient *not* to require $U_a \subset K_a$ or $K_a \subset K_b$. However, if $K_a \subset U$ with U open then $U_a \subset U$ because it contributes to the basis expansion. We shall call these the *primary* axioms for a basis because there are other (*secondary*) ones that it will also be convenient to impose (Definition 1.10).

Remark 1.4 Accounts differ on which way round to write the order relation $\sqsubseteq$. We choose the topological direction (as above and in Section 7) rather than the domain-theoretic one (*cf.* Proposition 5.14), but exponentiation reverses it (Proposition 8.15). Note, however, that we do *not* require $U_a \subset U_b \Longrightarrow a \sqsubseteq b$. Also, we may have both $a \sqsubseteq b$ and $b \sqsubseteq a$ without requiring $a = b$.

In fact, the preorder can be eliminated altogether (Lemma 3.7), but we feel that it is preferably conceptually to retain it. Theorems 6.12 and 9.1 characterise bases of the two kinds above in terms of subspaces of $\mathbf{Filt}(A, \sqsubseteq) \cong \mathbf{Idl}(A, \sqsubseteq^{\mathsf{op}})$. Also, keeping the preorder suggests how it could be generalised to a *category* that would encode a locally compact *topos*, *cf.* [**?**].

Notation 1.5 Because of the nature of compactness, we shall need to use unions of *finite* sets or lists ℓ of basic open–compact pairs. Everything that we do will be consistent with interpreting such ℓ *either* as a list *or* as a finite subset of A and there are computational advantages in maintaining this ambiguity.

The appropriate notion of finiteness here is that introduced by Kazimierz Kuratowski [Kur20], generated from the empty set $\circ$ by adding singletons. We write $\mathsf{Fin}\,(A)$ for either the set of lists or of finite subsets of A, $k \sqcup \ell$ for the union of two lists and $\bigsqcup L$ for the union of a list of lists.

Constructively, it is *decidable* whether any given $\ell \in \mathsf{Fin}\,(A)$ has $\ell = \circ$ or $\exists a.\, a \in \ell$. Also, a general subset of a finite set need not be finite, though it is iff it is decidable.

We adopt the convention that the early letters $(a, \ldots, e)$ of the alphabet denote *individual* members of the indexing set A, those (h, k, ℓ) in the middle are *lists* or *finite subsets* of A and the later ones $(p, \ldots, w)$ are *possibly infinite* subsets.

Notation 1.6 Then we define the ***way-below*** relation

$$a \prec\!\!\!\prec \ell \qquad \text{as} \qquad K_a \subset U_\ell \;\equiv\; \bigcup_{b \in \ell} U_b.$$

The principal goal of this paper is to give the complete axiomatisation of this relation, so that the set A, the preorder $\sqsubseteq$ and the way-below relation $\prec\!\!\!\prec$ will together be enough to describe the locally compact sober space up to isomorphism.

Even though we will not require the basis to have a lattice structure, it is useful to have some notation for it. The operations $\sqcap$ and $\sqcup$ act on *indices* and then we define

$$U_{a \sqcup b} \;\equiv\; U_a \cup U_b \qquad\qquad K_{a \sqcup b} \;\equiv\; K_a \cup K_b$$

$$U_{a \sqcap b} \;\equiv\; U_a \cap U_b \qquad \text{and (NB)} \qquad K_{a \sqcap b} \;\subset\; K_a \cap K_b,$$

along with $\quad U_\circ \equiv K_\circ \equiv \emptyset, \quad U_\bullet \equiv X \quad$ and (if X is compact) $\quad K_\bullet \equiv X$.

Notation 1.7 We extend $\sqsubseteq$ and $\prec\!\!\!\prec$ to lists or finite subsets by writing

$$\begin{array}{lcl}
a \sqsubseteq \ell & \equiv & \exists b \in \ell.\, a \sqsubseteq b \\
a \sqsubseteq \ell_1 \sqcap \ell_2 & \equiv & a \sqsubseteq \ell_1 \;\wedge\; a \sqsubseteq \ell_2 \\
 & \equiv & \exists b_1 \in \ell_1.\, \exists b_2 \in \ell_2.\, b_1 \sqsupseteq a \sqsubseteq b_2 \\
k \sqsubseteq \ell & \equiv & \forall a \in k.\, a \sqsubseteq \ell \;\equiv\; \forall a \in k.\, \exists b \in \ell.\, a \sqsubseteq b \\
a \prec\!\!\!\prec b & \equiv & a \prec\!\!\!\prec \{b\} \\
k \prec\!\!\!\prec \ell & \equiv & \forall a \in k.\, a \prec\!\!\!\prec \ell \\
a \prec\!\!\!\prec \ell_1 \sqcap \ell_2 & \equiv & \exists k.\, a \prec\!\!\!\prec k \;\wedge\; \forall b \in k.\, b \sqsubseteq \ell_1 \sqcap \ell_2 \\
 & \equiv & \exists k.\, a \prec\!\!\!\prec k \;\wedge\; \forall b \in k.\, \exists c_1 \in \ell_1.\, \exists c_2 \in \ell_2.\, c_1 \sqsupseteq b \sqsubseteq c_2 \\
a \prec\!\!\!\prec^1 \ell & \equiv & \exists b \in \ell.\, a \prec\!\!\!\prec b \\
k \prec\!\!\!\prec^1 \ell & \equiv & \forall a \in k.\, a \prec\!\!\!\prec^1 \ell \;\equiv\; \forall a \in k.\, \exists b \in \ell.\, a \prec\!\!\!\prec b.
\end{array}$$

This structure makes $\mathsf{Fin}\,(A)$ with $k \sqsubseteq \ell$ into the free join semilattice on $(A, \sqsubseteq)$.

We are now ready to state our ***primary axioms***.

Definition 1.8 An ***abstract basis*** is a structure $(A, \sqsubseteq, \prec\!\!\!\prec)$ such that

$$\begin{array}{lcll}
 & & a \sqsubseteq a & \textit{reflexivity} \\
a \sqsubseteq b \sqsubseteq c & \Longrightarrow & a \sqsubseteq c & \textit{transitivity} \\
a \sqsubseteq b \prec\!\!\!\prec k \sqsubseteq \ell & \Longrightarrow & a \prec\!\!\!\prec \ell & \textit{co- \& contravariance} \\
(a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1) \;\wedge\; (k \prec\!\!\!\prec \ell_2) & \Longrightarrow & a \prec\!\!\!\prec \ell_1 \sqcap \ell_2 & \textit{weak intersection} \\
a \prec\!\!\!\prec \ell & \Longrightarrow & \exists k.\, a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell. & \textit{Wilker}
\end{array}$$

The final condition honours Peter Wilker's [Wil70] identification of a property like this as a key part of his study of topological function-spaces (*cf.* Section 8). He also anticipated many of the ideas of Locale Theory and continuous lattices that we will use in Section 6. The frequency with which similar properties appear in print without attribution indicates its importance. It allows ***interpolation*** of some k between given $a \prec\!\!\!\prec \ell$, but it is stronger than this because it says that each $b \in k$ is covered by a *single* c with $b \prec\!\!\!\prec c \in \ell$, whereas interpolation only says that the list ℓ covers *collectively*, $b \prec\!\!\!\prec \ell$.

Conversely, the special case of the weak intersection rule with $\ell_1 \equiv \ell_2$ is ***transitivity***:

$$(a \prec\!\!\!\prec k \prec\!\!\!\prec \ell) \;\equiv\; (a \prec\!\!\!\prec k) \wedge (\forall b \in k.\, b \prec\!\!\!\prec \ell) \quad\Longrightarrow\quad (a \prec\!\!\!\prec \ell).$$

In the next section we show that concrete bases in Point–Set Topology obey these primary axioms. Later we shall prove that any abstract basis ***presents*** a locally compact space, *i.e.* it arises from some basis on some such space. We do this in four different formulations of topology, for which respectively different foundational settings are appropriate.

The elements of the set A are intended to be codes that we can use for computation:

Example 1.9 The real line $\mathbb{R}$ has a familiar basis of intervals with endpoints. These are indexed by the set $A \equiv \{\langle d, u\rangle \mid d < u\}$ with $\langle d, u\rangle \sqsubseteq \langle e, t\rangle \equiv (e \le d < u \le t)$, where we may perhaps choose d and u to be dyadic rationals. Then

$$U_{\langle d,u\rangle} \;\equiv\; (d, u) \qquad\text{and}\qquad K_{\langle d,u\rangle} \;\equiv\; [d, u].$$

A typical instance of $a \prec\!\!\!\prec \ell$ in this basis is

$$[d, u] \;\subset\; (e_1, t_1) \cup \cdots \cup (e_n, t_n).$$

We can characterise this arithmetically, *without considering the intervals as sets or quantifying over the real numbers inside them*: up to permutation of the indices and elimination of redundancy, the condition is

$$e_1 < d \;\wedge\; e_2 < t_1 \;\wedge\; e_3 < t_2 \;\wedge\; \cdots \;\wedge\; e_n < t_{n-1} \;\wedge\; u < t_n.$$

In this example, the Wilker property says that we may shrink each of the (e_i, t_i) slightly but maintain the "way-below" property amongst them. On the other hand, the single interpolation rule below says that we may also enlarge $[d, u]$.

This formula for $a \prec\!\!\!\prec \ell$ is clearly very awkward and its analogue for balls in $\mathbb{R}^n$ would be quite unwieldy. However, this is not in practice a difficulty for computation, because we get to *choose* how to divide up a region. There needs to be further investigation of how to specify how $\prec\!\!\!\prec$ is *generated* by such divisions, particularly for product spaces (Remark 8.10), taking account of geometry as well as certain esoteric logical issues (Proposition 7.22). However, for the purposes of this paper we shall stick with the *canonical* relation that arises directly from topology.

We will nevertheless go beyond the fundamental soundness and completeness result for the axioms to represent continuous functions, which will be used for applications such as computation in future work.

Definition 1.10 Even in the present study we often find ourselves wanting to assume that there are enough *individual* basis elements for certain purposes, instead of using *unions* of them. The following ***secondary*** or ***roundedness*** conditions on concrete bases allow us to interpolate *single* basis elements such that

$$(K_a \subset U_\ell) \implies \exists b.\, (K_a \subset U_b) \wedge (K_b \subset U_\ell)$$

$$(K_{b_1} \subset U_a) \wedge (K_{b_2} \subset U_a) \implies \exists b.\, (K_{b_1} \subset U_b) \wedge (K_{b_2} \subset U_b) \wedge (K_b \subset U_a)$$

$$\exists b.\, K_a \subset U_b \qquad\text{and}\qquad \exists b.\, K_b \subset U_a.$$

These are called ***single interpolation***, ***rounded union*** and ***boundedness above*** and ***below***. The equivalent axioms for abstract bases are

$$a \prec\!\!\!\prec \ell \implies \exists b.\, a \prec\!\!\!\prec b \prec\!\!\!\prec \ell$$

$$(b_1 \prec\!\!\!\prec a) \wedge (b_2 \prec\!\!\!\prec a) \implies \exists b. (b_1 \prec\!\!\!\prec b \prec\!\!\!\prec a) \wedge (b_2 \prec\!\!\!\prec b)$$

$$\exists b.\, a \prec\!\!\!\prec b \qquad \text{and} \qquad \exists b.\, b \prec\!\!\!\prec a.$$

It seems to be very difficult to make progress beyond the basic results in this subject — and very easy to make errors — without the single interpolation rule. For example, with it, the list k in the Wilker rule may be taken to be bijective with ℓ, but otherwise k may have to be longer. Even in the simple case of $a \prec\!\!\!\prec b$, we would need to interpolate a *list* in $a \prec\!\!\!\prec k \prec\!\!\!\prec b$, rather than a single member of the basis.

In Section 3 we will show that, given a concrete basis satisfying the primary axioms, there is another basis for the same space that also satisfies the secondary ones. Similarly, any abstract basis has an equivalent one that also obeys the secondary axioms. We may therefore "assume without loss of generality" that our bases have all of these properties.

Definition 1.11 Any basis that uses compact subspaces (Definition 1.3) actually satisfies the ***strong intersection*** rule,

$$(a \prec\!\!\!\prec \ell_1) \wedge (a \prec\!\!\!\prec \ell_2) \implies a \prec\!\!\!\prec \ell_1 \sqcap \ell_2,$$

which is equivalent to the weak rule above together with ***rounded intersection***,

$$(K_a \subset U_{b_1}) \wedge (K_a \subset U_{b_2}) \implies \exists b. (K_a \subset U_b) \wedge (K_b \subset U_{b_1}) \wedge (K_b \subset U_{b_2})$$

or

$$(a \prec\!\!\!\prec b_1) \wedge (a \prec\!\!\!\prec b_2) \implies \exists b. (a \prec\!\!\!\prec b \prec\!\!\!\prec b_1) \wedge (b \prec\!\!\!\prec b_2).$$

Note that, although the *weak* intersection rule implies transitivity, the latter must be stated explicitly alongside the *strong* intersection rule.

It is likely that any *natural* choice of basis will obey all of the rules:

Examples 1.12 Various subsets of the basis of intervals for $\mathbb{R}$ in Example 1.9 illustrate the secondary axioms or their failure:

(a) the basis with *all* bounded intervals obeys all of the secondary axioms with strong intersection and is closed under binary unions and intersections;

(b) the basis with intervals of length < 1 obeys the secondary axioms but does not admit binary unions;

(c) the basis with intervals of length ≤ 1 fails single interpolation and boundedness above;

(d) if the intervals are required to have length 2^n with $n \in \mathbb{Z}$, the single interpolation and rounded union properties fail, *e.g.* for $[1,3] \cup [5,7] \subset (0,8)$; whilst

(e) adding a basis element $*$ with $U_* \equiv \emptyset$ but $K_* \equiv \{0\}$ destroys boundedness below.

The lesson for computation with intervals represented by their centres and radii is that the latter should have arbitrary, not fixed, precision (mantissa). □

Lemmas 4.3 and 8.13 and the following syntactic result show why the roundedness axioms are convenient for working with abstract bases:

Proposition 1.13 Let $(A, \sqsubseteq, \prec\!\!\!\prec)$ be an abstract basis that satisfies the primary, secondary and rounded intersection rules. Let $\phi(a)$ be a formula built from variables of type A, $\mathsf{Fin}\,(A)$, $\mathsf{Fin}\,(\mathsf{Fin}\,(A))$, ..., $\prec\!\!\!\prec$, $\wedge$, $\vee$, $\exists$, membership of finite sets and universal quantification over them (*e.g.* $\forall a \in \ell$). Suppose that $\phi(a)$ holds for a particular value of $a \in A$. Then there are values $a^-, a^+ \in A$ with $a^- \prec\!\!\!\prec a \prec\!\!\!\prec a^+$ such that $\forall a' \in A.\, a^- \prec\!\!\!\prec a' \prec\!\!\!\prec a^+ \implies \phi(a')$.

Proof The base cases $a \prec\!\!\!\prec \ell$ and $b \prec\!\!\!\prec a$ follow from the single interpolation rule and transitivity. For conjunction and universal quantification we use the rounded union and intersection rules. The other logical connectives require a straightforward structural recursion. □

Remark 1.14 We shall need the secondary axioms in Definition 1.10 almost from the outset, but we shall not assume the strong or rounded intersection rules in most of this paper. One reason for this is that we fully embrace non-Hausdorff spaces. In a Hausdorff space, the intersection of two compact spaces is closed in either of them and therefore compact. This need no longer be the case in a non-Hausdorff space, so the space is called ***stably locally compact*** if it is (and ***stably compact*** if the whole space is compact too).

Another is that, in the passage from Point–Set Topology to the formulations in weaker logics that we shall consider, it will be easier to make the analogy amongst them by considering the neighbourhood filter $\mathcal{K}_a \equiv \{U \mid K_a \subset U\}$ instead of the compact subspace K_a. We then find that the filter requirement is not really necessary.

We will explain these issues in the next section.

Finally, whilst it is possible to turn a basis with the weak intersection property into one obeying the strong rule, the construction in Section 12 requires the Axiom of Dependent Choice, which may be undesirable in certain foundational settings.

This completes our introduction to the axiomatisation of bases for locally compact *spaces*, so we do the same for *continuous functions*.

Definition 1.15 Let $f : X \to Y$ be a continuous function between locally compact sober spaces X and Y with concrete bases $\{(U_a, K_a) \mid a \in A\}$ and $\{(V_b, L_b) \mid b \in B\}$ respectively that obey the primary and secondary rules. We define a binary relation between the indices of the bases,

$$\langle\, a \,|\, f \,|\, b \,\rangle \qquad \text{by} \qquad K_a \subset f^{-1}V_b \qquad \text{or equivalently} \qquad fK_a \subset V_b.$$

In particular, $$\langle\, a \,|\, \mathsf{id} \,|\, a' \,\rangle \iff (a \prec\!\!\prec a').$$

We call $\langle\, a \,|\, f \,|\, b \,\rangle$ the ***concrete matrix*** of f, following the loose analogy between bases in topology and in linear algebra that we have already made in Definition 1.3. (The notation was inspired by that of Paul Dirac in Quantum Mechanics, whereas [G] used the notation $\widehat{H}^b_a$ from Albert Einstein's General Relativity.)

The matrix ***represents*** f in the sense that

$$fx \in V_b \iff \exists a.\, (x \in U_a) \;\wedge\; \langle\, a \,|\, f \,|\, b \,\rangle,$$

using the basis expansion of $f^{-1}V_b$. Such matrices are characterised as follows:

Definition 1.16 An ***abstract matrix*** between bases $(A, \sqsubseteq, \prec\!\!\prec)$ and $(B, \sqsubseteq, \prec\!\!\prec)$, is a binary relation $\langle\, a \,|\, f \,|\, b \,\rangle$ between the sets A and B that is ***contravariant*** and ***rounded*** in a,

$$(a \sqsubseteq a') \;\wedge\; \langle\, a' \,|\, f \,|\, b \,\rangle \implies \langle\, a \,|\, f \,|\, b \,\rangle \iff \exists a'.\, (a \prec\!\!\prec a') \;\wedge\; \langle\, a' \,|\, f \,|\, b \,\rangle,$$

and ***covariant*** and ***rounded*** in b,

$$\langle\, a \,|\, f \,|\, b' \,\rangle \;\wedge\; (b' \sqsubseteq b) \implies \langle\, a \,|\, f \,|\, b \,\rangle \iff \exists b'.\, \langle\, a \,|\, f \,|\, b' \,\rangle \;\wedge\; (b' \prec\!\!\prec b),$$

it has the ***partition property***,

$$\langle\, a \,|\, f \,|\, b \,\rangle \;\wedge\; (b \prec\!\!\prec \ell) \implies \exists k.\, (a \prec\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b' \in \ell.\, \langle\, a' \,|\, f \,|\, b' \,\rangle,$$

it is ***bounded***, $\quad \exists k.\, (a \prec\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \,|\, f \,|\, b \,\rangle,$

and ***weakly filtered***, $\quad (a \prec\!\!\prec a') \;\wedge\; \langle\, a' \,|\, f \,|\, b_1 \,\rangle \;\wedge\; \langle\, a' \,|\, f \,|\, b_2 \,\rangle \implies$

$$\exists k\ell.\, (a \prec\!\!\prec k) \;\wedge\; \big(\forall a' \in k.\, \exists b \in \ell.\, \langle\, a' \,|\, f \,|\, b \,\rangle\big) \;\wedge\; \big(\forall b \in \ell.\, b_1 \sqsupseteq b \sqsubseteq b_2\big),$$

or ***strongly*** so if the same holds without $(a \prec\!\!\prec a')$, and it is ***saturated***,

$$(a \prec\!\!\prec k) \;\wedge\; \forall a' \in k.\, \langle\, a' \,|\, f \,|\, b \,\rangle \implies \langle\, a \,|\, f \,|\, b \,\rangle.$$

Beware, however, that we also use the word *saturated* in an unrelated sense in Definition 3.16. The ***saturated composite*** of two such matrices is given by

$$\langle\, a \,|\, f \,;\, g \,|\, c \,\rangle \;\equiv\; \exists k.\, (a \prec\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \,|\, f \,|\, b \,\rangle \;\wedge\; \langle\, b \,|\, g \,|\, c \,\rangle.$$

The partition axiom is a combinatorial form of a very familiar property from real analysis:

Example 1.17 For $f : \mathbb{R} \to \mathbb{R}$ with the interval basis, the partition property expresses uniform ε–δ continuity *à la* Weierstrass: If ℓ is a list of intervals each of width ε that together cover the range of a function, there is a list k of intervals of width δ covering its argument. Then these have the property that, if x_1 and x_2 belong to the same δ-interval, then fx_1 and fx_2 will belong to the same ε-interval. □

Remark 1.18 In conclusion, the definition of an abstract basis that we intend to be used in future work includes *all* of the primary, secondary and strong intersection axioms. You may therefore ask why we did not give them all in Definitions 1.3 and 1.8. This is because

(a) the correspondences between concrete and abstract *bases* in all of the accounts of topology (Sections 6, 7 and 11) use only the primary axioms; although

(b) the direct construction in Point–Set Topology in Section 5 assumes that the abstract basis is countable and has single intersection; whereas

(c) the correspondence between continuous *functions* and matrices requires the secondary axioms too (*cf.* Lemma 4.3); and

(d) including exponentials (function-spaces) in this also needs the strong intersection rule (*cf.* Lemma 8.13).

We will show that the category of locally compact sober spaces and continuous functions is equivalent to the category of bases and matrices that have all of the above properties.

2 Point–Set Topology

We show in this section that any concrete basis using compact subspaces for a locally compact space in traditional Point–Set Topology gives rise to an abstract basis that satisfies the primary axioms. We also introduce a more general form of concrete basis, using Scott-open families, that identifies more precisely the criterion whereby a basic open subspace should contribute to the basis expansion.

We begin with the issues concerning intersections that give rise to the need for the weaker definition:

Lemma 2.1 Any basis using compact subspaces (Definition 1.3) satisfies the ***boundedness*** and ***strong intersection rules*** (Definition 1.11),

$$\exists \ell.\, a \prec\!\!\!\prec \ell \qquad \text{and} \qquad a \prec\!\!\!\prec \ell_1 \wedge a \prec\!\!\!\prec \ell_2 \implies a \prec\!\!\!\prec \ell_1 \sqcap \ell_2,$$

where $\quad a \prec\!\!\!\prec \ell_1 \sqcap \ell_2 \quad$ means $\quad \exists h.\, a \prec\!\!\!\prec h \wedge \forall b \in h.\, \exists c_1 \in \ell_1.\, \exists c_2 \in \ell_2.\, c_1 \sqsupseteq b \sqsubseteq c_2.$

Proof For boundedness, consider the basis expansion of the whole space *quâ* open subspace. This covers the given basic compact subspace K_a, but some finite subset ℓ of this cover suffices.

The hypotheses $a \prec\!\!\!\prec \ell_1$ and $a \prec\!\!\!\prec \ell_2$ of the intersection rule say that

$$K_a \subset U_{\ell_1} \cap U_{\ell_2} \equiv \bigcup \{U_{b_1} \mid b_1 \in \ell_1\} \cap \bigcup \{U_{b_1} \mid b_1 \in \ell_1\}.$$

Using distributivity and part (c) of Definition 1.3, this union is

$$\bigcup \{U_{b_1} \cap U_{b_2} \mid b_1 \in \ell_1,\ b_2 \in \ell_2\} = \bigcup \{U_c \mid \exists b_1 \in \ell_1.\, \exists b_2 \in \ell_2.\, b_1 \sqsupseteq c \sqsubseteq b_2\}.$$

Since K_a is compact, a finite set h of such c suffices to cover it, so

$$K_a \subset U_h \equiv a \prec\!\!\!\prec h \qquad \text{and} \qquad \forall c \in h.\, \exists b_1 \in \ell_1.\, \exists b_2 \in \ell_2.\, b_1 \sqsupseteq c \sqsubseteq b_2,$$

which is the definition of $a \prec\!\!\!\prec \ell_1 \sqcap \ell_2$. □

Definition 2.2 Definition 1.3 and this lemma would have been simpler if the preorder $\sqsubseteq$ had had a formal intersection *operation*, $\sqcap$, satisfying

$$a \sqsupseteq a \sqcap b \sqsubseteq b \qquad \text{and} \qquad a \sqsupseteq c \sqsubseteq b \implies c \sqsubseteq a \sqcap b,$$

whilst the basic subspaces would satisfy

$$U_{a \sqcap b} = U_a \cap U_b \qquad \text{but} \qquad K_{a \sqcap b} \subset K_a \cap K_b,$$

where the containment of compact subspaces need not be an equality. A ***stable abstract basis*** is one that has such a $\sqcap$ operation and also satisfies the boundedness and strong intersection rules.

Examples 2.3 Many important examples do have such an operation:
(a) intervals in $\mathbb{R}$ and *cuboids* in $\mathbb{R}^n$, with geometric intersection for $\sqcap$; and
(b) lists of constraints on data, with conjunction or concatenation for $\sqcap$.

On the other hand,
(c) it is more common to use *balls* as bases for $\mathbb{R}^n$ and other metric spaces, but they need not intersect in balls; but
(d) more fundamentally, the intersection of two compact subspaces in a non-Hausdorff space need not be compact. Consider, for example, an interval $[0,1]$ together with an extra $1'$, or more formally the cokernel of $[0,1) \hookrightarrow [0,1]$.

Besides this, the subspaces need not overlap at all, so we would need a name ($\circ$) for the empty subspace. Keeping track of empty subspaces creates some quite absurd difficulties. For example, in the Tychonov basis for the product of two spaces,

$$(a,b) \prec\!\!\!\prec (a',b') \iff (a \prec\!\!\!\prec a') \wedge (b \prec\!\!\!\prec b') \vee (a \prec\!\!\!\prec \circ) \vee (b \prec\!\!\!\prec \circ)$$

since $K \times L \subset U \times \emptyset$ for *any* compact K and L and open U. In order to avoid this complication when we construct the Tychonov product of two abstract bases, in [work in progress] we shall restrict to the case where $a \prec\!\!\!\prec \circ$ is forbidden, *cf.* Section 13.

Remark 2.4 There are two ways of proceeding without assuming stable local compactness:
(a) in applications we generally prefer to use compact subspaces for the dual basis, but not intersections of them; whilst
(b) in proving the equivalence of various notions in this paper, we replace compact subspaces by something weaker, which does allow us to use intersections of basis elements.

We may easily pass from the first method to the second. The other direction is rather more difficult, so we defer it to Section 12.

Lemma 2.5 For any compact space K, the family $\mathcal{K} \equiv \{V \mid K \subset V\}$ is a ***Scott-open filter***:
(a) if $\mathcal{K} \ni V \subset W$ then $\mathcal{K} \ni W$;
(b) if $\mathcal{K} \ni \bigcup_{i \in I} V_i$ then there is some finite subset $\ell \subset I$ for which $\mathcal{K} \ni \bigcup_{i \in \ell} V_i$;
(c) $\mathcal{K} \ni X$; and
(d) $\mathcal{K} \ni V, W \Longrightarrow \mathcal{K} \ni V \cap W$. $\square$

In fact, so long as the space is sober, *every* Scott-open filter of open subspaces arises in this way (Lemma 3.15). The difficulty in the non-stable case is that there is a conflict between the two uses of intersections: in Definition 2.2 involving compact subspaces and of the open ones in this Lemma.

However, for many purposes, it is unnecessary to use filters. So we can sacrifice the *subspaces* but retain the essence of *compactness*. ***Scott-open families*** satisfy parts (a) and (b). We adopt the habit of writing $\mathcal{K} \ni U$ rather than $U \in \mathcal{K}$, so, if you are not familiar with using Scott-open families, you may pretend that this says $K \subset U$ instead.

We then rewrite Definition 1.3:

Definition 2.6 A ***concrete basis using Scott-open families*** consists of
(a) for each $a \in A$, an open subspace U_a and a Scott-open family $\mathcal{K}_a$ of open subspaces;
(b) if $a \sqsubseteq b$ then $U_a \subset U_b$ and $\mathcal{K}_a \supset \mathcal{K}_b$;
(c) $U_a \cap U_b = \bigcup \{U_c \mid a \sqsupseteq c \sqsubseteq b\}$; and
(d) $V = \bigcup \{U_a \mid \mathcal{K}_a \ni V\}$.

We amend Notation 1.6 by writing

$$a \prec\!\!\!\prec \ell \quad \text{for} \quad \mathcal{K}_a \ni U_\ell \qquad \text{and} \qquad \mathcal{K}_\ell \equiv \bigcap \{\mathcal{K}_b \mid b \in \ell\}.$$

Having such a basis provides an alternative definition of local compactness, in fact the one that we shall use throughout this paper. This is *â priori* weaker, but we show in Section 12 that they are equivalent.

Remark 2.7 It is easy to add intersections ($\sqcap$) to a basis, but at the cost of using Scott-open families instead of compact subspaces. If $(U_a, \mathcal{K}_a)$ is a basis of either kind then, using lists to serve as the formal intersections,

$$U_{(a,b)} \equiv U_a \cap U_b \qquad \text{and} \qquad \mathcal{K}_{(a,b)} \equiv \mathcal{K}_a \cup \mathcal{K}_b,$$

but this union is unlikely to be a filter even if $\mathcal{K}_a$ and $\mathcal{K}_b$ were. In general

$$U_{(\ell)} \equiv \bigcap \{U_a \mid a \in \ell\} \qquad \text{and} \qquad \mathcal{K}_{(\ell)} \equiv \bigcup \{\mathcal{K}_a \mid a \in \ell\}$$

define another basis for the same space such that $\sqcap$ is given by union of lists. Take care not to confuse this construction with the preceding notation; we use parentheses on the subscripts to distinguish them. There seems to be no easy formula for $\prec\!\!\!\prec$.

Lemma 2.8 Any basis using Scott-open families obeys the ***weak intersection rule***,

$$a \prec\!\!\!\prec k \quad \wedge \quad k \prec\!\!\!\prec \ell_1 \quad \wedge \quad k \prec\!\!\!\prec \ell_2 \quad \Longrightarrow \quad a \prec\!\!\!\prec \ell_1 \sqcap \ell_2.$$

Proof The hypothesis $k \prec\!\!\!\prec \ell_1$ says that, for each $b \in k$,

$$\mathcal{K}_b \ni U_{\ell_1} \equiv \bigcup \{U_c \mid c \in \ell_1\},$$

so U_b contributes to the basis expansion of U_ℓ and $U_b \subset U_{\ell_1}$. Since $a \prec\!\!\!\prec k$, it follows that

$$\mathcal{K}_a \ni U_k \equiv \bigcup \{U_b \mid b \in k\} \subset U_{\ell_1} \cap U_{\ell_2},$$

but a Scott-open family $\mathcal{K}_a$ must be closed upwards, so $\mathcal{K}_a \ni U_{\ell_1} \cap U_{\ell_2}$ too. By a similar argument as in Lemma 2.1, but using Scott-openness of $\mathcal{K}_a$ in place of compactness of K_a, there is some finite set h with

$$(\mathcal{K}_a \ni U_h) \equiv (a \prec\!\!\!\prec h) \qquad \text{and} \qquad \forall c \in h.\, \exists b_1 \in \ell_1.\, \exists b_2 \in \ell_2.\, (b_1 \sqsupseteq c \sqsubseteq b_2),$$

which is the definition of $a \prec\!\!\!\prec \ell_1 \sqcap \ell_2$. □

We also need a rule to govern unions, which comes from the following observation:

Lemma 2.9 If a compact subspace is covered by two open ones, $K \subset U_1 \cup U_2$, then there are compact L_1 and L_2 and open V_1, V_2 with $K \subset V_1 \cup V_2$, $V_1 \subset L_1 \subset U_1$ and $V_2 \subset L_2 \subset U_2$. □

Lemma 2.10 A basis of either kind also obeys the ***Wilker rule*** that

$$a \prec\!\!\!\prec \ell \implies \exists k.\ a \prec\!\!\!\prec k \ \wedge\ \forall b \in k.\, \exists c \in \ell.\, b \prec\!\!\!\prec c.$$

Proof Given $\mathcal{K}_a \ni U_\ell \equiv \bigcup \{U_b \mid b \in \ell\}$, the basis expansion of U_b for each $b \in \ell$ yields

$$\mathcal{K}_a \ni U_\ell = \bigcup_{b \in \ell} U_b = \bigcup_{b \in \ell} \bigcup_{c} \{U_c \mid \mathcal{K}_c \ni U_b\} = \bigcup_{c} \{U_c \mid \exists b \in \ell.\, \mathcal{K}_c \ni U_b\}.$$

Since $\mathcal{K}_a$ is a Scott-open family, there is some finite set k of such c for which we still have

$$\mathcal{K}_a \ni \bigcup \{U_c \mid c \in k\} \equiv U_k \qquad \text{and} \qquad \forall c \in k.\, \exists b \in \ell.\, (\mathcal{K}_c \ni U_b),$$

which is what the conclusion says. □

In the rest of the paper we will make heavy use of Scott-open families and it will not surprise you to learn that they are part of a bigger picture:

Proposition 2.11 The Scott-open subsets of any complete lattice form a topology, called the ***Scott topology***. A function $M^* : \Omega_2 \to \Omega_1$ between complete lattices is ***Scott-continuous***, *i.e.* with respect to this topology, iff it preserves ***directed joins***, written $\bigvee^{\uparrow}$ or $\bigcup^{\uparrow}$. These are joins of families $\{U_i \mid i \in I\}$ for which

$$\exists i.\, i \in I \qquad \text{and} \qquad i_1, i_2 \in I \implies \exists i \in I.\ U_{i_1} \leqslant U_i \geqslant U_{i_1}.$$ □

In fact, we shall see in Proposition 8.15 that this is the topology on the topology on a locally compact space X that defines the exponential (function-space) Σ^X.

3 Manipulating bases

In this section we show how to "upgrade" a concrete basis satisfying the primary axioms to one that obeys the secondary ones too. As corollaries, we obtain bases for open or closed subspaces and show how to eliminate the preorder $\sqsubseteq$ from an abstract basis. We also show how "formal" points and compact subspaces may be derived from the lattice of open subspaces.

Although these constructions also upgrade abstract bases, it is difficult to define how the new bases are equivalent to the given ones, *cf.* Remark 4.22. It seems to be necessary to go *via* the spaces and concrete bases that we will construct, which fortunately rely only on the primary axioms for abstract bases.

Although it is a major goal of this paper to develop bases that do *not* have be closed under unions or intersections, some of the issues that we shall discuss do need the former. In the following results, let $\{(U_a, \mathcal{K}_a) \mid a \in A\}$ be any basis using Scott-open families for a locally compact space X.

Proposition 3.1 The ***directed basis*** consists of

$$U_\ell \;\equiv\; \bigcup\{U_a \mid a \in \ell\} \qquad \text{and} \qquad \mathcal{K}_\ell \;\equiv\; \bigcap\{\mathcal{K}_a \mid a \in \ell\}.$$

Proof For the filtered condition on basic opens (Definition 1.3(c)),

$$\begin{aligned} x \in U_k \;\wedge\; x \in U_\ell \quad &\equiv\quad \exists a \in k.\, \exists b \in \ell.\, x \in U_a \;\wedge\; x \in U_b \\ &\Rightarrow\quad \exists abc.\, x \in U_c \;\wedge\; k \ni a \sqsupseteq c \sqsubseteq b \in \ell \\ &\Rightarrow\quad \exists h.\, x \in U_h \;\wedge\; k \sqsupseteq h \sqsubseteq \ell, \end{aligned}$$

where $h \equiv \{c\}$. The basis expansion (Definition 1.3(d)) is

$$\begin{aligned} x \in V \quad &\Leftrightarrow\quad \exists a.\, x \in U_a \;\wedge\; \mathcal{K}_a \ni V \;\Longrightarrow\; \exists \ell.\, x \in U_\ell \;\wedge\; \mathcal{K}_\ell \ni V \\ &\equiv\quad \exists \ell a.\, a \in \ell \;\wedge\; x \in U_a \;\wedge\; \forall b \in \ell.\, \mathcal{K}_b \ni V \;\Longrightarrow\; \exists a.\, x \in U_a \;\wedge\; \mathcal{K}_a \ni V. \end{aligned}$$

Using Notation 1.5, the way-below relation is

$$\begin{aligned} k \prec\!\!\!\prec_{\mathrm{dir}} L \quad &\equiv\quad \mathcal{K}_k \ni \bigcup\{U_a \mid \exists \ell.\, a \in \ell \in L\} \\ &\Leftrightarrow\quad \forall b \in k.\, \mathcal{K}_b \ni U_{\bigsqcup L} \;\equiv\; k \prec\!\!\!\prec_A \bigsqcup L. \end{aligned}$$

This inherits co- and contravariance, the Wilker and intersection rules, essentially as they stand. □

Lemma 3.2 The directed basis obeys the single interpolation and rounded union rules.

Proof The interpolation property for $(U_a, \mathcal{K}_a)$ gives *single* interpolation for $(U_\ell, \mathcal{K}_\ell)$,

$$k \prec\!\!\!\prec_{\mathrm{dir}} L \;\equiv\; k \prec\!\!\!\prec_A \bigsqcup L \;\Longrightarrow\; \exists h.\, k \prec\!\!\!\prec_A h \prec\!\!\!\prec_A \bigsqcup L \;\equiv\; \exists h.\, k \prec\!\!\!\prec_{\mathrm{dir}} \{h\} \prec\!\!\!\prec_{\mathrm{dir}} L.$$

For rounded binary unions,

$$\begin{aligned} \{\ell_1, \ell_2\} \prec\!\!\!\prec_{\mathrm{dir}} k \quad &\equiv\quad \ell_1 \sqcup \ell_2 \prec\!\!\!\prec_A k \\ &\Rightarrow\quad \ell_1 \sqcup \ell_2 \prec\!\!\!\prec_A h \prec\!\!\!\prec_A k \\ &\equiv\quad \{\ell_1, \ell_2\} \prec\!\!\!\prec_{\mathrm{dir}} h \prec\!\!\!\prec_{\mathrm{dir}} k \end{aligned}$$

using the interpolation property that we already have. □

It follows that, for any locally compact space with a basis satisfying the primary axioms, *there exists* another that also obeys single interpolation and rounded union. It is a little unsatisfying that the only way that we know how to construct a basis having these useful extra properties is to sacrifice the one that is the main purpose of the paper, but, as Examples 1.12 illustrated, "naturally occurring" bases probably already come with these properties anyway and only with perverse choices do they fail.

The boundedness properties, on the other hand, are easy to achieve, in a canonical way, just by discarding the redundant members.

Lemma 3.3 Let $\underline{A} \equiv \{b \mid \exists a.\, a \prec\!\!\!\prec b\}$. Then $\{(U_a, \mathcal{K}_a) \mid a \in \underline{A}\}$ is a basis for the same space and is bounded below.

Proof Only $\underline{A}$ contributes to the concrete basis expansion, because

$$x \in U_b \iff \exists a \in A.\, x \in U_a \ \wedge\ K_a \subset U_b \implies \exists a \in A.\, a \prec\!\!\!\prec b \ \equiv\ b \in \underline{A},$$

so $\underline{A}$ still satisfies parts (c,d) of Definition 1.3. Then $\underline{A}$ is bounded below using either the same argument again or single interpolation. Alternatively, we have $a \prec\!\!\!\prec d \implies \exists bc.\, b \prec\!\!\!\prec c \prec\!\!\!\prec d$ because the Wilker property gives $a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell \prec\!\!\!\prec d$ and then

- either $k = \circ$, in which case $a \prec\!\!\!\prec \circ \sqsubseteq a \prec\!\!\!\prec d$, so we take $b \equiv c \equiv a$, or
- there are $b \in k$ and $c \in \ell$ with $b \prec\!\!\!\prec c \prec\!\!\!\prec d$. □

Lemma 3.4 Let $\overline{A} \equiv \{a \mid \exists \ell.\, a \prec\!\!\!\prec \ell\}$. Then $\{(U_a, \mathcal{K}_a) \mid a \in \overline{A}\}$ is a basis for the same space and is bounded above.

Proof First observe that

$$\mathcal{K}_a \ni U \ = \ \bigcup^{\uparrow}\{U_\ell \mid U_\ell \subset U\} \implies \exists \ell.\, \mathcal{K}_a \ni U_\ell \subset U \implies \exists \ell.\, a \prec\!\!\!\prec \ell \ \equiv\ a \in \overline{A},$$

since the family (U_ℓ) provides a directed basis and $\mathcal{K}_a$ is Scott-open. Hence the basis expansion is

$$\begin{aligned} x \in U \quad &\Leftrightarrow \quad \exists a.\, (x \in U_a) \ \wedge\ (\mathcal{K}_a \ni U) \\ &\Leftrightarrow \quad \exists a.\, (x \in U_a) \ \wedge\ (\mathcal{K}_a \ni U) \ \wedge\ (\exists \ell.\, a \prec\!\!\!\prec \ell) \\ &\equiv \quad \exists a \in \overline{A}.\, (x \in U_a) \ \wedge\ (\mathcal{K}_a \ni U). \end{aligned}$$

The subset $\overline{A}$ is downwards-closed with respect to $\sqsubseteq$ and $\prec\!\!\!\prec$ because of contravariance and transitivity of $\prec\!\!\!\prec$. Hence the concrete basis still has the filtered property and the abstract one still obeys the Wilker and intersection rules:

$$a \prec\!\!\!\prec \ell \subset \overline{A} \implies \exists k.\, a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell \ \wedge\ k \subset \overline{A}$$

$$a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1 \subset \overline{A} \ \wedge\ k \prec\!\!\!\prec \ell_2 \subset \overline{A} \implies \exists \ell'.\, a \prec\!\!\!\prec \ell' \sqsubseteq \ell_1 \ \wedge\ \ell' \sqsubseteq \ell_2 \ \wedge\ \ell' \subset \overline{A}. \qquad \square$$

Boundedness above is related to open subspaces:

Lemma 3.5 A concrete basis for an open subspace $V \subset X$ is given by

$$U_a^V \ \equiv\ U_a \cap V \qquad \text{and} \qquad \mathcal{K}_a^V \ \equiv\ \mathcal{K}_a \cap {\downarrow} V.$$

If the given basis for X uses compact subspaces then that for V has

$$a \prec\!\!\!\prec^V \ell \iff a \prec\!\!\!\prec^X \ell \ \wedge\ (K_a \subset V)$$

and then $\overline{A} \equiv \{a \mid K_a \subset V\}$ provides a basis for V that is bounded above.

Proof The basis expansion of $x \in U \subset V$ is

$$\begin{aligned} x \in U \quad &\Leftrightarrow \quad \exists a.\, x \in U_a \ \wedge\ \mathcal{K}_a \ni U \\ &\Leftrightarrow \quad \exists a.\, x \in (U_a \cap V) \ \wedge\ (\mathcal{K}_a \ni U \subset V). \end{aligned}$$

The filter property is

$$\begin{aligned} x \in U_a^V \ \wedge\ x \in U_b^V \quad &\Leftrightarrow \quad x \in U_a \ \wedge\ x \in U_b \ \wedge\ x \in V \\ &\Leftrightarrow \quad \exists c.\, x \in (U_c \cap V) \ \wedge\ (a \sqsupseteq c \sqsubseteq b). \end{aligned} \qquad \square$$

Roughly speaking, the basis for the complementary closed subspace consists of the members of the basis that we discarded to obtain the open subspace. Unfortunately, this is not constructive and in Section 13 we investigate when it is possible to eliminate empty covers.

Lemma 3.6 A basis for a closed subspace $C \subset X$ is given by

$$U_a^C \ \equiv\ U_a \cup V \qquad \text{and} \qquad \mathcal{K}_a^C \ \equiv\ \mathcal{K}_a,$$

where V is the complementary open subspace to C (*cf.* Proposition 8.2). Hence

$$a \prec\!\!\prec^C \ell \iff \exists k.\ (a \prec\!\!\prec k \sqcup \ell) \wedge (\mathcal{K}_k \ni V).$$

Proof If $x \in C$, so $x \notin V$, then

$$x \in (U_a \cup V) \wedge x \in (U_b \cup V) \iff \exists c.\, x \in (U_c \cup V) \wedge (a \sqsupseteq c \sqsubseteq b)$$

and
$$x \in W \iff \exists a.\, x \in (U_a \cup V) \wedge \mathcal{K}_a \ni W.$$

Notice in particular that $(a \prec\!\!\prec^C \circ)$ if $\mathcal{K}_a \ni V$. □

Next we have some applications of single interpolation. The first eliminates the preorder $\sqsubseteq$, more or less just by replacing it with $\prec\!\!\prec$:

Lemma 3.7 Any abstract basis $(A, \sqsubseteq, \prec\!\!\prec)$ with single interpolation satisfies

$$a \prec\!\!\prec k \prec\!\!\prec \ell \implies a \prec\!\!\prec \ell \implies \exists b.\, a \prec\!\!\prec b \prec\!\!\prec \ell$$

and
$$a \prec\!\!\prec k \prec\!\!\prec \ell_1, \ell_2 \implies \exists k'.\, a \prec\!\!\prec k' \prec\!\!\prec^1 \ell_1, \ell_2.$$

If $a \prec\!\!\prec b$ then $U_a \subset U_b$ and $\mathcal{K}_a \supset \mathcal{K}_b$ in the concrete basis, where the filter property is

$$x \in U_a \wedge x \in U_b \implies \exists d.\, x \in U_d \wedge (a \succ\!\!\succ d \prec\!\!\prec b).$$

Conversely, any relation $\prec\!\!\prec$ with these properties defines an abstract basis $(A, \sqsubseteq, \prec\!\!\prec)$ by

$$a \sqsubseteq b \equiv a \prec\!\!\prec b \vee a = b.$$

Proof We deduce the second property from the weak intersection, Wilker and covariance rules:

$$a \prec\!\!\prec k \prec\!\!\prec \ell_1, \ell_2 \implies \exists k'\ell.\, a \prec\!\!\prec k' \prec\!\!\prec^1 \ell \sqsubseteq \ell_1, \ell_2 \implies \exists k'.\, a \prec\!\!\prec k' \prec\!\!\prec^1 \ell_1, \ell_2.$$

In a concrete basis, $a \prec\!\!\prec b \equiv \mathcal{K}_a \ni U_b \implies U_a \subset U_b$ since U_a contributes to the basis expansion of U_b. Similarly, $a \prec\!\!\prec b \wedge \mathcal{K}_b \ni U \implies \mathcal{K}_a \ni U_b \subset U \implies \mathcal{K}_a \ni U$ since $\mathcal{K}_a$ is upper.

If $x \in U_a$ and $x \in U_b$ then $x \in U_c$ for some $c \in A$ with $a \sqsupseteq c \sqsubseteq b$, then the basis expansion of U_c gives some $d \in A$ with $x \in U_d$ and $\mathcal{K}_d \ni U_c$, so $d \prec\!\!\prec c$ and $a \succ\!\!\succ d \prec\!\!\prec b$.

For the converse, we prove transitivity of $\sqsubseteq$ by an easy case analysis, the extension of which to (Kuratowski) finite sets or lists gives covariance of $\prec\!\!\prec$ with respect to $\sqsubseteq$:

$$\begin{aligned} b \prec\!\!\prec k \sqsubseteq \ell &\Rightarrow \exists k_1 k_2 k'.\, (b \prec\!\!\prec k' \prec\!\!\prec^1 k = k_1 \sqcup k_2) \wedge (k_1 \prec\!\!\prec^1 \ell) \wedge (k_2 \subset \ell) \\ &\Rightarrow \exists k'.\, (b \prec\!\!\prec k' \prec\!\!\prec^1 \ell). \end{aligned}$$ □

The following technical result sharpens the Wilker and weak intersection rules.

Lemma 3.8 If $(a \prec\!\!\prec b \prec\!\!\prec \ell)$ then $\exists k.\, (a \prec\!\!\prec k \prec\!\!\prec b) \wedge (k \prec\!\!\prec^1 \ell)$.

Proof By the Wilker and single interpolation rules (twice), there are a', b' and ℓ' with

$$a \prec\!\!\prec a' \prec\!\!\prec b' \prec\!\!\prec b \prec\!\!\prec \ell' \prec\!\!\prec^1 \ell, \qquad \text{so} \qquad a' \prec\!\!\prec \ell'.$$

Then $a \prec\!\!\prec b' \sqcap \ell'$ by the weak intersection rule, *i.e.* there is k such that

$$a \prec\!\!\prec k \sqsubseteq b' \prec\!\!\prec b \qquad \text{and} \qquad k \sqsubseteq \ell' \prec\!\!\prec^1 \ell'.$$

Then $k \prec\!\!\prec b$ and $k \prec\!\!\prec^1 \ell$ as required. □

Turning to the rules for binary intersections, first we observe that the strong and rounded rules are equivalent:

Lemma 3.9 Suppose that $(A, \sqsubseteq, \prec\!\!\prec)$ satisfies the covariance, transitivity and single interpolation rules. Then it obeys the strong intersection rule,

$$(a \prec\!\!\prec \ell_1) \wedge (a \prec\!\!\prec \ell_2) \implies a \prec\!\!\prec \ell_1 \sqcap \ell_2$$

iff it obeys both the weak intersection rule

$$(a \prec\!\!\!\prec b \prec\!\!\!\prec \ell_1) \ \wedge\ (b \prec\!\!\!\prec \ell_2) \implies a \prec\!\!\!\prec \ell_1 \sqcap \ell_2$$

(with a singleton b instead of a set k) and the ***rounded intersection*** rule

$$(a \prec\!\!\!\prec c_1) \ \wedge\ (a \prec\!\!\!\prec c_2) \implies \exists b.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec c_1) \ \wedge\ (b \prec\!\!\!\prec c_2).$$

Proof The weak rule follows from the strong one by transitivity. The strong rule, single interpolation and covariance give

$$(a \prec\!\!\!\prec c_1) \ \wedge\ (a \prec\!\!\!\prec c_2) \implies \exists kb.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec k \sqsubseteq c_1 \sqcap c_2) \implies \exists b.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec c_1) \ \wedge\ (b \prec\!\!\!\prec c_2).$$

Conversely, single interpolation, rounded intersection, transitivity and weak intersection give

$$\begin{array}{rcl} (a \prec\!\!\!\prec \ell_1) \ \wedge\ (a \prec\!\!\!\prec \ell_2) & \Rightarrow & \exists c_1 c_2.\, (a \prec\!\!\!\prec c_1 \prec\!\!\!\prec \ell_1) \ \wedge\ (a \prec\!\!\!\prec c_2 \prec\!\!\!\prec \ell_2) \\ & \Rightarrow & \exists b c_1 c_2.\, (a \prec\!\!\!\prec b \prec\!\!\!\prec c_1 \prec\!\!\!\prec \ell_1) \ \wedge\ (b \prec\!\!\!\prec c_2 \prec\!\!\!\prec \ell_2) \\ & \Rightarrow & a \prec\!\!\!\prec \ell_1 \sqcap \ell_2. \end{array}$$

□

We need to check that Proposition 3.1 preserves these rules:

Lemma 3.10 If the given basis has the strong intersection property then so does the directed basis.

Proof The rounded intersection property for the directed basis,

$$h \prec\!\!\!\prec \ell_1 \ \wedge\ h \prec\!\!\!\prec \ell_2 \implies \exists k.\, h \prec\!\!\!\prec k \ \wedge\ k \sqsubseteq \ell_1 \sqcap \ell_2,$$

is the same as the strong intersection property for the given one and we deduce strong intersection for the directed basis using Lemma 3.9. □

The key idea for converting a concrete basis that uses Scott-open families into one that uses compact subspaces and for imposing the strong intersection rule on an abstract basis is due to Jimmie Lawson [GHK+80, §I 3.3] and depends on the axiom of Dependent Choice. We present the argument for abstract bases because shall want to adapt it. Where the following results use abstract bases, they only rely on the primary axioms, not single interpolation.

Lemma 3.11 Let $a \in r \subset A$ where r is ***rounded***,

$$r \ni b \iff \exists c.\, r \ni c \prec\!\!\!\prec b.$$

Then there is a ***$\prec\!\!\!\prec$-filter*** s with $a \in s \subset r$, *i.e.*

$$\exists a.\, a \in s, \qquad a \in s \ni b \iff \exists c \in s.\, a \succ\!\!\!\succ c \prec\!\!\!\prec b.$$

Proof By repeated use of roundedness of r and Dependent Choice, there is a sequence

$$\cdots \prec\!\!\!\prec a_3 \prec\!\!\!\prec a_2 \prec\!\!\!\prec a_1 \prec\!\!\!\prec a_0 \ \equiv\ a$$

all of whose members belong to r. Then let $s \equiv \{b \mid \exists i.\, a_i \prec\!\!\!\prec b\}$.

Then $a \in s$ because $a_1 \prec\!\!\!\prec a_0 \equiv a$.

Also s is upper because if $b \in s$ with $b \prec\!\!\!\prec b'$ or $b \sqsubseteq b'$ then $\exists i.\, a_i \prec\!\!\!\prec b \prec\!\!\!\prec b'$ and $b' \in s$.

Also s is a $\prec\!\!\!\prec$-filter because if $a_{i_1} \prec\!\!\!\prec b_1$ and $a_{i_2} \prec\!\!\!\prec b_2$ then with $i = \mathsf{max}(i_1, i_2)+1$, $a_i \prec\!\!\!\prec a_{i_1} \prec\!\!\!\prec b_1$ and $a_i \prec\!\!\!\prec a_{i_2} \prec\!\!\!\prec b_2$. □

Now recall from Lemma 2.5 that any compact subspace $K \subset X$ gives rise to a Scott-open filter $\mathcal{K} \equiv \{U \mid K \subset U\}$. Filters in the abstract basis with respect to $\prec\!\!\!\prec$ also give rise to Scott-open filter of open subspaces; Corollary 8.16 will prove the converse, if the basis is directed.

Lemma 3.12 Let $(U_a, \mathcal{K}_a)$ be a basis using Scott-open families and let $s \subset A$ a $\prec\!\!\!\prec$-filter as in the previous result. Then $\mathcal{K} \ \equiv\ \{U \mid \exists a \in s.\, \mathcal{K}_a \ni U\}$ is a Scott-open filter of open subspaces.

Proof It is Scott-open since the $\mathcal{K}_a$ are and we have $\mathcal{K} \ni X$ because s is inhabited. If $\mathcal{K} \supset \mathcal{K}_a \ni V$ and $\mathcal{K} \supset \mathcal{K}_b \ni V$ then there are $d \prec\!\!\prec c \prec\!\!\prec a, b$ in s and so by the weak intersection rule there is some k with $d \prec\!\!\prec k \sqsubseteq a, b$, so

$$\mathcal{K}_d \ \ni\ U_k \ \subset\ U_a \cap U_b \ \subset\ U \cap V$$

and $\mathcal{K} \supset \mathcal{K}_d \ni U \cap V$ since it's upper. □

Before we can show that *every* Scott-open filter arises from a compact subspace, we need to know how to express points in terms of open subspaces. Of course, any singleton is a compact subspace, so we have to add a condition to Scott-open filters.

Definition 3.13 A ***completely co-prime filter*** or ***formal point*** for the topology on X is a family $\mathcal{P}$ of open subspaces of X such that

$$\mathcal{P} \ni X, \qquad \mathcal{P} \ni U, V \iff \mathcal{P} \ni U \cap V \qquad \text{and} \qquad \mathcal{P} \ni \bigcup U_i \iff \exists i.\, \mathcal{P} \ni U_i.$$

In particular, for every ordinary point $x \in X$, the ***neighbourhood filter*** $\mathcal{P}_x \equiv \{U \mid x \in U\}$ is a formal point. Hence we say that a formal point $\mathcal{P}$ ***lies in*** an open subspace U if $\mathcal{P} \ni U$, inverting the traditional membership relation.

Then a space X is ***sober*** if every formal point is of this form for some unique ordinary point $x \in X$. Sobriety is often stated as requiring that every irreducible closed subspace C is the closure of a unique point p. This is equivalent to our definition, with

$$\mathcal{P} \ \equiv\ \{U \mid U \cap C = \emptyset\} \qquad \text{and} \qquad C \ \equiv\ X \setminus \bigcup \{U \mid \mathcal{P} \not\ni U\},$$

so that $U \cap C = \emptyset \iff \mathcal{P} \ni U \iff x \in U$.

Containment, $\mathcal{P}_1 \subset \mathcal{P}_2$, of one formal point in another is called the ***specialisation order***, as is the corresponding relation between ordinary points.

We are now ready to give the characterisation of compact subspaces, due to Karl Hofmann and Michael Mislove [HM81]. Beware that it requires the space to be sober, though not necessarily locally compact.

Lemma 3.14 Let $\mathcal{K} \subset \Omega$ be a Scott-open filter with $\mathcal{K} \not\ni U$. Then there is a maximal Scott-open filter $\mathcal{P}$ with $\mathcal{K} \subset \mathcal{P} \subset \Omega$ but $\mathcal{P} \not\ni U$, and then $\mathcal{P}$ is completely coprime.

Proof This is based on a well known argument for commutative rings, using Zorn's Lemma, but see [Joh82, Lemma VII 4.3] for an explicit proof for lattices of open subspaces. □

Proposition 3.15 Any Scott-open filter $\mathcal{K}$ of open subspaces of a sober space satisfies

$$\mathcal{K} \ni U \iff K \subset U \qquad \text{where} \qquad K \ \equiv\ \bigcap \mathcal{K} \quad \text{is compact.}$$

Proof If $\mathcal{K} \ni U$ then $K \subset U$ by definition of $\bigcap \mathcal{K}$. Conversely, by Lemma 3.14, if $\mathcal{K} \not\ni U$ then there is a formal point $\mathcal{P}$ with $\mathcal{K} \subset \mathcal{P} \not\ni U$, so by sobriety (Definition 3.13) there is a (concrete) point p with $p \in V \iff \mathcal{P} \ni V$. Hence $p \in K$ but $p \notin U$, as required. The subspace K is compact because its neighbourhood filter $\mathcal{K}$ is Scott-open. □

Definition 3.16 We therefore call any Scott-open filter $\mathcal{K}$ a ***formal compact subspace***. However, Proposition 5.14 illustrates that not every (concrete) compact subspace is the intersection of its neighbourhoods like this; one that does so is called ***saturated***, although this use of the word is unrelated to that in Definition 1.16. We say that a formal point $\mathcal{P}$ ***lies in*** a (saturated) formal compact subspace $\mathcal{K}$ if $\mathcal{P} \supset \mathcal{K}$, whilst an open subspace U ***covers*** $\mathcal{K}$ if $\mathcal{K} \ni U$.

Proposition 3.17 If the abstract basis satisfies the boundedness and strong intersection rules then each Scott-open family $\mathcal{K}_a$ is a filter and $K_a \equiv \bigcap \mathcal{K}_a$ is a compact subspace with $K_a \subset U \iff \mathcal{K}_a \ni U$. Then the basis expansion is

$$x \in U \iff \exists a.\, x \in U_a \wedge K_a \subset U \qquad \text{or} \qquad U \ =\ \bigcup \{U_a \mid K_a \subset U\}.$$

We will describe K_a more explicitly in terms of the abstract basis in Theorem 5.12.

Proof Each $a \in A$ has some $a \prec\!\!\prec b$ by boundedness and $U_b \subset X$, so $\mathcal{K}_a \ni X$.

If $\mathcal{K}_a \ni U, V$ then $U_k \subset U$ and $U_\ell \subset V$ with $a \prec\!\!\prec k, \ell$, so $a \prec\!\!\prec h \sqsubseteq k, \ell$ for some h by the strong intersection rule, but then $U_h \subset U_k \subset U$ and similarly $U_h \subset V$ and $U_h \subset U \cap V$, making $\mathcal{K}_a \ni U \cap V$. So $\mathcal{K}_a \ni U \iff K_a \subset U$ by Proposition 3.15 and the basis expansion follows. □

For most of the rest of this paper it will be more convenient to define a locally compact space to be one that has a basis using Scott-open families. In fact, we have just given most of the proof that such a space also has a basis using compact subspaces, but we defer the rest of the argument to Section 12, where we formulate it in terms of the abstract basis instead.

4 Continuous maps

In this section (alone) it would be possible throughout to use either compact subspaces ($K \subset$) or Scott-open families ($\mathcal{K} \ni$) and the LaTeX source has a switch to allow both of them. I would appreciate the views of readers on which would be clearer. It is currently set to use Scott-open families.

Having described concrete and abstract *bases* for locally compact *spaces*, we shall now do the same for *continuous functions*, which we shall characterise using binary relations that we call *matrices*. The results in this section make essential use of the secondary axioms.

Notation 4.1 Let $f : X \to Y$ be a continuous function between locally compact sober spaces that have bases (U_a, K_a) and (V_b, L_b) respectively using compact subspaces. The ***concrete matrix*** for f is the binary relation $\langle\ | f |\ \rangle$ that is defined by

$$\langle a \,|\, f \,|\, b \rangle \;\equiv\; (f K_a \subset V_b) \;\equiv\; (K_a \subset f^{-1} V_b) \;\equiv\; (\mathcal{K}_a \ni f^{-1} V_b),$$

where the last form is the one that we use for Scott-open families. In particular,

$$\langle a \,|\, \mathsf{id} \,|\, b \rangle \;\equiv\; (a \prec\!\!\prec b).$$

We will characterise matrices for continuous functions by the axioms in Definition 1.15. In fact, we can replace f^{-1} in this notation by any Scott-continuous operator $M^* : \Omega Y \to \Omega X$ (Proposition 2.11):

$$\langle a \,|\, M \,|\, b \rangle \;\equiv\; (\mathcal{K}_a \ni M^* V_b),$$

although the correspondence only works properly when either the bases are directed or M^* preserves all unions.

Lemma 4.2 For any Scott-continuous operator M^*, the concrete matrix $\langle a \,|\, M \,|\, b \rangle$ is contravariant and saturated in a and covariant in b. It also satisfies

$$M^* V_b \;=\; \bigcup_a \{U_a \mid \langle a \,|\, M \,|\, b \rangle\} \;=\; \bigcup^{\uparrow}_k \{U_k \mid \forall a \in k.\, \langle a \,|\, M \,|\, b \rangle\}.$$

Proof The variance properties follow from those of $\mathcal{K}_a$ and V_b (Definition 1.3(b)) and monotonicity of M^*. The last part is the basis expansion of $M^* V_b$, from which we deduce

$$\mathcal{K}_a \ni M^* V_b \iff \exists k.\, \mathcal{K}_a \ni U_k \;\wedge\; \forall a' \in k.\, \mathcal{K}_{a'} \ni M^* V_b$$

since $\mathcal{K}_a$ is Scott-open. Hence the matrix is ***saturated*** in a:

$$\langle a \,|\, M \,|\, b \rangle \iff \exists k.\, (a \prec\!\!\prec k) \;\wedge\; \forall a' \in k.\, \langle a' \,|\, M \,|\, b \rangle. \qquad \square$$

We can improve on this using the ideas of the previous section:

Lemma 4.3 If the bases obey the single interpolation, rounded union and boundedness below properties (Definition 1.10) then the concrete matrix is rounded on both sides.

Proof By single interpolation for a within the saturation property of the previous result,

$$\begin{aligned}
\langle\, a \,|\, M \,|\, b \,\rangle \quad & \Leftrightarrow \quad \exists k.\, (a \prec\!\!\prec k) \;\wedge\; \forall a'' \in k.\, \langle\, a'' \,|\, M \,|\, b \,\rangle \\
& \Leftrightarrow \quad \exists a'k.\, (a \prec\!\!\prec a' \prec\!\!\prec k) \;\wedge\; \forall a'' \in k.\, \langle\, a'' \,|\, M \,|\, b \,\rangle \\
& \Leftrightarrow \quad \exists a'.\, (a \prec\!\!\prec a') \;\wedge\; \langle\, a' \,|\, M \,|\, b \,\rangle,
\end{aligned}$$

we deduce roundedness in a.

The expansion of V_b with respect to the directed basis (Proposition 3.1) is

$$V_b \;=\; \bigcup_{\ell}{}^{\uparrow} \{V_\ell \mid \forall b' \in \ell.\, \mathcal{L}_{b'} \ni V_b\} \;\equiv\; \bigcup_{\ell}{}^{\uparrow} \{V_\ell \mid \ell \prec\!\!\prec b\},$$

so, since M^* is Scott-continuous and $\mathcal{K}_a$ is Scott-open,

$$\begin{aligned}
\langle\, a \,|\, M \,|\, b \,\rangle \quad & \equiv \quad \mathcal{K}_a \ni M^* V_b \;=\; \bigcup_{\ell}{}^{\uparrow} \{M^* V_\ell \mid \ell \prec\!\!\prec b\} \\
& \Leftrightarrow \quad \exists \ell.\, \mathcal{K}_a \ni M^* V_\ell \;\wedge\; (\ell \prec\!\!\prec b) \\
& \Leftrightarrow \quad \exists \ell b'.\, \mathcal{K}_a \ni M^* V_\ell \;\wedge\; (\ell \prec\!\!\prec b' \prec\!\!\prec b) \\
& \equiv \quad \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; (b' \prec\!\!\prec b),
\end{aligned}$$

where b' comes from the rounded union and boundedness below properties for b. Hence the matrix is rounded in b. □

It is tempting to try to enforce roundedness by redefining

$$\langle\, a \,|\, M \,|\, b \,\rangle \qquad \text{as} \qquad \exists a'b'.\, (a \prec\!\!\prec a') \;\wedge\; \mathcal{K}_{a'} \ni M^* V_{b'} \;\wedge\; (b' \prec\!\!\prec b),$$

but to prove that this is rounded still needs single interpolation, whilst saturation requires rounded unions.

Here is the converse transformation:

Lemma 4.4 For any abstract matrix $\langle\, a \,|\, M \,|\, b \,\rangle$ that is rounded in b, the operator $M^\dagger$ defined by

$$\begin{aligned}
M^\dagger V \quad & \equiv \quad \bigcup_{a} \{U_a \mid \exists b.\, \langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; \mathcal{L}_b \ni V\} \\
& = \quad \bigcup_{k}{}^{\uparrow} \{U_k \mid \forall a \in k.\, \exists b.\, \langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; \mathcal{L}_b \ni V\}
\end{aligned}$$

is Scott-continuous in V and

$$M^\dagger V_b \;=\; \bigcup_{a} \{U_a \mid \langle\, a \,|\, M \,|\, b \,\rangle\} \;=\; \bigcup_{k}{}^{\uparrow} \{U_k \mid \forall a \in k.\, \langle\, a \,|\, M \,|\, b \,\rangle\}.$$

Hence if the matrix $\langle\, a \,|\, M \,|\, b \,\rangle$ had been a concrete one defined from an operator M^* then

$$M^\dagger V \;\subset\; M^* V \qquad \text{and} \qquad M^\dagger V_b \;=\; M^* V_b.$$

We say that M^* is ***representable*** if $M^\dagger = M^*$.

Proof Scott continuity is immediate from Scott-openness of $\mathcal{L}_b$, whilst roundedness gives

$$\begin{aligned}
M^\dagger V_b \quad & \equiv \quad \bigcup_{a} \{U_a \mid \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; \mathcal{L}_{b'} \ni V_b\} \\
& \equiv \quad \bigcup_{a} \{U_a \mid \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; (b' \prec\!\!\prec b)\} \\
& \Leftrightarrow \quad \bigcup_{a} \{U_a \mid \langle\, a \,|\, M \,|\, b \,\rangle\}.
\end{aligned}$$

To show that $M^\dagger V \subset M^* V$ it suffices to observe that

$$\langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; \mathcal{L}_b \ni V \;\Longrightarrow\; \mathcal{K}_a \ni M^* V_b \;\wedge\; \mathcal{L}_b \ni V \;\Longrightarrow\; U_a \subset M^* V_b,$$

by the basis expansion of $M^* V_b$. Equality in the case $V \equiv V_b$ follows from Lemma 4.2. $\square$

Lemma 4.5 If the abstract matrix $\langle\ | M |\ \rangle$ is co- and contravariant, rounded on both sides and saturated in its input then it is recovered from the operator $M^\dagger$.

Proof By the previous lemma, the derived matrix is

$$\mathcal{K}_a \ni M^\dagger V_b \iff \exists k.\, \mathcal{K}_a \ni U_k \ \wedge\ \forall a' \in k.\, \langle a' \,|\, M \,|\, b \rangle,$$

but the right hand side of this is just $\langle a \,|\, M \,|\, b \rangle$ because this is saturated by hypothesis. $\square$

Notation 4.6 Given abstract matrices $\langle\ | M |\ \rangle$ and $\langle\ | N |\ \rangle$,

$$\begin{array}{rcl} M^\dagger(N^\dagger W) & = & \bigcup^{\uparrow}\{U_k \mid \forall a \in k.\, \exists b.\, \langle a \,|\, M \,|\, b \rangle \ \wedge\ \mathcal{L}_b \ni N^\dagger W\}, \\ \text{so} \quad \langle a \,|\, M\,;N \,|\, c \rangle & \equiv & \mathcal{K}_a \ni M^\dagger(N^\dagger W_c) \\ & = & \exists k.\, (a \prec\!\!\!\prec k) \ \wedge\ \forall a' \in k.\, \exists b.\, \langle a' \,|\, M \,|\, b \rangle \ \wedge\ \langle b \,|\, N \,|\, c \rangle, \end{array}$$

which we call the ***saturated composite***. However, this definition is not yet safe to use:

Example 4.7 Even when Scott-continuous operators M^* and N^* are representable in the sense of Lemma 4.4, their composite $P^* \equiv M^* \cdot N^*$ not not be.

Proof Let $X \equiv \mathbf{1} \equiv \{\bullet\}$ with prime basis $A \equiv \{\bullet\}$, $Y \equiv \mathbf{2} \equiv \{0,1\}$ with directed basis $B \equiv \{0,1,\bullet\}$ and $Z \equiv \mathbf{2} \times \mathbf{2}$ with prime basis $C \equiv \{(0,0),(0,1),(1,0),(1,1)\}$.

Let $M^* : \Sigma^Y \to \Sigma^X$ be conjunction, so its only true matrix element is $\langle \bullet \,|\, M \,|\, \bullet \rangle$.

Let $N^* : \Sigma^Z \to \Sigma^Y$ be disjunction on the second component, its true matrix elements being $\langle 0 \,|\, N \,|\, (0,0) \rangle$, $\langle 0 \,|\, N \,|\, (0,1) \rangle$, $\langle 1 \,|\, N \,|\, (1,0) \rangle$ and $\langle 1 \,|\, N \,|\, (1,1) \rangle$.

Then $M^* N^* \{(0,1),(1,0)\} = \{\bullet\}$ but $M^* N^* \{(z_1,z_2)\} = \emptyset$ for any of the four singletons. Therefore, since these singletons provide the basis for Z, the matrix $\langle \bullet \,|\, P \,|\, c \rangle$ for $P^* \equiv M^* \cdot N^*$ is everywhere false and $P^\dagger V = \emptyset$. The relational and saturated composite matrices are also everywhere false. $\square$

It is not this failure that should surprise you but that we ever suggested that we could define matrices using *singletons* instead of *lists*, when we needed to use lists in bases to capture the way-below relation for locally compact spaces other than domains. It can in fact be done, so long as we have some control over finite unions. It is sufficient to do this *either* by using directed bases *or* by restricting our attention to operators M^* that preserve all joins.

Lemma 4.8 If the basis $(V_b, \mathcal{L}_b)$ is directed then every Scott-continuous operator is represented by its concrete matrix.

Proof The hypothesis means that the basis expansion $V = \bigcup\{V_b \mid \mathcal{L}_b \ni V\}$ is a directed union, so M^* preserves it. By Lemma 4.4, the operator $M^\dagger$ that is derived from the matrix $\langle a \,|\, M \,|\, b \rangle$ that was obtained from M^* also preserves this union, whilst $M^\dagger V_b = M^* V_b$. Hence $M^\dagger V = M^* V$ for any V. $\square$

In this case, Lemmas 4.2–4.4 define a bijection between these operators and matrices that are co- and contravariant, rounded and saturated. It follows that the category of locally compact spaces and Scott-continuous operators is equivalent to one of bases and matrices with saturated composition. Given this equivalence, such composition must be associative, although this is far from obvious from the formula.

It is, however, the express purpose of this paper *not* to use directed bases. Then rounded saturated matrices just correspond to *some* of the Scott-continuous operators between open-set lattices, but unfortunately not even to a subcategory of them.

We therefore retreat regarding the generality that we are trying to capture, by one step towards the inverse images of continuous functions:

Lemma 4.9 If M^* preserves all joins then it is represented by its concrete matrix.

Proof We have $M^\dagger V \subset M^* V$ and $M^\dagger V_b = M^* V_b$ from Lemma 4.4 and the basis expansion gives

$$\begin{aligned} M^\dagger V \;=\; M^\dagger \bigcup\{V_b \mid \mathcal{K}_b \ni V\} \;&\supset\; \bigcup\{M^\dagger V_b \mid \mathcal{K}_b \ni V\} \\ &=\; \bigcup\{M^* V_b \mid \mathcal{K}_b \ni V\} \\ &=\; M^* \bigcup\{V_b \mid \mathcal{K}_b \ni V\} \;=\; M^* V. \end{aligned}$$
□

We want to identify the property of the matrix that characterises when M^* preserves all joins. First we extend the definition of the concrete matrix to unions in the output:

Lemma 4.10 If M^* preserves all unions then

$$\mathcal{K}_a \ni M^* V_\ell \iff \exists k.\,(a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b \in \ell.\, \langle\, a' \,|\, M \,|\, b \,\rangle.$$

Proof Using the directed basis expansion of $M^* V_b$,

$$\begin{aligned} M^* V_\ell \;\equiv\; M^* \bigcup_{b\in\ell} V_b \;&=\; \bigcup_{b\in\ell} M^* V_b \;=\; \bigcup_{b\in\ell}\bigcup_{a'} \{U_{a'} \mid \mathcal{K}_{a'} \ni M^* V_b\} \\ &=\; \bigcup_{a'} \{U_{a'} \mid \exists b \in \ell.\, \mathcal{K}_{a'} \ni M^* V_b\} \\ &=\; \bigcup_{k}^{\uparrow} \{U_k \mid \forall a' \in k.\, \exists b \in \ell.\, \mathcal{K}_{a'} \ni M^* V_b\}. \end{aligned}$$

Then, since $\mathcal{K}_a$ is Scott-open,

$$\mathcal{K}_a \ni M^* V_\ell \iff \exists k.\, \mathcal{K}_a \ni U_k \;\wedge\; \forall a' \in k.\, \exists b \in \ell.\, \mathcal{K}_{a'} \ni M^* V_b,$$

whence the result follows by the definitions of $(a \prec\!\!\!\prec k)$ and $\langle\, a' \,|\, M \,|\, b \,\rangle$. □

This brings us to the matrix characterisation of operators that preserve arbitrary unions:

Lemma 4.11 If M^* preserves unions then the concrete matrix $\langle\;|\, M \,|\;\rangle$ has the ***partition property***,

$$\langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; (b \prec\!\!\!\prec \ell) \implies \exists k.\,(a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b \in \ell.\, \langle\, a' \,|\, M \,|\, b \,\rangle.$$

Proof Since $(b \prec\!\!\!\prec \ell) \equiv (\mathcal{L}_b \ni V_\ell) \Rightarrow (V_b \subset V_\ell) \Rightarrow (M^* V_b \subset M^* V_\ell)$, the previous result gives

$$\begin{aligned} \langle\, a \,|\, M \,|\, b \,\rangle \;\wedge\; (b \prec\!\!\!\prec \ell) \;&\Rightarrow\; \mathcal{K}_a \ni M^* V_b \subset M^* V_\ell \\ &\Rightarrow\; \exists k.\,(a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b' \in \ell.\, \langle\, a' \,|\, M \,|\, b' \,\rangle. \end{aligned}$$
□

Lemma 4.12 For any predicate ϕ on the indexing set of the basis,

$$\bigcup\{U_a \mid \exists k.\,(a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \phi a'\} \;\subset\; \bigcup\{U_{a'} \mid \phi a'\}.$$

Proof If $a \prec\!\!\!\prec k$ then $\mathcal{K}_a \ni U_k$, so $U_a \subset U_k \equiv \bigcup\{U_{a'} \mid a' \in k\}$. Hence if also $\forall a' \in k.\, \phi a'$ then $U_a \subset \{U_{a'} \mid \phi a'\}$ and the result follows. □

Lemma 4.13 If the abstract matrix $\langle\;|\, M \,|\;\rangle$ has the partition property then $M^\dagger$ preserves unions.

Proof If $b \in \ell$ then $V_b \subset V_\ell$ and $M^\dagger V_b \subset M^\dagger V_\ell$, so $\bigcup\{M^\dagger V_b \mid b \in \ell\} \;\subset\; M^\dagger V_\ell$.

For the reverse inclusion, by the partition property and Lemma 4.12,

$$\begin{aligned} M^\dagger V_\ell \;&=\; \bigcup\{U_a \mid \exists b'.\, \langle\, a \,|\, M \,|\, b' \,\rangle \;\wedge\; (b' \prec\!\!\!\prec \ell)\} \\ &\subset\; \bigcup\{U_a \mid \exists k.\,(a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b \in \ell.\, \langle\, a' \,|\, M \,|\, b \,\rangle\} \\ &\subset\; \bigcup\{U_{a'} \mid \exists b \in \ell.\, \langle\, a' \,|\, M \,|\, b \,\rangle\} \\ &=\; \bigcup\{M^\dagger V_b \mid b \in \ell\}. \end{aligned}$$

Then, since $M^\dagger$ also preserves directed unions, it preserves all of them. $\square$

Proposition 4.14 If the bases obey the single interpolation, rounded union and boundedness below properties then the correspondence above defines a bijection between union-preserving operators and matrices that are co- and contravariant, rounded and saturated and have the partition property. $\square$

It remains to find the properties of matrices that correspond to the fact that inverse image maps preserve the whole space and intersections. We will say that a matrix is ***bounded*** and ***filtered*** respectively if it has the relevant properties. As before, we cannot do this independently of the unions: we must assume either that the bases are directed or that the matrices have the partition property. Unfortunately, the resulting condition is rather complicated.

Remark 4.15 Suppose first that the bases are stable (Definition 2.2). More precisely, the concrete basis for the source space X needs to use compact subspaces K_a, whilst that for the target Y has a greatest element $\bullet$ and an intersection operation $\sqcap$ with

$$V_\bullet = Y \qquad \text{and} \qquad V_{b_1\sqcap b_2} = V_{b_1} \cap V_{b_2}.$$

Then the matrix for a continuous function satisfies

$$\langle\, a \,|\, f \,|\, \bullet \,\rangle \;\equiv\; (fK_a \subset Y) \;\Leftrightarrow\; \top$$

and $$(fK_a \subset V_{b_1}) \;\wedge\; (fK_a \subset V_{b_2}) \;\Longleftrightarrow\; (fK_a \subset V_{b_1\sqcap b_2}),$$

which is $$\langle\, a \,|\, f \,|\, b_1 \,\rangle \;\wedge\; \langle\, a \,|\, f \,|\, b_2 \,\rangle \;\Longleftrightarrow\; \langle\, a \,|\, f \,|\, b_1 \sqcap b_2 \,\rangle.$$

However, as we discussed in Examples 2.3, we do not want to assume that our bases carry this semilattice structure. In some cases we may replace the *actual* top element or intersection above with an existentially quantified variable b:

Definition 4.16 A matrix is ***uniformly*** bounded and filtered respectively if

$$\exists b.\, \langle\, a \,|\, f \,|\, b \,\rangle$$

and $$\langle\, a \,|\, f \,|\, b_1 \,\rangle \;\wedge\; \langle\, a \,|\, f \,|\, b_2 \,\rangle \;\Longrightarrow\; \exists b.\, \langle\, a \,|\, f \,|\, b \,\rangle \;\wedge\; (b \prec\!\!\!\prec b_1) \;\wedge\; (b \prec\!\!\!\prec b_2).$$

However, matrices generally only have these properties if the bases are closed under finite unions, which we do not want to assume any more than we did intersections. We really need $\langle\, a \,|\, f \,|\, \ell \,\rangle \equiv (\mathcal{K}_a \ni f^{-1}V_\ell)$, but this was not defined in Notation 1.15. However, Lemma 4.10 gave a formula for it that is related to saturation. So, instead of requiring *uniform* boundedness and filteredness as above, we ask that these properties hold *after they have been saturated.*

Lemma 4.17 If the bases obey the secondary rules including boundedness, M^* preserves unions and $M^*Y = X$ then the concrete matrix $\langle\; |\, M \,|\; \rangle$ is ***bounded*** in the sense that

$$\exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \,|\, M \,|\, b \,\rangle.$$

Conversely, if $\langle\; |\, M \,|\; \rangle$ is bounded in this sense then $M^\dagger Y = X$.

Proof Boundedness of the basis means that each a has some k with $a \prec\!\!\!\prec k$, so $\mathcal{K}_a \ni U_k \subset X$ and $\mathcal{K}_a \ni X$. Then $\mathcal{K}_a \ni X = M^*Y = M^* \bigcup^{\uparrow} V_\ell = \bigcup^{\uparrow} M^*V_\ell$, so $\exists \ell.\, \mathcal{K}_a \ni M^*V_\ell$. Using Lemma 4.10, this amounts to the given formula for boundedness of the matrix.

Conversely, by Lemmas 4.4, and 4.12 and a similar observation about $\mathcal{L}_b \ni Y$,

$$\begin{aligned} M^\dagger Y \;&=\; \bigcup \{U_{a'} \mid \exists b.\, \langle\, a' \,|\, M \,|\, b \,\rangle \;\wedge\; \mathcal{L}_b \ni Y\} \\ &\supset\; \bigcup \{U_a \mid \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \,|\, M \,|\, b \,\rangle\} \\ &\supset\; \bigcup \{U_a \mid \top\} \;=\; X. \end{aligned}$$

$\square$

This complicated property reduces to the simpler ones if we have the relevant structure:

Lemma 4.18 Let the abstract matrix $\langle\ |\,M\,|\ \rangle$ be covariant, rounded, bounded and saturated. Then
(a) if the basis B has a top element $\bullet$ with respect to $\sqsubseteq$ then $\langle\, a \,|\, M \,|\, \bullet\,\rangle \Leftrightarrow \top$;
(b) if B is directed then $\exists b.\,\langle\, a \,|\, M \,|\, b\,\rangle$; and
(c) if A is prime (Proposition 5.14) then $\exists b.\,\langle\, a \,|\, M \,|\, b\,\rangle$.
On the other hand, if we drop the requirement that the bases be bounded above but keep the other secondary axioms, the formula becomes

$$a \prec\!\!\!\prec a'' \implies \exists k.\,(a \prec\!\!\!\prec k)\ \wedge\ \forall a' \in k.\,\exists b\ell.\,\langle\, a' \,|\, M \,|\, b\,\rangle. \qquad \square$$

Turning to binary intersections, we have different results for bases that use compact subspaces or Scott-open families:

Lemma 4.19 Let $M^* : \Omega Y \to \Omega X$ be an operator that preserves all unions and binary intersections. If the basis for X uses compact subspaces then the concrete matrix $\langle\ |\,M\,|\ \rangle$ is ***strongly filtered***:

$$\langle\, a \,|\, M \,|\, b_1\,\rangle\ \wedge\ \langle\, a \,|\, M \,|\, b_2\,\rangle \implies$$
$$\exists k\ell.\,(a \prec\!\!\!\prec k)\ \wedge\ \big(\forall a' \in k.\,\exists b \in \ell.\,\langle\, a' \,|\, M \,|\, b\,\rangle\big)\ \wedge\ \big(\forall b \in \ell.\,b_1 \sqsupseteq b \sqsubseteq b_2\big).$$

If instead it uses Scott-open families then $\langle\ |\,M\,|\ \rangle$ is ***weakly filtered***:

$$(a \prec\!\!\!\prec a')\ \wedge\ \langle\, a' \,|\, M \,|\, b_1\,\rangle\ \wedge\ \langle\, a' \,|\, M \,|\, b_2\,\rangle \implies$$
$$\exists k\ell.\,(a \prec\!\!\!\prec k)\ \wedge\ \big(\forall a' \in k.\,\exists b \in \ell.\,\langle\, a' \,|\, M \,|\, b\,\rangle\big)\ \wedge\ \big(\forall b \in \ell.\,b_1 \sqsupseteq b \sqsubseteq b_2\big).$$

Proof The hypotheses for the strong rule are $K_a \subset M^*V_{b_1}$ and $K_a \subset M^*V_{b_1}$. Then

$$K_a \subset M^*V_{b_1} \cap M^*V_{b_2} = M^*(V_{b_1} \cap V_{b_2}) \qquad \text{and so} \qquad K_a \subset M^*V_\ell$$

for some ℓ with $\ell \sqsubseteq b_1$ and $\ell \sqsubseteq b_2$. By Lemma 4.10, this is the stated conclusion.

In the weak case, we are given $\mathcal{K}_a \ni U_{a'}$, $\mathcal{K}_{a'} \ni M^*V_{b_1}$ and $\mathcal{K}_{a'} \ni M^*V_{b_1}$. Then we deduce $\mathcal{K}_a \ni M^*V_{b_1} \cap M^*V_{b_2}$ as in Lemma 2.8 and the rest of the argument is the same as in the strong case. $\square$

Lemma 4.20 If the abstract matrix $\langle\ |\,M\,|\ \rangle$ is weakly or strongly filtered and has the partition property then $M^\dagger$ preserves binary intersections.

Proof By Lemma 4.4, the filter property of a concrete basis, contravariance, the basis expansion of U_a (for roundedness), the weak intersection rule and Lemma 4.12,

$$\begin{aligned}
& M^\dagger V_{b_1} \cap M^\dagger V_{b_2} \\
\subset\ & \bigcup\{U_a \mid \exists a_1 a_2.\,(a_1 \sqsupseteq a \sqsubseteq a_2)\ \wedge\ \langle\, a_1 \,|\, M \,|\, b_1\,\rangle\ \wedge\ \langle\, a_2 \,|\, M \,|\, b_2\,\rangle\} \\
\subset\ & \bigcup\{U_a \mid \langle\, a \,|\, M \,|\, b_1\,\rangle\ \wedge\ \langle\, a \,|\, M \,|\, b_2\,\rangle\} \\
\subset\ & \bigcup\{U_{a'} \mid \exists a.\,(a' \prec\!\!\!\prec a)\ \wedge\ \langle\, a \,|\, M \,|\, b_1\,\rangle\ \wedge\ \langle\, a \,|\, M \,|\, b_2\,\rangle\} \\
\subset\ & \bigcup\{U_{a'} \mid \exists k.\,(a' \prec\!\!\!\prec k)\ \wedge\ \forall a'' \in k.\,\exists b.\,\langle\, a'' \,|\, M \,|\, b\,\rangle\ \wedge\ (b_1 \sqsupseteq b \sqsubseteq b_2)\} \\
\subset\ & \bigcup\{U_{a''} \mid \exists b.\,\langle\, a'' \,|\, M \,|\, b\,\rangle\ \wedge\ (V_{b_1} \supset V_b \subset V_{b_2})\} \\
\subset\ & M^\dagger(V_{b_1} \cap V_{b_2}),
\end{aligned}$$

where the fourth line is not needed if $\langle\ |\,M\,|\ \rangle$ is strongly filtered. Then $M^\dagger V_1 \cap M^\dagger V_2 = M^\dagger(V_1 \cap V_2)$ since $M^\dagger$ also preserves arbitrary unions by Lemma 4.13. $\square$

We use sobriety (Definition 3.13) to complete the characterisation of matrices for continuous functions:

Theorem 4.21 Let X and Y be locally compact sober spaces with concrete bases $(U_a, \mathcal{K}_a)$ and $(V_b, \mathcal{L}_b)$ that obey the primary and secondary axioms. Then the formulae

$$\langle\, a \,|\, f \,|\, b \,\rangle \;\equiv\; \mathcal{K}_a \ni f^{-1}V_b \qquad \text{and} \qquad fx \in V_b \iff \exists a.\, x \in U_a \;\wedge\; \langle\, a \,|\, f \,|\, b \,\rangle$$

define bijections amongst
(a) a continuous function $f : X \to Y$;
(b) an operator $f^* : \Omega Y \to \Omega X$ that preserves finite intersections and arbitrary unions; and
(c) a matrix $\langle\, a \,|\, f \,|\, b \,\rangle$ that is co- and contravariant, rounded, saturated, bounded and filtered and has the partition property.

Proof The correspondence between (a) and (b) is Definition 3.13 of sobriety and that between (b) and (c) was the subject of this section. Lemmas 4.5 and 4.9 gave the bijection between operators and matrices, relying on preservation of joins and roundedness and therefore on the secondary axioms for the bases. The subsequent results matched the other properties. Then, for each ordinary point $x \in X$,

$$\mathcal{P}_x \;\equiv\; \{V \in \Omega Y \mid x \in f^*V\} \;\equiv\; \{V \mid \exists ab.\, x \in U_a \;\wedge\; \langle\, a \,|\, f \,|\, b \,\rangle \;\wedge\; \mathcal{L}_b \ni V\}$$

is a formal point of Y, because f^* preserves finite intersections and arbitrary unions. Hence by sobriety $\mathcal{P}_x \ni V \iff y \in V$ for some unique $y \in Y$ and we put $fx \equiv y$. This defines a continuous function because $f^{-1}V = f^*V \subset X$ and this is open by construction, for any open $V \subset Y$. □

This result provides us with the definition of the category of *abstract* bases and matrices, but the full details of the abstract construction of this category and its structure will take another whole paper.

In order to check that you understand the axioms for bases and matrices and how to use them, you should verify that $\langle\, a \,|\, \mathsf{id} \,|\, b \,\rangle \equiv (a \prec\!\!\!\prec b)$ has all of the properties of a matrix and is a unit for saturated composition. Single interpolation is needed for roundedness and boundedness of the basis for that of the matrix.

Remark 4.22 Isomorphisms in the abstract category should also define what it means for abstract bases to be equivalent. In particular, we use the way-below relation $a \prec\!\!\!\prec b$ for the matrices in both directions to show that the "upgraded" bases in Lemmas 3.3, 3.4 and 3.7 are equivalent to the given ones, whilst $a \prec\!\!\!\prec \ell$ and $\ell \prec\!\!\!\prec a$ do so for the directed basis in Proposition 3.1.

Unfortunately, however, we run into the *reason* for upgrading the bases, namely that the secondary properties of bases were needed in Lemma 4.3 to prove the fundamental properties of matrices. What we would like to be an equivalence of categories becomes an adjunction between 2-categories, so we leave the interested reader to investigate this.

Equivalence of the directed basis also illustrates another point about the way in which the properties of matrices have been defined: the matrix $a \prec\!\!\!\prec \ell$ is *uniformly* bounded but its inverse $\ell \prec\!\!\!\prec a$ is not, *cf.* Lemma 4.18(b). □

We may sum up what we have achieved so far in categorical language by saying that there is a full and faithful functor from the category of locally compact sober spaces with given concrete bases and continuous functions to the category of abstract bases and matrices satisfying the conditions that we have identified. In order to make these categories equivalent we therefore have to show that this functor is essentially surjective.

5 Classical completeness

We now embark on the recovery of a space from any given abstract basis. We start, in the traditional way, with the points. These are continuous functions from the singleton, which has just one basis element $\bullet$, with $\bullet \prec\!\!\!\prec \bullet$, so points correspond to matrices of the form $\langle\, \bullet \,|\, f \,|\, b \,\rangle$. Hence, in the axioms in the previous section, contravariance, saturation and roundedness in the argument are trivial, whilst by Lemma 4.18(c) boundedness and filteredness are uniform. The partition property is also simplified, to one that we call locatedness by analogy with the Dedekind real line below. Writing $p \equiv \{b \mid \langle \bullet \mid f \mid b \rangle\} \subset A$, we have

Definition 5.1 A ***formal point*** for an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ is a (typically infinite) subset $p \subset A$ such that

$$
\begin{array}{lcll}
a \sqsupseteq b \in p & \Rightarrow & a \in p & \text{upper} \\
a \in p & \Leftrightarrow & \exists b.\,(b \prec\!\!\!\prec a) \ \wedge \ b \in p & \text{rounded} \\
 & & \exists a.\, a \in p & \text{bounded} \\
(a \in p) \ \wedge \ (b \in p) & \Rightarrow & \exists c.\,(a \sqsupseteq c \sqsubseteq b) \ \wedge \ c \in p & \text{filtered} \\
(a \in p) \ \wedge \ (a \prec\!\!\!\prec k) & \Rightarrow & k \between p \ \equiv \ \exists b.\,(b \in k) \ \wedge \ (b \in p). & \text{located}
\end{array}
$$

We write X for the set of formal points and $\mathsf{Spec}\,(A, \sqsubseteq, \prec\!\!\!\prec)$ for the space that we shall construct. Beware that this notion of formal point is related to the abstract basis, whereas the one in Definition 3.13 is defined by the topology, which we now describe. We will show that the two notions are isomorphic in Lemma 5.11. The specialisation order is given by inclusion.

In the simplest case of a discrete space we already see that sobriety corresponds to a logical principle:

Example 5.2 Any set N (maybe, but not necessarily, $\mathbb{N}$) provides a concrete basis for itself, considered as a discrete locally compact space, where $U_n \equiv K_n \equiv \{n\}$. The abstract basis is $(N, =, \in)$. A formal point $p \subset N$ is a ***description***, satisfying

$$\exists n.\, n \in p \qquad \text{and} \qquad n \in p \ni m \implies n = m.$$

Then N is sober iff every description is a singleton, $\{n\}$. This principle of Definition by Description was first correctly identified by Giuseppe Peano [Pea97, §22]; for the connection with sobriety see [A]. □

The term *located* is derived from our running example:

Example 5.3 A formal point p for the basis of intervals on $\mathbb{R}$ (Example 1.9) corresponds to a ***Dedekind cut*** (δ, υ) by

$$\delta \ \equiv \ \{d \mid \exists u.\, \langle d, u\rangle \in p\}, \quad \upsilon \ \equiv \ \{u \mid \exists d.\, \langle d, u\rangle \in p\} \quad \text{and} \quad p \ \equiv \ \{\langle d, u\rangle \mid d \in \delta \wedge u \in \upsilon\},$$

where δ and υ are characterised by

$$
\begin{array}{cc}
u \in \upsilon \iff \exists t.\, t \in \upsilon \ \wedge \ (t < u) & \qquad d \in \delta \iff \exists e.\,(d < e) \ \wedge \ e \in \delta \\
\exists u.\, u \in \upsilon & \qquad \exists d.\, d \in \delta \\
d \in \delta \ \wedge \ u \in \upsilon \implies (d < u) & \qquad (d < u) \implies d \in \delta \ \vee \ u \in \upsilon.
\end{array}
$$

Proof The bijection, roundedness and boundedness properties are easy. The filter property of p amounts to $\langle d, u\rangle \in p \ni \langle e, t\rangle \implies (d < t)$, so it is equivalent to the fifth axiom (disjointness) for (δ, υ).

Let p be located in the sense of the Definition and $e < t$. Then the other properties provide $c < d < e < t < u < v$ with $\langle d, u\rangle \in p$. Since $[d, u] \subset (e, v) \cup (c, t)$, locatedness of p gives $\langle e, v\rangle \in p \vee \langle c, t\rangle \in p$, whence $e \in \delta \vee t \in \upsilon$. The converse is more complicated since it involves arbitrarily many intervals, but is essentially Lemma 6.16 of [I].

Hence sobriety for $\mathbb{R}$ is Dedekind completeness. □

Now we return to the general situation and define its basis.

Definition 5.4 For each $a \in A$ and $u \subset A$, the basic and general open subsets of X are

$$U_a \ \equiv \ \{p \mid a \in p\} \qquad \text{and} \qquad U_u \ \equiv \ \{p \mid p \between u \ \equiv \ \exists a.\, a \in p \ \wedge \ a \in u\}.$$

Lemma 5.5 If $a \sqsubseteq b$ or $a \prec\!\!\!\prec b$ then $U_a \subset U_b$. The whole set X of formal points is open, *i.e.* it is expressible as a union of basic open subsets, as is the intersection of any two subsets that are so expressible.

Proof The first three parts follow from the requirements that formal points be upper, rounded and bounded respectively, whilst the filteredness property of formal points says that

$$U_a \cap U_b \;=\; \bigcup \{U_c \mid a \sqsupseteq c \sqsubseteq b\}$$

and the property for intersections of general unions follows from this. $\square$

Lemma 5.6 The family of open subspaces given by

$$\mathcal{K}_a \;\equiv\; \{U \mid \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; U_k \subset U\}$$

is Scott-open. If $a \prec\!\!\!\prec k$ then $\mathcal{K}_a \ni U_k$ and if $a \sqsubseteq b$ then $\mathcal{K}_a \supset \mathcal{K}_b$.

Proof The second part is immediate and the third follows directly from contravariance of $\prec\!\!\!\prec$. We then deduce Scott-openness:

$$\mathcal{K}_a \ni U \;\equiv\; \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; U_k \subset U \implies \exists k.\, \mathcal{K}_a \ni U_k \subset U. \qquad \square$$

Lemma 5.7 The system $(U_a, \mathcal{K}_a)$ satisfies the basis expansion

$$p \in U \iff \exists a.\, p \in U_a \;\wedge\; \mathcal{K}_a \ni U \qquad \text{or} \qquad U \;=\; \bigcup \{U_a \mid \mathcal{K}_a \ni U\}$$

and is therefore a concrete basis for X using Scott-open families.

Proof [$\Rightarrow$] Since general open subsets are unions of basic ones, $p \in U_b \subset U$ for some b. Then $b \in p$ and by roundedness of p there is some $a \in p$ with $a \prec\!\!\!\prec b$. Hence $p \in U_a$ and $\mathcal{K}_a \ni U_b \subset U$, so $\mathcal{K}_a \ni U$ too.

[$\Leftarrow$] For some a and k, we have $a \in p$ and $a \prec\!\!\!\prec k$ with $U_k \subset U$, so by locatedness of p there is some $b \in k \cap p$ and $p \in U_b \subset U_k \subset U$. $\square$

This is all very well, but the problem was to find a space with a concrete basis that induces the *given* abstract basis, *i.e.* such that $\mathcal{K}_a \ni U_k$ if *and only if* $a \prec\!\!\!\prec k$. Proving such things in *point*–set topology involves *finding* points with specific properties. In particular, if $\mathcal{K}_a$ is of the form $\{U \mid K_a \subset U\}$ but $a \not\prec\!\!\!\prec k$ then we need to find a point that is in K_a but not in U_k.

For us, a "point" is a certain kind of subset of A (Definition 5.1) and we need one that includes some elements of A but excludes others. Lawson's Lemma 3.11 provides a $\prec\!\!\!\prec$-filter, so we need a way of obtaining rounded located subsets of the basis. The following arguments require it to be countable and satisfy single interpolation and assume Excluded Middle and Dependent Choice.

Lemma 5.8 For any subset $r \subset A$, we obtain a rounded located subset $\underline{r} \subset r$ by

$$\underline{r} \;\equiv\; \{a \in A \mid \exists a'.\, (a' \prec\!\!\!\prec a) \;\wedge\; a' \bullet r\}$$

where $$a' \bullet r \;\equiv\; (\forall k.\, a' \prec\!\!\!\prec k \implies k \between r).$$

Indeed, $r \mapsto \underline{r}$ is coclosure operation for which $r = \underline{r}$ iff r is rounded and located.

Proof The operation is decreasing ($\underline{r} \subset r$), by putting $k \equiv \{a\}$, so $a \in k \cap r$.

It also preserves order: if $r \subset r'$ then $a' \bullet r \Rightarrow a' \bullet r'$ and so $\underline{r} \subset \underline{r'}$.

If r is already rounded and located then $\underline{r} = r$: given $a \in r$, by roundedness there is some $a' \in r$ with $a' \prec\!\!\!\prec a$ and if $a' \prec\!\!\!\prec k$ then $k \between r$ by locatedness.

For general r, the subset $\underline{r}$ is rounded: if $a \in \underline{r}$ then by the definition of $\underline{r}$ and single interpolation there are $a'' \prec\!\!\!\prec a' \prec\!\!\!\prec a$ with $a'' \bullet r$, so $a' \in \underline{r}$.

The difficult part is locatedness of $\underline{r}$. Let $a \in \underline{r}$ with $a \prec\!\!\!\prec \ell$, so there are a' and k with $a' \prec\!\!\!\prec k \prec\!\!\!\prec a$ and $k \prec\!\!\!\prec^1 \ell$ by Lemma 3.8. We need to find $b \in k$ with $b \bullet r$, from which we obtain c with $b \prec\!\!\!\prec c \in \ell$ since $k \prec\!\!\!\prec^1 \ell$ and then $c \in \ell \cap \underline{r}$.

Suppose that there is no such $b \in k$, so

$$\forall b \in k.\, \neg(b \bullet r) \quad\equiv\quad \forall b \in k.\, \exists h_b.\, (b \prec\!\!\!\prec h_b) \wedge (h_b \cap r = \emptyset).$$

Then $$a \prec\!\!\!\prec k \prec\!\!\!\prec h \equiv \bigcup \{h_b \mid b \in k\} \qquad \text{with} \qquad h \cap r = \emptyset,$$

which contradicts $a \bullet r$. Hence there is some $b \in k$ with $b \bullet r$ as required. $\square$

Now we want to find a point p such that $s \subset p \subset r \subset A$, where s is a $\prec\!\!\!\prec$-filter and r a rounded located subset. One way of making a formal point from a filter is to incorporate instances of locatedness into the proof of (Lawson's) Lemma 3.11, which we can do if the basis is countable:

Lemma 5.9 Let $(A, \sqsubseteq, \prec\!\!\!\prec)$ be a countable abstract basis and $a \in r \subset A$, where r is rounded and located. Then there is a point p with $a \in p \subset r$.

Proof Let k_i be an enumeration of $\mathsf{Fin}\,(A)$ such each finite set k occurs infinitely often, so for any $k \in \mathsf{Fin}\,(A)$ and $i \in \mathbb{N}$ there is some $j > i$ with $k = k_j$.

As in Lemma 3.11, we put $a_0 \equiv a$ and define a descending sequence with $a_{i+1} \prec\!\!\!\prec a_i$, but we use locatedness to modify the choice of the terms.

As before, at each stage $i \in \mathbb{N}$, we first let $a' \prec\!\!\!\prec a_i$ with $a' \in r$ since r is rounded. If $a_i \not\prec\!\!\!\prec k_i$ then just let $a_{i+1} \equiv a'$.

If $a' \prec\!\!\!\prec a_i \prec\!\!\!\prec k_i$ then by Lemma 3.8 there is some k' with $a' \prec\!\!\!\prec k' \prec\!\!\!\prec^1 a_i, k_i$. Since $a' \in r$ and r is located, there is some $a'' \in r \cap k'$, so $a'' \prec\!\!\!\prec a_i$ and $a'' \prec\!\!\!\prec b \in k_i$, so $b \in r$ since r is upper. We put $a_{i+1} \equiv a''$.

Again as before, the subset $p \equiv \{b \mid \exists i.\, a_i \prec\!\!\!\prec b\}$ is a $\prec\!\!\!\prec$-filter with $a \in p \subset r$.

But p is also located. If $a_i \prec\!\!\!\prec a' \prec\!\!\!\prec k$ then, by assumption on the enumeration of $\mathsf{Fin}\,(A)$, $k \equiv k_j$ for some j with $i < j$. By construction, $a_j \prec\!\!\!\prec a_i \prec\!\!\!\prec a' \prec\!\!\!\prec k \equiv k_j$ and then $a_{j+1} \prec\!\!\!\prec b \in k_j$, so $b \in k \cap p$ as required.

Then p is a filter with respect to $\sqsubseteq$ as well as $\prec\!\!\!\prec$: If $a \in p \ni b$ then there is $d \in p$ with $a \succ\!\!\!\succ d \prec\!\!\!\prec b$ and a further $e \in p$ with $e \prec\!\!\!\prec d$. Then by the weak intersection rule there is some k with $e \prec\!\!\!\prec k \sqsubseteq a, b$. Since p is located, there is some $c \in k \cap p$, so $a \sqsupseteq c \sqsubseteq b$.

Hence p has all the properties of a formal point. $\square$

The statement of this result is very similar to Lemma 3.14, so with some ingenuity you may be able to adapt that to the uncountable case. In fact, we will see how to do this in the next two sections, with the benefit of the point-free view of topology. But for the moment we accept the countability restriction and use the result that we possess to recover $a \prec\!\!\!\prec k$:

Lemma 5.10 If the basis is countable and $\mathcal{K}_a \ni U_k$ then $a \prec\!\!\!\prec k$.

Proof We claim first that

$$(b \prec\!\!\!\prec c) \ \wedge\ (U_c \subset U_k) \ \equiv\ (b \prec\!\!\!\prec c) \ \wedge\ (\forall p.\, c \in p \Rightarrow p \between k) \implies (b \prec\!\!\!\prec k).$$

Otherwise, by Lemma 5.8, there is a rounded located subset $r \subset A$ with $c \in r \subset A \setminus k$. Then by Lemma 5.9 there is a point p with $c \in p \subset r$. This means that $p \in U_c \subset U_k$, so $p \between k$, contradicting $p \cap k = \emptyset$ from the construction.

We generalise this to covers by lists using the Wilker and transitivity properties for $\prec\!\!\!\prec$:

$$\begin{aligned} \mathcal{K}_a \ni U_k \quad &\Rightarrow \quad \exists \ell\ell'.\, (a \prec\!\!\!\prec \ell' \prec\!\!\!\prec^1 \ell) \ \wedge\ \forall c \in \ell.\, (U_c \subset U_k) \\ &\Rightarrow \quad \exists \ell'.\, (a \prec\!\!\!\prec \ell') \ \wedge\ \forall b \in \ell'.\, \exists c.\, (b \prec\!\!\!\prec c) \ \wedge\ (\forall p.\, c \in p \Rightarrow p \between k) \\ &\Rightarrow \quad \exists \ell'.\, (a \prec\!\!\!\prec \ell') \ \wedge\ \forall b \in \ell'.\, (b \prec\!\!\!\prec k) \quad \implies \quad a \prec\!\!\!\prec k. \end{aligned}$$ $\square$

Now we can at last return to the topological ideas.

Lemma 5.11 If the basis is countable then the space X is sober.

Proof Let $\mathcal{P}$ be a formal point in the sense of Definition 3.13, *i.e.* a family of open subspaces of X such that

$$\mathcal{P} \ni X, \qquad \mathcal{P} \ni U, V \iff \mathcal{P} \ni U \cap V \qquad \text{and} \qquad \mathcal{P} \ni \bigcup U_i \iff \exists i.\, \mathcal{P} \ni U_i.$$

We claim that $p \equiv \{a \mid \mathcal{P} \ni U_a\}$ is a formal point in the sense of Definition 5.1 and satisfies $\mathcal{P} = \{U \mid p \in U\}$. Indeed, $p \in U_a \iff a \in p \iff \mathcal{P} \ni U_a$ and this extends to $p \in U \equiv U_u \iff \mathcal{P} \ni U_u$ by the third property of $\mathcal{P}$.

We leave it to the reader to show that $\mathcal{P}$ is a filter, *i.e.* bounded, filtered and upper.

It is located: if $a \in p$ and $a \prec\!\!\!\prec \ell$ then $\mathcal{P} \ni U_a$ and $\mathcal{K}_a \ni U_\ell$, so $\mathcal{P} \ni U_\ell \supset U_a$ from the basis expansion, but then $\mathcal{P} \ni U_b$ by the third property of $\mathcal{P}$, for some $b \in \ell$, for which $b \in p$.

Finally, using Lemma 5.10, the basis expansion $U_a = \bigcup\{U_b \mid \mathcal{K}_b \ni U_a\}$ gives the roundedness property $a \in p \iff \exists b.\, b \in p \wedge b \prec\!\!\!\prec a$.

Alternatively, $q \equiv \{a \mid \exists b.\, \mathcal{P} \ni U_b \wedge b \prec\!\!\!\prec a\}$ is easily seen to be rounded and upper, whilst the proof that p is filtered and located can be adapted to q, but then showing that $q \in U \iff \mathcal{P} \ni U$ depends on Lemma 5.10. $\square$

Theorem 5.12 Every countable abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ with single interpolation presents a concrete basis using Scott-open families for some locally compact sober topological space $X \equiv \mathsf{Spec}\,(A, \sqsubseteq, \prec\!\!\!\prec)$, assuming Excluded Middle and Dependent Choice. If the abstract basis satisfies the boundedness and strong intersection rules then the concrete basis uses compact subspaces, where

$$K_a \;\equiv\; \bigcap \mathcal{K}_a \;\equiv\; \{p \mid \forall k.\, (a \prec\!\!\!\prec k) \Longrightarrow p \between k\}.$$

Proof We have already completed the proof for Scott-open families, so it only remains to identify the points of the compact subspace in the strong case, using Proposition 3.17:

$$\begin{aligned} p \in \bigcap \mathcal{K}_a &\equiv \forall U \in \mathcal{K}_a.\, p \in U \\ &\equiv \forall k.\, \forall U.\, (a \prec\!\!\!\prec k) \;\wedge\; U_k \subset U \;\Rightarrow\; p \in U \\ &\Leftrightarrow \forall k.\, (a \prec\!\!\!\prec k) \;\Rightarrow\; p \in U_k \\ &\equiv \forall k.\, (a \prec\!\!\!\prec k) \;\Rightarrow\; (p \between k). \end{aligned}$$ $\square$

Remark 5.13 If $a \prec\!\!\!\prec c$ but $a \not\prec\!\!\!\prec k$ then $K_a \subset U_c$ but $K_a \not\subset U_k$, so there is a point p with $p \in K_a \subset U_c$ but $p \notin U_k$, so $c \in p$ but $p \cap k = \emptyset$. However, this begs the question, because we used this property to prove sobriety and so to characterise compact subspaces.

Examining the place where we needed the partial result (Lemma 5.9), we notice first that the topology on X is not actually being used: the arguments just concern the relationship between the abstract basis and its formal points. In fact the difficulty was in translating the containment of subspaces $U_c \subset U_k$ in Lemma 5.10 and the basis expansion $U_a = \bigcup\{U_b \mid \mathcal{K}_b \ni U_a\}$ in Lemma 5.11 from their definition in terms of points in Definition 5.4 back into the properties of $\prec\!\!\!\prec$. Indeed it was the $U_k \subset U$ in Lemma 5.6 (which was needed to make $\mathcal{K}_a$ upper) that obliged us to do this.

In the next two sections we shall define the open subspaces *directly* from the abstract basis without this diversion *via* formal points, and thereby solve the problem.

Before doing that, however, we show how to use a preorder with a trivial way-below relation to present spaces that are important in theoretical computer science and will provide the starting point for our general construction. There is no countability restriction. The discrete case of this was Example 5.2.

Proposition 5.14 For any preorder $(A, \sqsubseteq)$, the relation

$$a \prec\!\!\!\prec^0 \ell \;\equiv\; \exists b.\, a \sqsubseteq b \in \ell$$

defines a ***prime*** abstract basis that satisfies the secondary axioms and strong intersection. It presents a locally compact space with a basis using compact subspaces.

Proof The formal points are (upper, bounded) filters $p \subset A$, so

$$b \sqsupseteq a \in p \Longrightarrow b \in p, \quad \exists a.\, a \in p \quad \text{and} \quad a \in p \ni b \Longrightarrow \exists c.\, c \in p \wedge a \sqsupseteq c \sqsubseteq b.$$

In particular, each $a \in A$ defines a so-called ***compact point*** $p \;\equiv\; \uparrow a \;\equiv\; \{b \mid a \sqsubseteq b\}$, for which the specialisation order is the reverse of the usual one in domain theory: if $a \sqsubseteq b$ then $\uparrow a \supset \uparrow b$.

This space carries the ***Scott topology*** (Proposition 2.11) on all of the points or the ***Alexandrov topology*** on the compact ones, in which the basic open and compact subspaces are

$$U_a \;\equiv\; \{p \mid a \in p\} \qquad \text{and} \qquad K_a \;\equiv\; \{\uparrow a\} \quad \text{or} \quad K_a \;\equiv\; \uparrow\uparrow a \;\equiv\; \{p \mid a \in p\};$$

and the basis expansion is

$$p \in U \iff \exists a.\, p \in U_a \;\wedge\; K_a \subset U \iff \exists a.\;\, a \in p \;\wedge\; \uparrow a \in U,$$

so the way-below relation is, as required,

$$K_a \subset U_\ell \iff \uparrow a \in \{p \mid p \between \ell\} \iff \exists b.\, a \sqsubseteq b \in \ell \;\equiv\; b \prec\!\!\!\prec^0 \ell. \qquad \square$$

This space is called $\mathbf{Filt}(A, \sqsubseteq)$ or $\mathbf{Idl}\,(A, \sqsubseteq^{\mathsf{op}})$ and is (the typical example of) an ***algebraic dcpo*** (***directed-complete partial order***) or ***(pre)domain***. Notice that we have a choice for the basic compact subspaces between singletons and their saturations, *cf.* the ambiguity in Definitions 1.3 and 3.16.

6 Point-free general topology

The applications of topology to other disciplines are often called ***spectra***, in which the "points" are structures such as prime ideals that have fairly complicated definitions (*cf.* Definitions 3.13 and 5.1) and can be difficult to find (*cf.* Lemma 5.9). On the other hand, the "open subspaces" typically correspond directly to much simpler features of the mathematical system under study.

Peter Johnstone's book [Joh82] explores many examples of this phenomenon. This book is the standard text for Locale Theory, except that another approach, called Formal Topology, offers more efficient technology for constructing locales from bases.

Foundationally, one advantage of Locale Theory is that it largely avoids the Axiom of Choice and (if we are exceptionally careful) even Excluded Middle, so it is valid in the logic of an elementary topos. We consider a further tightening of our foundational belt, called predicativity, in Section 7.14.

This section summarises the techniques that we require from Locale Theory and Formal Topology for *general* topology. In the next section we show how local compactness is formulated in these settings and solve the problem of reconstructing a locally compact space (in any of the formulations) from an abstract basis in the sense of this paper.

Definition 6.1 A ***frame*** Ω is a lattice with arbitrary joins ($\bigvee$) over which meets ($\wedge$) distribute,

$$U \wedge \bigvee V_i = \bigvee (U \wedge V_i),$$

so the lattice ΩX of open subspaces of any topological space X is an example. Accordingly, a ***frame homomorphism*** $f^* : \Omega_2 \to \Omega_1$ is a function that preserves $\bigvee$, $\top$ and $\wedge$, just as the inverse image operator $f^{-1} : \Omega Y \to \Omega X$ does for any continuous function $f : X \to Y$. Frames and homomorphisms form a category, but when we want to use them to discuss topological ideas we use the names ***locale*** and ***continuous map*** instead, when referring to the objects and morphisms of the *opposite* category.

For compatibility with Point–Set Topology, we shall (sometimes) continue to use capital letters for elements of a frame. However, we write $U \leq V$ instead of $U \subset V$ for the order, because it is abstract and not necessarily represented by an inclusion (*cf.* Warning 6.17). As we have already done, we also use $\wedge$ and $\bigvee$ instead of $\cap$ and $\bigcup$ for the operations.

There are no points or sets of them in the definition of a locale, but Definitions 3.13 and 3.16 provide substitutes for these features:

Definition 6.2 In the locale defined by a frame Ω,
(a) a ***formal point*** is a completely coprime filter $\mathcal{P} \subset \Omega$;
(b) a ***formal open subspace*** is an element $U \in \Omega$ of the frame;
(c) a formal point $\mathcal{P}$ ***lies in*** a formal open subspace U if $\mathcal{P} \ni U$;
(d) a ***formal Scott-open family*** is a Scott-open subset $\mathcal{K} \subset \Omega$ of the frame;
(e) a ***formal compact subspace*** is a Scott-open filter $\mathcal{K} \subset \Omega$;
(f) a formal open subspace U ***covers*** a formal compact subspace $\mathcal{K}$ if $\mathcal{K} \ni U$; and
(g) a formal point $\mathcal{P}$ ***lies in*** a formal compact subspace $\mathcal{K}$ if $\mathcal{P} \supset \mathcal{K}$.
Note that these "compact" subspaces are also saturated in the sense of Definition 3.16.

Some aspects of Locale Theory owe more to its algebraic formulation than to topology, the following being an important example:

Definition 6.3 Following standard categorical usage, a ***sublocale*** is one that arises as the equaliser of some pair of continuous maps between locales, which means that the frame is the coequaliser of some pair of frame homomorphisms. In *universal* algebra, a coequaliser is calculated as the quotient by a congruence, but in *particular* algebraic theories there are sometimes more concise ways of describing congruences, such as a *normal* subgroup or an *ideal* of a ring.

In our case, any continuous function $i : X \to Y$ has direct and inverse image operations, i_* and $i^* \equiv i^{-1}$ respectively, which both preserve finite meets and also satisfy $\mathsf{id}_{\Omega Y} \leqslant i_* \cdot i^*$ and $i^* \cdot i_* \leqslant \mathsf{id}_{\Omega X}$. These arise from a monomorphism iff $i^* \cdot i_* = \mathsf{id}_{\Omega X}$. The situation therefore is captured by the composite $j \equiv i_* \cdot i^*$, which is called a ***nucleus*** on ΩY (a pun on *kernel*) and satisfies

$$\mathsf{id} \leqslant j = j^2 \quad \text{and} \quad j(U \wedge V) = jU \wedge jV.$$

Beware that there are rather more sublocales than there are subspaces in Point–Set Topology, but the familiar cases of the open and closed sublocales named by the element $U \in \Omega$ are given by the nuclei

$$U \Rightarrow (-) \quad \text{and} \quad U \vee (-)$$

respectively [Joh82, Exercise II 2.4].

Granted, there are conceptual differences like this, whilst there are hard problems like the one that blocked our progress in the last section that really do depend on finding points. On the other hand, there are a great many arguments in general topology where the only role of the points is to say how one formula involving finite unions and intersections of open subspaces compares with another. It is a straightforward exercise to rewrite these in Locale Theory.

In particular, we can translate Definition 1.1 for bases:

Definition 6.4 A ***concrete basis using open subspaces*** indexed by a preorder $(A, \sqsubseteq)$ for a frame or locale Ω has

(a) for each $a \in A$, an element $U_a \in \Omega$ of the frame, such that

(b) if $a \sqsubseteq b$ then $U_a \leq U_b$;

(c) $U_a \wedge U_b \ = \ \bigvee \{U_c \mid a \sqsupseteq c \sqsubseteq b\}$; and

(d) $U = \bigvee \{U_a \mid U_a \leq U\}$ for any $U \in \Omega$.

Remark 6.5 Since frames are algebras, we present them by means of generators and equations. A set of generators for a frame is a basis using open subspaces, whilst quotients are captured by nuclei. We therefore need a convenient way of constructing nuclei from equations between generators of frames.

Such equations relate expressions using finite meets and arbitrary joins, but these can be simplified using distributivity. The following technique, called Formal Topology, seems to be the most efficient way of expressing them.

In this notation the elements of the frame are written as *lower case* letters. Whilst these stand for (possibly infinite) subsets, beware that they are subsets of the *basis* A (*cf.* the first line of Definition 5.4) and not of the set of *points* (*cf.* the displayed equations there) as in Point–Set Topology. We shall give the connection between these two subsets at the end of this section.

Definition 6.6 A ***formal cover*** $(A, \sqsubseteq, \lhd)$ consists of a preorder $\sqsubseteq$ on a set A together with a relation $a \lhd u$ between elements and (possibly infinite) subsets of A such that

$$a \in u \Longrightarrow a \lhd u, \qquad b \sqsubseteq c \lhd u \sqsubseteq v \Longrightarrow b \lhd v,$$

$$a \lhd u \lhd v \Longrightarrow a \lhd v \qquad \text{and} \qquad c \lhd u \wedge c \lhd v \iff c \lhd u \sqcap v,$$

where $\qquad u \lhd v \ \equiv \ \forall b \in u.\, b \lhd v, \qquad u \sqsubseteq v \ \equiv \ \forall b \in u.\, \exists c \in v.\, b \sqsubseteq c$

and $\qquad u \sqcap v \ \equiv \ \{b \mid u \sqsupseteq b \sqsubseteq v\} \ \equiv \ \{b \mid (\exists c \in u.\, b \sqsubseteq c) \ \wedge \ (\exists d \in v.\, b \sqsubseteq d)\}.$

Therefore $\sqcap$ and not $\cap$ is the meet operation corresponding to the preorder $\sqsubseteq$. In particular, $u \sqcap v$ itself is meaningful as a (possibly infinite) subset of A, whereas Notation 1.7 only defined the whole phrase $a \prec\!\!\!\prec k \sqcap \ell$. However, we shall see that these usages agree where we need it in Proposition 7.3 and Lemma 7.7.

Warning 6.7 You will be relieved to learn that there are no secondary axioms for formal covers. However, in the literature on Formal Topology, it is commonly assumed that the preorder $(A, \sqsubseteq)$ is a $\sqcup$-semilattice, *but often without saying so*. Also, formal topologists write a capital U where we have lower case $u \subset A$.

We leave the soundness of these axioms to the assiduous student:

Lemma 6.8 For any A-indexed concrete basis using open subspaces for a topological space or frame, the relation

$$a \lhd u \quad \text{defined by} \quad U_a \;\le\; U_u, \quad \text{where} \quad U_u \;\equiv\; \bigvee \{U_b \mid b \in u\}$$

is a formal cover with the same $\sqsubseteq$. $\square$

Lemma 6.9 Given any formal cover $(A, \sqsubseteq, \lhd)$, the map j on subsets of A that takes

$$u \subset A \qquad \text{to} \qquad ju \;\equiv\; \{a \mid a \lhd u\} \subset A$$

is a nucleus on $\mathcal{D}(A, \sqsubseteq)$, since

$$ja \subset ju \iff a \lhd u \qquad \text{and} \qquad ju \cap jv \;=\; j(u \sqcap v).$$

Conversely, any nucleus defines a cover by $a \lhd u \iff a \in ju$ and these processes are inverse.

Proof If $u \subset v$ then $\forall a.\, a \lhd u \Longrightarrow a \lhd v$ so $ju \subset jv$.
If $a \in u$ then $a \lhd u$, so $u \subset ju$.
Therefore $a \lhd u \iff a \in ju \iff ja \subset ju$ and $u \lhd v \iff u \subset jv \iff ju \subset jv$.
If $a \in ju$ then $a \lhd u$ so $ju \lhd u$ and $j(ju) = ju$.
For the intersection, $ju \cap jv = j(u \sqcap v)$ because $a \lhd u \wedge a \lhd v \iff a \lhd u \sqcap v$.
The arguments for the converse and bijection are similar, noting that j acts on lower and not arbitrary subsets of A. $\square$

Theorem 6.10 Every formal cover $(A, \sqsubseteq, \lhd)$ presents a frame or locale

$$\Omega \;\equiv\; \{u \subset A \mid u = ju \equiv \{a \mid a \lhd u\}\}$$

where $\qquad (u \le v) \;\equiv\; (u \subset v) \iff (u \lhd v), \quad \text{and} \quad U_a \;\equiv\; \{b \mid b \lhd a\} \;\in\; \Omega.$

provides a concrete basis using open subspaces.

Conversely, any locale with a concrete basis using open subspaces is recovered up to isomorphism from the formal cover that it defines, where

$$u \mapsto \bigvee \{U_b \mid b \lhd u\} \qquad \text{and} \qquad U \mapsto \{a \mid U_a \le U\}$$

and the basic open subspaces are U_a and $ja \;=\; \{b \mid b \lhd a\} \;=\; \{b \mid U_b \le U_a\}$.

Note that we have put no countability restriction on this result as we did in Lemmas 5.9ff and Theorem 5.12: it holds for *any* formal cover.

Proof The lattice operations are

$$\top \;\equiv\; A \;\in\; \Omega \qquad u \wedge v \;\equiv j(u \sqcap v) \qquad \text{and} \qquad \bigvee u_i \;\equiv\; j(\bigcup u_i)$$

are in Ω whenever $u, v, u_i \in \Omega$. We have $\bigvee (u \wedge v_i) \;\le\; u \wedge \bigvee v_i$ trivially. Conversely, writing $v \equiv \bigcup v_i$,

$$\begin{aligned}
u \sqcap v \;\equiv\; u \sqcap \bigcup v_i \quad &= \quad \{d \mid \exists a \in u.\, \exists i.\, \exists b \in v_i.\, a \sqsupseteq d \sqsubseteq b\} \\
&= \quad \bigcup (u \sqcap v_i) \;\lhd\; \bigcup j(u \sqcap v_i) \;\equiv\; \bigcup (u \wedge v_i). \\
\text{and so} \qquad c \in u \wedge \bigvee v_i \quad &\Rightarrow \quad c \lhd u \sqcap \bigvee v_i \Longrightarrow c \lhd u \;\wedge\; c \lhd \bigvee v_i \\
&\Rightarrow \quad c \lhd u \;\wedge\; c \lhd v \Longrightarrow c \lhd u \sqcap v \lhd \bigcup (u \sqcap v_i) \\
&\Rightarrow \quad c \in \bigvee (u \wedge v_i).
\end{aligned}$$

Hence Ω is a frame. The concrete basis using open subspaces is $U_a \equiv ja$. This is covariant in a, filtered:

$$U_a \wedge U_b \;\equiv\; \{d \mid a \rhd d \lhd b\} \quad \lhd \quad \{c \mid a \sqsupseteq c \sqsubseteq b\} \;\lhd\; \bigvee \{U_c \mid a \sqsupseteq c \sqsubseteq b\},$$

and has the basis expansion $U = \bigvee \{U_a \mid U_a \leq U\}$ because $u \lhd \{a \mid ja \lhd u\} \equiv ju$.

We recover the formal $\lhd$ relation because

$$ja \leq \bigvee u_i \;\equiv\; ja \subset j\big(\bigcup u_i\big) \iff a \lhd \bigcup u_i$$

by the Lemma. $\square$

Whilst we have introduced Ω here as a *subset* of the powerset $\mathcal{P}(A)$, in fact it is a retract and really it should be seen as a *quotient* of the lattice $\mathcal{D}(A, \sqsubseteq)$ of $\sqsubseteq$-lower subsets of A. That is, we are using a general subset $u \subset A$ to denote an element $ju \in \Omega$ of the frame.

We can sum all of this up by saying that the various structures that we have considered all express a general locale as a sublocale of one of a particular simple kind. We constructed this as a topological space in Proposition 5.14:

Lemma 6.11 For any preorder $(A, \sqsubseteq)$, the relation

$$a \lhd^0 u \;\equiv\; \exists b.\, a \sqsubseteq b \subset u$$

makes $(A, \sqsubseteq, \lhd)$ a formal cover that presents the frame $\mathcal{D}(A, \sqsubseteq)$ of lower subsets of the preorder, which is the topology on $\mathbf{Filt}(A, \sqsubseteq) \cong \mathbf{Idl}\,(A, \sqsubseteq^{\mathsf{op}})$. $\square$

Theorem 6.12 For any preorder $(A, \sqsubseteq)$, there is a bijective correspondence up to isomorphism amongst

(a) a locale Ω with basis using open subspaces (U_a) indexed by $(A, \sqsubseteq)$;

(b) a formal cover $(A, \sqsubseteq, \lhd)$;

(c) a nucleus j on the frame $\mathcal{D}(A, \sqsubseteq)$;

(d) a quotient frame of $\mathcal{D}(A, \sqsubseteq)$; and

(e) a sublocale of $\mathbf{Filt}(A, \sqsubseteq) \cong \mathbf{Idl}\,(A, \sqsubseteq^{\mathsf{op}})$.

Containment $X \overset{i}{\hookrightarrow} Y \hookrightarrow \mathbf{Filt}(A, \sqsubseteq)$ is expressed by $i^* : \Omega Y \twoheadrightarrow \Omega X$, $i_* : \Omega X \hookrightarrow \Omega Y$, $a \lhd_Y u \Longrightarrow a \lhd_X u$ and $j_Y \leqslant j_X$. $\square$

Using arguments analogous to those in Section 4, we can go on to express frame homomorphisms or continuous functions between locales in terms of a basis and therefore a formal cover:

Proposition 6.13 There is a bijective correspondence between continuous maps between locales and matrices, defined by

$$\big[\, a \,\big|\, f \,\big|\, b \,\big] \;\equiv\; \big(a \in f^*(jb)\big) \qquad \text{and} \qquad f^* v \;\equiv\; \{a \mid \exists b.\, \big[\, a \,\big|\, f \,\big|\, b \,\big] \;\wedge\; b \lhd v\},$$

where the matrices satisfy

$$\begin{array}{lll}
a \sqsubseteq a' \;\wedge\; \big[\, a' \,\big|\, f \,\big|\, b' \,\big] \;\wedge\; b' \sqsubseteq b & & \\
\qquad\Rightarrow\; \big[\, a \,\big|\, f \,\big|\, b \,\big] & & \text{co- \& contravariance} \\
\big[\, a \,\big|\, f \,\big|\, b \,\big] \;\wedge\; b \lhd v \;\Rightarrow\; \exists u.\, a \lhd u \;\wedge\; \forall a' \in u.\, \exists b' \in v.\, \big[\, a' \,\big|\, f \,\big|\, b' \,\big] & & \text{partition} \\
a \lhd u \;\wedge\; \forall a' \in u.\, \big[\, a' \,\big|\, f \,\big|\, b \,\big] & & \\
\qquad\Rightarrow\; \big[\, a \,\big|\, f \,\big|\, b \,\big] & & \text{saturation} \\
\big[\, a \,\big|\, f \,\big|\, b_1 \,\big] \;\wedge\; \big[\, a \,\big|\, f \,\big|\, b_2 \,\big] & & \text{filteredness} \\
\qquad\Rightarrow\; \exists u.\, a \lhd u \;\wedge\; \forall a' \in u.\, \exists b.\, \big[\, a' \,\big|\, f \,\big|\, b \,\big] \;\wedge\; b_1 \sqsupseteq b \sqsubseteq b_2 & & \\
\qquad\; \exists u.\, a \lhd u \;\wedge\; \forall a' \in u.\, \exists b.\, \big[\, a' \,\big|\, f \,\big|\, b \,\big]. & & \text{boundedness}
\end{array}$$

$\square$

Saturation is required on the right of the boundedness and filteredness rules for the same reason as in Remark 4.15ff. For example, let $f : X \to Y$ be $\mathsf{id} : \mathbb{R} \to \mathbb{R}$, but where the whole line is a basic open in X but not in Y.

We can deduce the characterisation of formal points from this as we did in Definition 5.1.

Definition 6.14 A ***formal point*** for a formal cover is a subset $p \subset A$ such that

$$\begin{array}{rcll} & & \exists a.\, a \in p & \text{bounded} \\ a \sqsupseteq b \in p & \Rightarrow & a \in p & \text{upper} \\ (a \in p) \wedge (b \in p) & \Rightarrow & \exists c.\, (a \sqsupseteq c \sqsubseteq b) \wedge c \in p & \text{filtered} \\ (a \in p) \wedge (a \lhd u) & \Rightarrow & u \between p \equiv \exists b.\, (b \in u) \wedge (b \in p). & \text{positive} \end{array}$$

Proposition 6.15 The correspondence with Definition 3.13 is

$$p \equiv \{a \mid \mathcal{P} \ni ja\} \subset A \qquad \text{and} \qquad \mathcal{P} \equiv \{u \mid p \between u = ju\} \subset \Omega,$$

so that $$p \text{ lies in } u \quad \text{iff} \quad p \between u.$$

Proof We prove this in detail because we intend to use it as part of our construction in Point–Set Topology. Given a completely coprime filter $\mathcal{P} \subset \Omega$, the set p is upper because $\mathcal{P}$ is and j preserves inclusions. Also p is bounded because $\mathcal{P} \ni A = \bigvee \{ja \mid a \in A\}$ so $\exists a.\, \mathcal{P} \ni ja$ since it is completely coprime. For the filter property of p,

$$\begin{array}{rcl} a \in p \ni b & \equiv & ja \in \mathcal{P} \ni jb \\ & \Rightarrow & \mathcal{P} \ni ja \cap jb = j(a \sqcap p) = \bigvee \{jc \mid c \in a \sqcap b\} \\ & \Rightarrow & \exists c.\, \mathcal{P} \ni jc \wedge (a \sqsupseteq c \sqsubseteq b) \Longrightarrow \exists c.\, p \ni c \in a \sqcap b. \end{array}$$

For positivity,

$$\begin{array}{rcl} p \ni a \lhd u & \Rightarrow & \mathcal{P} \ni ja \subset ja = \bigvee \{jb \mid b \in u\} \\ & \Rightarrow & \exists b.\, \mathcal{P} \ni jb \wedge b \in u \Longrightarrow \exists b.\, p \ni b \in u. \end{array}$$

Conversely, given p, the family $\mathcal{P}$ is upper since $p \between u \subset v \Rightarrow p \between v$ and bounded since p is and so $\exists a.\, a \in p \wedge \mathcal{P} \ni ja$. For the filter property of $\mathcal{P}$,

$$\begin{array}{rcl} u \in \mathcal{P} \ni v & \Rightarrow & u \between p \between v \Longrightarrow \exists ab.\, u \ni a \in p \ni b \in v \\ & \Rightarrow & \exists c.\, u \sqcap v \ni c \in p \Longrightarrow \mathcal{P} \ni u \cap v. \end{array}$$

We recover $\mathcal{P}$ from p because $\mathcal{P}$ is completely coprime and

$$\begin{array}{rcl} \{u \mid p \between u = ju\} & = & \{u \mid \exists a.\, \mathcal{P} \ni ja \wedge a \in u = ju\} \\ & = & \{u \mid u = \bigvee \{ja \mid \mathcal{P} \ni ja \wedge a \in u\}\} = \mathcal{P}. \end{array}$$

We recover p from $\mathcal{P}$ because it is positive and

$$\{a \mid \mathcal{P} \ni ja\} = \{a \mid p \between ja\} = \{a \mid \exists b.\, p \ni b \lhd a\} = p.$$

Finally, recall that $\mathcal{P}$ lies in U iff $\mathcal{P} \ni U$. □

When we make the connection between $\lhd$ and $\prec\!\!\!\prec$ in the next section, we will see that the notions of formal point for these two relations also agree (Lemma 7.11), with positivity playing the same role as locatedness and roundedness together.

Proposition 6.16 The function $u \mapsto U_u \equiv \{p \mid p \between u\}$, which is called the ***extent*** of u, is a frame homomorphism.

Proof From the first three axioms, $\top \between p$ and $u \between p \between v \Longrightarrow p \between (u \sqcap v)$, so extent preserves finite meets. By the last, $p \between ju \iff p \between u$, so $p \between \bigvee u_i \iff p \between \bigcup u_i \iff \exists i.\, p \between u_i$ and extent preserves joins. □

Warning 6.17 Although the formal opens $u \in \Omega$ in Theorem 6.10 are sets, they are sets of basis elements and not sets of (formal) *points* as they were in Section 5. Indeed, *the formal opens of a locale need not in general be faithfully representable as sets of points at all*, since the extent need not be an isomorphism [Joh82]. A frame, locale or formal cover for which this is an isomorphism is called ***spatial*** or is said to ***have enough points***.

Since we just need $U_a \leq U_u \Longrightarrow a \lhd u$, the characterisation in terms of $\lhd$ is this:

Proposition 6.18 A formal cover $\lhd$ has enough points iff

$$\big(\forall p.\, a \in p \Rightarrow p \between u\big) \Longrightarrow a \lhd u. \qquad \square$$

7 Local compactness without points

We now give our definitions of local compactness for Locale Theory and Formal Topology. We use these and the methods of the previous section to re-construct locally compact spaces from abstract bases in these formulations and also in Point–Set Topology. This solves the problem that we left open in Section 5.

We will show that an abstract basis $\prec\!\!\!\ll$ presents a locally compact space in a very similar manner to that in which a formal cover $\lhd$ presented a locale in the previous section. The difference is simply that $a \lhd u$ means $U_a \subset U_u$ whilst $a \prec\!\!\!\ll \ell$ means $K_a \subset U_\ell$, so we expect $a \prec\!\!\!\ll \ell \Longrightarrow a \lhd \ell$ but not the converse. This section gives the precise correspondence between these relations, using only the primary axioms for $\prec\!\!\!\ll$.

Our presentation is heavily influenced by that of Sara Negri [Neg02, Definition 4.10], although she used the term *locally Stone* and wrote $i(a)$ for our $\{\ell \mid a \prec\!\!\!\ll \ell\}$.

We begin by translating Definition 2.6, with the aid of Definitions 6.1 and 6.2.

Definition 7.1 A ***concrete basis using Scott-open families*** indexed by a preorder $(A, \sqsubseteq)$ for a locale or frame Ω has
(a) for each $a \in A$, an element $U_a \in \Omega$ and a Scott-open subset $\mathcal{K}_a \subset \Omega$; such that
(b) if $a \sqsubseteq b$ then $U_a \leq U_b$ and $\mathcal{K}_a \supset \mathcal{K}_b$;
(c) $U_a \wedge U_b \ = \ \bigvee \{U_c \mid a \sqsupseteq c \sqsubseteq b\}$; and
(d) the basis expansion, $U = \bigvee \{U_a \mid \mathcal{K}_a \ni U\}$ for any $U \in \Omega$.

We define a ***locally compact locale*** to be one that has a basis of this kind, where the Scott-open families are *arbitrary.* This is not, however, the definition that is standardly found in the literature on Locale Theory, which uses a *canonical* Scott-open family, namely the *largest* one. We discuss this and the candidate definitions in Formal Topology in the next section.

We first rewrite this definition using the cover and way-below relations:

Lemma 7.2 Any such basis $(U_a, \mathcal{K}_a)$ gives rise to a formal cover $(A, \sqsubseteq, \lhd)$ and a relation

$$a \prec\!\!\!\ll \ell \quad \equiv \quad \mathcal{K}_a \ni U_\ell$$

that satisfy

$$a \sqsubseteq b \prec\!\!\!\ll \ell \implies a \prec\!\!\!\ll \ell, \qquad a \ \lhd \ \Downarrow a \equiv \{b \mid b \prec\!\!\!\ll a\},$$

$$a \prec\!\!\!\ll \ell \implies a \lhd \ell \qquad \text{and} \qquad a \prec\!\!\!\ll \ell \lhd u \iff \exists k.\, a \prec\!\!\!\ll k \subset u.$$

Proof The basis *â fortiori* also uses open subspaces (Definition 6.4) and so defines an abstract formal cover $(A, \sqsubseteq, \lhd)$. Note that this uses the filter condition (c) for the concrete basis, which we do not otherwise mention.

The first property is contravariance of $\mathcal{K}_{(-)}$ (part (b) of the Definition), whilst the second and third follow from the basis expansion (d):

$$U_a \ \leq \ \bigcup \{U_b \mid \mathcal{K}_b \ni U_a\} \qquad \text{and} \qquad \mathcal{K}_a \ni U_\ell \implies U_a \leq U_\ell.$$

The last uses the fact that $\mathcal{K}_a$ is Scott open:

$$a \prec\!\!\!\ll \ell \lhd u \ \equiv \ \mathcal{K}_a \ni U_\ell \leq U_u \implies \exists k.\, \mathcal{K}_a \ni U_k \ \wedge \ k \subset u \ \equiv \ \exists k.\, a \prec\!\!\!\ll k \subset u.$$

The converse of this last property is easy because $k \subset u \Rightarrow k \lhd u$. □

Proposition 7.3 For any locale with a basis using Scott-open families, $(A, \sqsubseteq, \prec\!\!\!\ll)$ satisfies the primary axioms for an abstract basis and

$$b \lhd v \iff (\forall a.\, a \prec\!\!\!\ll b \Rightarrow \exists \ell.\, a \prec\!\!\!\ll \ell \subset v).$$

Proof The fourth property in the Lemma and covariance of $\lhd$ give that of $\prec\!\!\!\ll$:

$$a \prec\!\!\!\ll k \sqsubseteq \ell \implies a \prec\!\!\!\ll k \lhd \ell \implies a \prec\!\!\!\ll \ell.$$

The (weak) intersection property of $\prec\!\!\!\prec$ follows from that of $\lhd$ using the third and fourth parts of the Lemma:

$$\begin{aligned} a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1, \ell_2 &\implies a \prec\!\!\!\prec k \lhd \ell_1, \ell_2 \\ &\implies a \prec\!\!\!\prec k \lhd (\ell_1 \sqcap \ell_2) \;\equiv\; \{b \mid \ell_1 \sqsupseteq b \sqsubseteq \ell_2\} \\ &\implies (\exists \ell.\, a \prec\!\!\!\prec \ell \subset \ell_1 \sqcap \ell_2) \;\equiv\; (a \prec\!\!\!\prec \ell_1 \sqcap \ell_2). \end{aligned}$$

The Wilker rule comes from the second and fourth parts:

$$\begin{aligned} a \prec\!\!\!\prec \ell &\implies a \prec\!\!\!\prec \ell \lhd \{b \mid \exists c.\, b \prec\!\!\!\prec c \in \ell\} \\ &\implies \exists k.\, a \prec\!\!\!\prec k \subset \{b \mid \exists c.\, b \prec\!\!\!\prec c \in \ell\} \;\equiv\; \exists k.\, a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell. \end{aligned}$$

The forward direction of the formula for $\lhd$ is the fourth part of the Lemma. Conversely, let $u \equiv \{a \mid a \prec\!\!\!\prec b\}$, so $u \lhd v$ because $a \prec\!\!\!\prec \ell \subset v \implies a \lhd v$, and then $b \lhd u \lhd v$. □

Such bases may be upgraded to satisfy the secondary axioms, including the strong intersection rule. Continuous functions between locales also correspond bijectively to matrices. We leave the assiduous student to show these things by translating the arguments in Sections 3, 12 and 4.

Now we turn to the construction of concrete locally compact spaces and bases from abstract ones, starting with the formal cover relation $\lhd$.

Definition 7.4 A ***locally compact formal cover*** $(A, \sqsubseteq, \lhd)$ is one that arises from some abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ by

$$(b \lhd u) \;\equiv\; (\forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u),$$

although we still have to justify its properties.

Lemma 7.5 This cover relation satisfies

$$b \in u \implies b \lhd u, \qquad b \sqsubseteq c \lhd u \sqsubseteq v \implies b \lhd v,$$

$$b \;\lhd\; \downarrow\!\!\!\downarrow b \equiv \{a \mid a \prec\!\!\!\prec b\} \qquad a \prec\!\!\!\prec \ell \implies a \lhd \ell$$

and

$$a \prec\!\!\!\prec \ell \lhd u \iff \exists k.\, a \prec\!\!\!\prec k \subset u.$$

Proof

$$\begin{array}{llll}
b \lhd u & \equiv & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u & \\
 & \Leftarrow & \forall a.\, (a \prec\!\!\!\prec b \Rightarrow a \prec\!\!\!\prec b \in u) & \\
 & \Leftarrow & b \in u & \\
b \sqsubseteq c \lhd u & \equiv & b \sqsubseteq c \;\wedge\; \forall a.\, a \prec\!\!\!\prec c \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u & \\
 & \Rightarrow & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u & \\
 & \equiv & b \lhd u & \text{contravariance of } \prec\!\!\!\prec \\
c \lhd u \sqsubseteq v & \equiv & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists k.\, a \prec\!\!\!\prec k \subset u \sqsubseteq v & \\
 & \Rightarrow & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists k\ell.\, a \prec\!\!\!\prec k \sqsubseteq \ell \subset v & \\
 & \equiv & c \lhd v & \\
b \lhd \ell & \equiv & \forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists k.\, a \prec\!\!\!\prec k \subset \ell & \\
 & \Leftarrow & \forall a.\, a \prec\!\!\!\prec b \Rightarrow a \prec\!\!\!\prec \ell & \\
 & \Leftarrow & b \prec\!\!\!\prec \ell & \text{transitivity of } \prec\!\!\!\prec \\
c \lhd \downarrow\!\!\!\downarrow c & \equiv & \forall a.\, a \prec\!\!\!\prec c \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset \{b \mid b \prec\!\!\!\prec c\} & \\
 & \Leftarrow & \forall a.\, a \prec\!\!\!\prec c \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \prec\!\!\!\prec c. & \text{Wilker} \\
a \prec\!\!\!\prec \ell \lhd u & \Rightarrow & \exists h.\, a \prec\!\!\!\prec h \prec\!\!\!\prec^1 \ell \;\wedge\; (\forall b.\, \forall c \in \ell.\, b \prec\!\!\!\prec c \Rightarrow \exists k.\, b \prec\!\!\!\prec k \subset u) & \text{Wilker} \\
 & \Rightarrow & \exists h.\, a \prec\!\!\!\prec h \;\wedge\; (\forall b \in h.\, \exists k.\, b \prec\!\!\!\prec k \subset u) & \\
 & \Rightarrow & \exists hk.\, a \prec\!\!\!\prec h \prec\!\!\!\prec k \subset u & \\
 & \Rightarrow & \exists k.\, a \prec\!\!\!\prec k \subset u. & \text{transitivity} \quad \square
\end{array}$$

The proof of the other two properties of $\lhd$ of course depends on the Wilker and weak intersection rules for $\prec\!\!\!\prec$.

Lemma 7.6 If $c \lhd u \lhd v$ then $c \lhd v$.

Proof Suppose that $a \prec\!\!\!\prec c \lhd u \lhd v$. Since $c \lhd u$ means

$$\forall a.\, (a \prec\!\!\!\prec c) \Rightarrow \exists \ell.\, (a \prec\!\!\!\prec \ell \subset u),$$

there is some finite set ℓ with $a \prec\!\!\!\prec \ell \subset u$. Then by the Wilker rule there is another finite set k with

$$a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell \subset u \;\equiv\; (a \prec\!\!\!\prec k) \;\wedge\; \forall b \in k.\, \exists c \in \ell.\, (b \prec\!\!\!\prec c \in u).$$

We combine this with $u \lhd v \;\equiv\; \forall bc.\, (b \prec\!\!\!\prec c \in u \Rightarrow \exists h.\, b \prec\!\!\!\prec h \subset v)$ to give

$$a \prec\!\!\!\prec k \;\wedge\; \forall b \in k.\, \exists h_b.\, b \prec\!\!\!\prec h_b \subset v.$$

Taking $h \equiv \bigcup\{h_b \mid b \in k\} \subset v$, we obtain $a \prec\!\!\!\prec k \prec\!\!\!\prec h \subset v$, from which $a \prec\!\!\!\prec h \subset v$ follows by transitivity of $\prec\!\!\!\prec$. Hence $c \lhd v$. □

Lemma 7.7 If $c \lhd u$ and $c \lhd v$ then $c \lhd u \sqcap v$.

Proof Given $a \prec\!\!\!\prec c$, we first interpolate $a \prec\!\!\!\prec \ell \prec\!\!\!\prec c$, so $a \prec\!\!\!\prec \ell \;\wedge\; \forall b \in \ell.\, b \prec\!\!\!\prec c$.

Combining this with $c \lhd u$ and $c \lhd v$ gives

$$a \prec\!\!\!\prec \ell \;\wedge\; (\forall b \in \ell.\, \exists h_b.\, b \prec\!\!\!\prec h_b \subset u) \;\wedge\; (\forall b \in \ell.\, \exists k_b.\, b \prec\!\!\!\prec k_b \subset v).$$

Taking $h \equiv \bigcup\{h_b \mid b \in \ell\} \subset u$ and $k \equiv \bigcup\{k_b \mid b \in \ell\} \subset v$, we obtain

$$a \prec\!\!\!\prec \ell \prec\!\!\!\prec h \subset u \;\wedge\; \ell \prec\!\!\!\prec k \subset v.$$

Then the weak intersection rule gives $a \prec\!\!\!\prec h \sqcap k$, which means

$$\exists \ell'.\, a \prec\!\!\!\prec \ell' \;\wedge\; \forall b \in \ell'.\, (\exists c.\, b \sqsubseteq c \in h \subset u) \;\wedge\; (\exists d.\, b \sqsubseteq d \in k \subset v),$$

but this is $a \prec\!\!\!\prec \ell' \subset u \sqcap v$. Hence $c \lhd u \sqcap v$. □

Theorem 7.8 Any abstract basis satisfying the primary axioms presents a (locally compact) formal cover. □

That was easy because formal covers and abstract bases are so similar. Even so, we have at last solved the completeness problem for *some* version of topology, so now we can re-trace our steps *via* Locale Theory back to Point–Set Topology.

Lemma 7.9 The frame Ω constructed in Theorem 6.10 from a locally compact formal cover has as elements those subsets $u \subset A$ such that

$$b \prec\!\!\!\prec \ell \subset u \implies b \in u \qquad \text{and} \qquad \downarrow\!\!\!\downarrow b \;\equiv\; \{a \mid a \prec\!\!\!\prec b\} \subset u \implies b \in u. \qquad \square$$

Proof By construction, $u \in \Omega$ iff $u = ju \equiv \{b \mid b \lhd u\}$, iff $b \in u \iff b \lhd u$. If this holds then

$$a \prec\!\!\!\prec \ell \subset u \implies a \lhd \ell \lhd u \implies a \lhd u \implies a \in u$$

and
$$\downarrow\!\!\!\downarrow b \subset u \implies b \lhd \downarrow\!\!\!\downarrow b \lhd u \implies b \lhd u \implies b \in u.$$

Conversely, if $u \subset A$ has these two closure properties then

$$b \lhd u \implies (\forall a.\, a \prec\!\!\!\prec b \Rightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset u) \implies (\forall a.\, a \prec\!\!\!\prec b \Rightarrow a \in u) \implies b \in u. \qquad \square$$

Theorem 7.10 Any abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ obeying the primary axioms presents a locally compact locale, in which

$$U_a \;\equiv\; ja \;\equiv\; \{b \mid b \lhd a\} \;\in\; \Omega \qquad \text{and} \qquad \mathcal{K}_a \;=\; \{u \mid \exists \ell.\, a \prec\!\!\!\prec \ell \subset u\} \;\subset\; \Omega$$

provide a concrete basis using Scott-open families and $a \prec\!\!\!\prec \ell \iff \mathcal{K}_a \ni U_\ell$.

Proof (U_a) is already a concrete basis using open subspaces. The first three parts of Definition 7.1 follow easily from the properties of $\prec\!\!\!\prec$ and $\lhd$. Using the two closure properties in the previous Lemma,

$$\text{if} \quad b \lhd u \quad \text{then} \quad b \ \lhd \ {\Downarrow} b \ \equiv \ \{a \mid a \prec\!\!\!\prec b\} \ \subset \ \{a \mid \exists \ell.\, a \prec\!\!\!\prec \ell \subset u\},$$

so

$$u \ \lhd \ \{a \mid \exists \ell.\, a \prec\!\!\!\prec \ell \subset u\}.$$

Rewriting this using the definition of the basis gives

$$u \ \equiv \ ju \ = \ j\{a \mid \exists \ell.\, a \prec\!\!\!\prec \ell \subset u\} \ = \ \bigvee \{ja \mid \mathcal{K}_a \ni U\},$$

which is the required basis expansion of $u = ju \in \Omega$ using Scott-open families. Finally,

$$\mathcal{K}_a \ni U_\ell \ \equiv \ \exists k.\, a \prec\!\!\!\prec k \subset j\ell \ \equiv \ \exists k.\, a \prec\!\!\!\prec k \lhd \ell \iff \exists k'.\, a \prec\!\!\!\prec k' \subset \ell \iff a \prec\!\!\!\prec \ell$$

by the final part of Lemma 7.5. □

Again, this analogue of Lemma 5.10 is valid for locales in complete generality, not just countably based ones, and we have not used the Axiom of Choice, Excluded Middle or the secondary axioms. This is because we avoided using points, even formal ones.

On the other hand, we can re-introduce the points to prove the classical version of the theorem, but now without either the countability restriction or single interpolation.

Lemma 7.11 Definitions 3.13, 5.1 and 6.14 for formal points in terms of completely prime filters or the relations $\prec\!\!\!\prec$ and $\lhd$ agree.

Proof Proposition 6.15 showed that the definitions in terms of completely prime filters and $\lhd$ are equivalent.

The relations $\lhd$ and $\prec\!\!\!\prec$ share the properties of being upper, bounded and filtered. We therefore just have to show that a subset p is rounded and located (with respect to $\prec\!\!\!\prec$) iff it is positive (with respect to $\lhd$).

Substituting the definition of $\lhd$ from $\prec\!\!\!\prec$, $p \subset A$ is positive iff, for all $b \in A \supset u$,

$$b \in p \ \wedge \ b \lhd u \quad \equiv \quad b \in p \ \wedge \ (\forall a.\, a \prec\!\!\!\prec b \ \Rightarrow \ \exists \ell.\, a \prec\!\!\!\prec \ell \subset u) \quad \Longrightarrow \quad p \between u.$$

If p is positive and $b \in p$, put $u \equiv {\Downarrow} b \equiv \{a \mid a \prec\!\!\!\prec b\}$. Then $b \lhd u$ (the bracketed clause holds) because if $a \prec\!\!\!\prec b$ then $a \in u$ and we may interpolate $a \prec\!\!\!\prec \ell \prec\!\!\!\prec b$ by Wilker, so $a \prec\!\!\!\prec \ell \subset u$. Then by positivity there is some $d \in p \cap u$, so $p \ni d \prec\!\!\!\prec b$. Hence p is rounded.

If further $p \ni b \prec\!\!\!\prec \ell$ then $b \lhd u \equiv \ell$ because $a \prec\!\!\!\prec b \Longrightarrow a \prec\!\!\!\prec \ell$. So by positivity $p \between u \equiv \ell$ and p is located.

Conversely, suppose that p is rounded and located and $b \in p$, so we have $p \ni a \prec\!\!\!\prec b$ by roundedness. Then if $b \lhd u$ we have $a \prec\!\!\!\prec \ell \subset u$ by the bracketed clause and so $p \between \ell \subset u$ by locatedness. Hence p is positive. □

Although locales and formal covers in general need not have enough points (Warning 6.17), locally compact ones do. The underlying idea here is actually the one that we were unable to use in Remark 5.13, which we expressed there using compact subspaces. Here we exploit the way-below relation $\prec\!\!\!\prec$ instead, but we use that on the formal cover, whereas the same proof in [Joh82, Theorem VII 4.3] used the frame.

Proposition 7.12 Any locally compact locale or formal cover has enough points, assuming the axioms of choice and Excluded Middle.

Proof Recall from Proposition 6.18 that we need to show that

$$\big(\forall p.\, c \in p \Rightarrow p \between u\big) \ \Longrightarrow \ c \lhd u,$$

so suppose that $c \not\lhd u$. Then $c \in r \equiv A \setminus ju$ by Lemma 6.9 and we require $c \in p \subset r$.

By Theorem 7.10, r is rounded, so by Lemma 3.11 there is a $\prec\!\!\!\prec$-filter $s \equiv \{a \mid \exists i.\, c_i \prec\!\!\!\prec a\}$ with $s \subset r$, where $\cdots \prec\!\!\!\prec c_2 \prec\!\!\!\prec c_1 \prec\!\!\!\prec c_0 \equiv c$ and $c_i \in r$. Then by Lemma 3.12, $\mathcal{K} \equiv \{v \mid \exists a \in s.\, \mathcal{K}_a \ni v\} =$

$\{v \mid \exists i\ell.\, c_i \prec\!\!\!\prec \ell \subset v\} \subset \Omega$ is a Scott-open filter. If $\mathcal{K} \ni ju$ then $\exists i\ell.\, c_i \prec\!\!\!\prec \ell \subset ju$, so $c_i \in ju$ by Theorem 7.10, but by construction this is not the case, so $\mathcal{K} \not\ni ju$.

Now, by Lemma 3.14, which relies on the Axiom of Choice and applies to locales as well as traditional topology, there is a completely coprime filter $\mathcal{P}$ with $\mathcal{K} \subset \mathcal{P} \not\ni ju$.

By Proposition 6.15, $p \equiv \{d \mid \mathcal{P} \ni jd\}$ is a formal point in the sense of $\lhd$, which is the same as that of $\prec\!\!\!\prec$ by Lemma 7.11, and $\mathcal{P} = \{v \mid p \between v\}$.

If $d \in p$ then $\mathcal{P} \ni jd$ whilst $\mathcal{P} \not\ni ju$ and $\mathcal{P}$ is upper, so $jd \not\subset ju$ and $d \notin ju$ by Lemma 6.9, which means $d \in r$.

Also, $c_1 \prec\!\!\!\prec c_0 \equiv c \in jc \in \mathcal{K} \subset \mathcal{P}$, so $c \in p \subset r \equiv A \setminus ju$ as required. □

Theorem 7.13 Every abstract basis satisfying the primary axioms presents a concrete basis using Scott-open families on some locally compact sober topological space, assuming the Axiom of Choice and Excluded Middle. In particular,

$$(b \prec\!\!\!\prec c) \;\wedge\; (\forall p.\, c \in p \Rightarrow p \between k) \implies (b \prec\!\!\!\prec k).$$

Proof Combine the Proposition with Definition 7.4 and the results of Section 5. □

This completes the proof of the equivalence of categories between locally compact sober spaces or locales and continuous functions on the one hand and abstract bases and matrices on the other, although we defer the summary of this to the Conclusion.

The morphisms in the two settings are related like this:

Proposition 7.14 The correspondence between matrices with respect to $\lhd$ and $\prec\!\!\!\prec$ is:

$$\begin{array}{lcl} [\, a \mid f \mid b \,] & \Leftrightarrow & \big(\forall a'.\, a' \prec\!\!\!\prec a \implies \langle\, a' \mid f \mid b \,\rangle\big) \\ \langle\, a \mid f \mid b \,\rangle & \Leftrightarrow & \exists k.\, a \prec\!\!\!\prec k \;\wedge\; \forall a' \in k.\, [\, a' \mid f \mid b \,]. \end{array}$$

Proof Using the basis expansions of both kinds,

$$\begin{array}{lcl} [\, a \mid f \mid b \,] & \Leftrightarrow & U_a \;=\; \bigcup\{U_{a'} \mid \mathcal{K}_{a'} \ni U_a\} \;\subset\; f^{-1}V_b = \bigcup\{U_{a'} \mid \mathcal{K}_{a'} \ni f^{-1}V_b\} \\ & \Leftrightarrow & \forall a'.\, \mathcal{K}_{a'} \ni U_a \;\Rightarrow\; \mathcal{K}_{a'} \ni f^{-1}V_b \\ & \Leftrightarrow & \forall a'.\, a' \prec\!\!\!\prec a \;\Rightarrow\; \langle\, a' \mid f \mid b \,\rangle \\ \langle\, a \mid f \mid b \,\rangle & \Leftrightarrow & \mathcal{K}_a \ni f^{-1}V_b \;=\; \bigcup\{U_{a'} \mid U_{a'} \subset f^{-1}V_b\} \\ & \Leftrightarrow & \exists \ell.\, \mathcal{K}_a \ni U_\ell \;\wedge\; \forall a' \in \ell.\, U_{a'} \subset f^{-1}V_b \\ & \Leftrightarrow & \exists \ell.\, a \prec\!\!\!\prec \ell \;\wedge\; \forall a' \in \ell.\, [\, a' \mid f \mid b \,]. \end{array}$$

□

Continuous lattices and (im)predicativity

This section contains material that was removed from the previous one and needs to be developed. In order to preserve the numbering of later sections, this one has not been numbered.

We have already made one detour *via* Locale Theory and Formal Topology to prove correctness of the re-construction of a locally compact space from its abstract basis. This section takes the scenic route through continuous lattices and the foundational issue of predicativity.

This is important from a historical point of view in the development of both Locale Theory and Formal Topology.

If you are working in a complete lattice (such as a frame of open subspaces) and some property is preserved by joins, it is natural to consider the join of all instances, *i.e.* the greatest one. However, it is exactly that step that is objectionable, according to those who choose to work in Per Martin-Löf's Type Theory.

In the definition of locally compact locales that we chose in the previous section, the Scott-open families are *arbitrary*. However, it has been customary to make a *canonical* choice for them, in fact the *largest* one, where $\mathcal{K}_a \equiv \Uparrow U_a$ is determined order-theoretically by U_a:

Proposition 7.15 If a frame Ω has a basis $(U_a, \mathcal{K}_a)$ using Scott-open families then

$$\mathcal{K}_a \ni V \implies U_a \ll V \qquad \text{and so} \qquad a \prec\!\!\!\prec \ell \;\equiv\; \mathcal{K}_a \ni U_\ell \implies U_a \ll U_\ell,$$

where we say that U is ***way below*** V in Ω, written

$$U \ll W, \qquad \text{if} \qquad \forall (W_i).\ V \leq \bigvee_{i \in I} W_i \implies \exists \ell \subset I.\ U \leq \bigvee_{i \in \ell} W_i.$$

In such a frame, the subset $\Uparrow U \equiv \{V \mid U \ll V\} \subset \Omega$ is itself Scott-open. Hence a locale is locally compact iff the frame is ***continuous***,

$$V \;=\; \bigvee^{\uparrow} \{U \mid U \ll V\}.$$

Proof If $\mathcal{K}_a \ni V \leq \bigvee W_i$ then, since $\mathcal{K}_a$ is Scott-open, a finite subset $\ell \subset I$ will do, so $\mathcal{K}_a \ni W \equiv \bigvee\{W_i \mid i \in \ell\}$. The latter means that U_a contributes to the expansion of W, so $U_a \leq W$, as required for the definition of $U_a \ll V$. □

The whole of the previous section remains valid with $\ll$ in place of $\prec\!\!\!\prec$, simply because it is a special case.

Remark 7.16 The notion of a continuous lattice arose during the 1970s in theoretical computer science, topological lattice theory and spectral theory, leading to the six-author *Compendium* [GHK$^+$80]: see in particular the historical notes at the end of its Section I 1. In the case $\mathcal{K}_a \equiv \Uparrow U_a$, we have $(a \prec\!\!\!\prec \ell) \equiv (\mathcal{K}_a \ni U_\ell) \equiv (U_a \ll U_\ell)$. The axioms that we are using for *abstract* way-below relations, especially the interpolation property, were motivated by results that were first discovered for the one on a continuous lattice.

Remark 7.17 Some of the results about Σ-splittings for locales could maybe be moved here, although they need $\lhd$ and nuclei.

Give the construction using $Eu \equiv \{a \mid \exists \ell.\ a \prec\!\!\!\prec \ell \subset u\}$.

This gives a retract of a continuous lattice.

$\Omega_E \cong \Omega_j$; they're the same quotient but different subsets.

Turning to Formal Topology, Inger Sigstam was the first person to consider local compactness in that setting, before Sara Negri. She translated this (canonical) way-below relation $\ll$ into a formal cover [Sig95, Definition 4.1]:

Proposition 7.18 The frame presented by the formal cover $(A, \sqsubseteq, \lhd)$ is continuous iff

$$a \;\lhd\; \Downarrow a \equiv \{b \mid b \ll a\} \qquad \text{where} \qquad (u \ll v) \;\equiv\; (\forall w.\ v \lhd w \Rightarrow \exists \ell.\ u \lhd \ell \subset w).$$

We then say that A is a ***continuous formal cover***.

Proof By Lemma 6.9, this $u \ll v$ relation on subsets of the basis A is equivalent to

$$\forall w.\ jv \subset jw \implies \exists \ell.\ ju \subset j\ell \;\wedge\; \ell \subset w,$$

which is the lattice-theoretic way-below relation $ju \ll jv$ in Ω. However, we are claiming that it is enough to use *single* elements of the basis to test continuity of the frame. The set-wise continuity condition (Proposition 7.15), for $v \equiv \{a\}$, implies

$$a \;\lhd\; \bigcup\{u \mid u \ll a\} \;=\; \{b \mid \exists u.\ b \in u \ll a\} \;=\; \{b \mid b \ll a\},$$

as follows from the fact that $b \in u \ll a \Rightarrow b \ll a$. Conversely, the singleton condition gives

$$\forall a \in u.\ a \;\lhd\; \{b \mid b \ll a\} \;\subset\; \bigcup\{v \mid v \ll u\}, \qquad \text{so} \qquad u \;\lhd\; \bigcup\{v \mid v \ll u\},$$

since $b \ll a \in u \Longrightarrow v \equiv \{b\} \ll u$. □

Remark 7.19 If we construct a formal cover $\lhd$ and locale from an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ then this way-below relation $\ll$ satisfies

$$a \prec\!\!\!\prec \ell \implies a \ll \ell.$$

They are not equivalent, because $\prec\!\!\!\prec$ encodes more information, namely a basis using *arbitrary* Scott-open families, whereas $\ll$ corresponds to the largest one, $\Uparrow U_a$.

Remark 7.20 Negri's treatment and ours (up to Theorem 7.8) are predicative, although we defer further discussion of this issue to the end of this section. Our position in this debate is that Definition 7.4 should be taken as the definition of local compactness in Formal Topology, on grounds of topology, foundations and simplicity. Indeed, the examples that are usually given, in particular $\mathbb{R}$, are already of this form.

Remark 7.21 Discuss with the formal topologists. Rewrite. Invite readers to skip it.

Introduce inductively generated covers.

Show that the definition in the previous section is inductively generated.

Give the arguments from Peter Aczel, Giovanni Curi and Milly Maietti.

Foundationally, Formal Topology as a discipline goes a step further than Locale Theory by avoiding impredicative definitions and arguments, as well as the Axiom of Choice and Excluded Middle, so that it is valid in Martin-Löf Type Theory.

However, the $\forall w$ in Sigstam's formula for $\ll$ in terms of $\lhd$ makes it impredicative. This is something that often happens when we take the *largest* instance of something, in this case $\Uparrow U_a$ is the largest Scott-open family $\mathcal{K}_a$ that can be used with U_a in a basis. Giovanni Curi [Cur07, Section 7.3] gave a predicative formula that is equivalent to the one above, based on an observation of Peter Aczel [Acz06, Section 4.3].

The remaining remarks in this section are addressed to Formal Topologists and concern the definition and foundations of local compactness. There is an extensive discussion of the relevant proof-theoretic issues in [CSSV03], and abstract bases provide a very simple example of this:

Proposition 7.22 For any abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$, the families

$$I(a) \;\equiv\; \{k \mid a \prec\!\!\!\prec k\} + \{\downarrow\!\!\!\downarrow\}, \quad C(a,k) \;\equiv\; k \quad \text{and} \quad C(a,\downarrow\!\!\!\downarrow) \;\equiv\; \downarrow\!\!\!\downarrow a \;\equiv\; \{b \mid b \prec\!\!\!\prec a\}$$

inductively generate the cover $\lhd$ in the sense that $a \lhd u$ holds iff it is provable using just the axioms

$$a \in u \implies a \lhd u \qquad \text{and} \qquad C(a,i) \lhd u \implies a \lhd u,$$

which are called *reflexivity* and *infinity*. The purpose of this is to eliminate *transitivity*.

Proof Any such proof is sound by transitivity, because $a \lhd C(a,i)$. Conversely, these axioms are complete because we have the following deduction, using reflexivity and infinity but not transitivity:

$$\begin{array}{lcll}
\cdots & \vdash & \forall a \in \downarrow\!\!\!\downarrow b.\, \exists k.\, a \prec\!\!\!\prec k \subset u & \\
\cdots,\ a \in \downarrow\!\!\!\downarrow b & \vdash & \exists k \in I(a).\, k \subset u & \\
\cdots,\ a \in \downarrow\!\!\!\downarrow b & \vdash & \exists k \in I(a).\, C(a,k) \equiv k \lhd u & \text{reflexivity} \\
\cdots,\ a \in \downarrow\!\!\!\downarrow b & \vdash & a \lhd u & \text{infinity} \\
\cdots & \vdash & \forall a \in \downarrow\!\!\!\downarrow b.\ a \lhd u & \\
\cdots & \vdash & C(b,\downarrow\!\!\!\downarrow) \equiv \downarrow\!\!\!\downarrow b \ \lhd u & \\
\cdots & \vdash & b \lhd u. & \text{infinity} \quad \square
\end{array}$$

For the same issue in what we have just done, we leave the interested reader to use Lemmas 3.7 and 3.8 to show that if instead

$$C(a,k) \;\equiv\; \bigcup \{k' \mid \exists a'.\, a' \prec\!\!\!\prec k' \prec\!\!\!\prec a \;\wedge\; k' \prec\!\!\!\prec^1 k\}$$

then we obtain a *localised* inductive cover in the sense of [CSSV03, Definition 3.4].

Remark 7.23 Even for those who specifically wish to study $\lhd$ using Martin-Löf Type Theory, our account and those of Negri, Aczel and Curi make a compelling case for presenting $\lhd$ in terms of $\prec\!\!\!\prec$ whenever the space happens to be locally compact.

If a specific formal cover is inductively generated in some more complicated way and is locally compact in the sense of Curi [Cur07] then by our Propositions 7.3 and 7.22 it has an abstract basis and hence a *simple* inductive generation. In particular, the *proof* of the Curi property for the cover also serves to show that it satisfies our definition. On the other hand, if we wish to work with locally compact formal covers in general, is it more convenient to assume that they are presented in our simpler way.

The one situation where it is necessary to start in a more general setting is in order to show that local compactness is necessary for exponentiability [Mai05]. However, the real difficulty lies with the products (Remark 8.10) and this result can be deduced from a categorical observation (Theorem 9.9). Defining local compactness in terms of abstract bases also gives a much simpler formula than Maietti's for the cover on the exponential (Remark **??**).

The presentation of local compactness in Formal Topology that we have just given will also make a much clearer connection between that subject and the one that we introduce in Section 10,

Remark 7.24 Similar methods could be used to say how some more manageable sparser system might generate $\prec\!\!\!\prec$ in the way that we wanted in $\mathbb{R}^{\mathsf{n}}$ (Example 1.9). We would need to consider how intersections are managed.

8 Products and exponentials

This section requires a lot more work.

First we need to study a fundamental object that often has a derisory treatment in point-set topology; it is an example of Proposition 5.14.

Definition 8.1 The ***Sierpiński space*** is the locally compact space, locale or formal cover $\Sigma \equiv$ **Filt**$(\odot \sqsubseteq \bullet)$, for which the way-below $a \prec\!\!\!\prec \ell$ and cover $a \lhd \ell$ relations are both $a \in \ell \vee \bullet \in \ell$. Classically, Σ has an open point $\top$ and a closed one $\bot$, which are

$$\top \;\equiv\; \{\odot, \bullet\} \qquad \text{and} \qquad \bot \;\equiv\; \{\bullet\}$$

as formal points (Definition 5.1), and three open subspaces,

$$U_\circ \;\equiv\; \emptyset, \qquad U_\odot \;\equiv\; \{\top\} \qquad \text{and} \qquad U_\bullet \;\equiv\; \Sigma.$$

The basic compact subspaces are $K_\odot \equiv \{\top\}$ and either $K_\bullet \equiv \Sigma$ or $\{\bot\}$.

Proposition 8.2 In both point-set topology and intuitionistic Locale Theory, the Sierpiński space has the (double) universal property that, for any space X, there is a three-way bijective correspondence amongst
(a) an open subspace $U \subset X$,
(b) a continuous function $\phi : X \to \Sigma$ and
(c) a closed subspace $C \subset X$,
where we say that ϕ ***classifies*** $U = \phi^{-1}(\top)$ and ***co-classifies*** $C = \phi^{-1}(\bot)$.

It is a topological distributive lattice, with respect to which

$$(\sigma \in U) \iff (\bot \in U) \;\vee\; \sigma \;\wedge\; (\top \in U)$$

for any point $\sigma \in \Sigma$ and open subspace $U \subset \Sigma$.

Proof The classical version is easy. The locale (or topology on) Σ is the lattice of lower subsets of $(\odot \sqsubseteq \bullet)$, which is also the free frame on one generator $(\odot)$, so frame homomorphisms $\Omega\Sigma \to \Omega X$ correspond to elements of ΩX. This means that in Locale Theory continuous maps $X \to \Sigma$ are given by elements $U \in \Omega X$ of the frame corresponding to X.

Definition 6.3 explained how open and closed sublocales are defined by the nuclei $U \Rightarrow (-)$ and $U \vee (-)$ respectively, so the question is uniqueness of U.

If $V \in \Omega$ gives rise to an isomorphic sublocale of *either* kind then the corresponding nuclei are equal as endofunctions, but by applying them both to $\emptyset$, U and V, we deduce that $U = V$. Hence the Sierpiński locale enjoys the same universal property as its classical analogue for both open and closed sublocales. □

The characterisation of products is unfortunately rather complicated in all formulations of topology, so we shall concentrate on the traditional version.

Proposition 8.3 Let X and Y be sober topological spaces with bases (U_a) and (V_b) respectively using open subspaces. Then, in the category of sober topological spaces and continuous functions, the product $X \times Y$ has as points the pairs (x, y) and as basis $(U_a \times V_b)$ using open subspaces. Hence $W \subset X \times Y$ is open iff it satisfies

$$(x, y) \in W \iff \exists ab.\, x \in U_a \ \wedge\ y \in V_b \ \wedge\ U_a \times V_b \subset W. \qquad \square$$

We are going to characterise open subspaces of the product of two locally compact sober spaces in the same way that we did continuous functions in Section 4.

Remark 8.4 We will assume that the bases (U_a, K_a) and (V_b, L_b) use *non-empty compact* subspaces with single interpolation.

(a) They need to be non-empty because from

$$K_a \times \emptyset \ \subset\ U_k \times V_\ell$$

we can deduce nothing at all about a.

(b) If we used Scott-open families in the concrete bases and the weak intersection rule in the abstract ones, we would not be able to treat a and b separately in the roundedness and saturation properties below.

These features are essential to proving the simple form for λ-abstraction (the universal property of the exponential) in Theorem 8.14 below.

Definition 8.5 Given abstract bases $(A, \sqsubseteq_A, \prec\!\!\!\prec_A)$ and $(B, \sqsubseteq_B, \prec\!\!\!\prec_B)$, a subset $w \subset A \times B$ is called

(a) ***lower*** in a if

$$(a' \sqsubseteq_A a) \ \wedge\ (a, b) \in w \implies (a', b) \in w,$$

(b) ***rounded*** in a if

$$(a, b) \in w \implies \exists a'.\, (a \prec\!\!\!\prec_A a') \ \wedge\ (a', b) \in w$$

(c) and ***saturated*** in a if

$$(a \prec\!\!\!\prec_A \ell) \ \wedge\ \forall a' \in \ell.\, (a', b) \in w \implies (a, b) \in w.$$

Properties for b are defined in the same way, *cf.* Definition 1.16.

Lemma 8.6 If $K_a \times L_b \subset W$ then $U_a \times L_b,\ K_a \times V_b,\ U_a \times V_b \subset W$.

Proof This would be trivial if the basic compact subspaces K_a and L_b had been chosen to be saturated (in the sense of Definition 3.16, not the one that we have just given), so that $U_a \subset K_a$ and $V_b \subset L_b$.

For any $y \in Y$, the subspace $y^*W \equiv \{x \mid (x, y) \in W\} \subset X$ is open, so by its basis expansion, $K_a \subset y^*W \Longrightarrow U_a \subset y^*W$. Therefore, quantifying over $y \in L_b$ and $x \in U_a$,

$$\begin{array}{lcl} K_a \times \{y\} \subset W & \Longrightarrow & U_a \times \{y\} \subset W \\ K_a \times L_b \subset W & \Longrightarrow & U_a \times L_b \subset W \\ \{x\} \times L_b \subset W & \Longrightarrow & \{x\} \times V_b \subset W \\ U_a \times L_b \subset W & \Longrightarrow & U_a \times V_b \subset W. \end{array} \qquad \square$$

Lemma 8.7 If the subset $w \subset A \times B$ is lower and saturated in a and b and

$$K_a \times L_b \ \subset\ \bigcup \{U_{a'} \times V_{b'} \mid (a', b') \in w\}$$

then $(a, b) \in w$. There are also $a \prec\!\!\!\prec_A a'$ and $b \prec\!\!\!\prec_B b'$ with $(a', b), (a, b'), (a', b') \in w$.

Proof For each $x \in K_a$, the compact slice of the rectangle $K_a \times L_b$ is covered,

$$L_b \ \cong\ \{x\} \times L_b \ \subset\ K_a \times L_b \ \subset\ \bigcup \{U_{a'} \times V_{b'} \mid (a', b') \in w)\}.$$

The inverse image of this under the inclusion $Y \cong \{x\} \times Y \subset X \times Y$ is

$$L_b \;\subset\; \bigcup\{V_{b'} \mid \exists a'.\,(a',b') \in w \;\wedge\; x \in U_{a'}\},$$

so there is some non-empty finite subset $h \subset w$ with

$$L_b \subset V_\ell \equiv \bigcup\{V_{b'} \mid b \in \ell\} \quad \text{where} \quad b \;\prec\!\!\!\prec\; \ell \;\equiv\; \{b' \mid \exists a'.\,(a',b') \in h\}$$

and $$\forall (a',b') \in h. \quad x \in U_{a'}.$$

The filter property of the basis for X gives some lower bound $a'' \in A$ around x:

$$x \in U_{a''} \quad \text{and} \quad \forall (a'.\,b') \in h.\,(a'' \sqsubseteq a').$$

Since w is lower in a we may trim the open rectangles down to width a''. Hence

$$\{a''\} \times \ell \subset w, \quad \text{whilst} \quad b \prec\!\!\!\prec \ell,$$

but then $(a'',b) \in w$ since it is saturated in b.

We have shown that, for each $x \in K_a$, there is some $a'' \in A$ such that $x \in U_{a''}$ and $(a'',b) \in w$. Thus

$$K_a \;\subset\; \bigcup\{U_{a''} \mid (a'',b) \in w\},$$

so there is some finite $k \subset A$ with

$$K_a \subset U_k \equiv \bigcup\{U_{a''} \mid a'' \in k\}, \quad \text{so} \quad a \;\prec\!\!\!\prec\; k \quad \text{where} \quad k \times \{b\} \subset w,$$

but then $(a,b) \in w$ since it is saturated.

Just before the final step we could have used single interpolation of $a \prec\!\!\!\prec a' \prec\!\!\!\prec k$ to deduce $(a',b) \in w$. We cannot introduce $b \prec\!\!\!\prec b' \prec\!\!\!\prec \ell$ at the earlier stage of the argument, because it would depend on x. However, we could of course consider X and Y the other way round to obtain $(a,b') \in w$ and then $(a',b') \in w$ by using both ways. □

Lemma 8.8 There is a bijection defined by

$$w \;\equiv\; \{(a,b) \mid K_b \times L_a \subset W\} \quad \text{and} \quad W \;\equiv\; \bigcup\{U_a \times V_b \mid (a,b) \in w\}$$

between an open subspace $W \subset X \times Y$ and a subset $w \subset A \times B$ that is lower, rounded and saturated in each argument.

Proof The subspace W is open for any w, whilst w is lower for any W.

Given open $W \subset X \times Y$, suppose that $a \prec\!\!\!\prec k$ and $\forall a' \in k.\,(a',b) \in w$. Then for each $a' \in k$, we have $K_{a'} \times L_b \subset W$, so $U_{a'} \times L_b \subset W$ by Lemma 8.6. Since $a \prec\!\!\!\prec k$ we have

$$K_a \times L_b \;\subset\; \bigcup\{U_{a'} \times L_b \mid a' \in k\} \;\subset\; W,$$

so $(a,b) \in w$, Thus w is saturated in a and similarly in b.

Hence Lemma 8.7 is applicable for any $(a,b) \in w$, so by its final part there are $a \prec\!\!\!\prec_A a'$ and $b \prec\!\!\!\prec_B b'$ with $(a',b),(a,b') \in w$ too. So w is rounded in both arguments.

Now, from $W \subset X \times Y$ we derive $w \subset A \times B$ and thence $W' \subset X \times Y$, where

$$W' \;\equiv\; \bigcup\{U_a \times V_b \mid K_a \times L_b \subset W\}.$$

Then $W' \subset W$ by Lemma 8.6. Conversely, by Proposition 8.3 and the basis expansions of U_a and V_b, we have

$$\begin{aligned}
(x,y) \in W \quad &\Rightarrow \quad \exists ab.\,x \in U_a \;\wedge\; y \in V_b \;\wedge\; U_a \times V_b \subset W \\
&\Rightarrow \quad \exists aa'bb'.\,x \in U_{a'} \;\wedge\; K_{a'} \subset U_a \\
&\qquad\qquad \wedge\, y \in V_{b'} \;\wedge\; L_{b'} \subset V_b \;\wedge\; U_a \times V_b \subset W \\
&\Rightarrow \quad \exists a'b'.\,(x,y) \in U_{a'} \times V_{b'} \;\wedge\; K_{a'} \times L_{b'} \subset W \\
&\Rightarrow \quad (x,y) \in W'.
\end{aligned}$$

Given $w \subset A \times B$ we derive $W \subset X \times Y$ and thence $w' \subset A \times B$, where

$$w' \equiv \{(a,b) \mid K_b \times L_a \subset \bigcup \{U_{a'} \times V_{b'} \mid (a',b') \in w\}\}.$$

Lemma 8.7 says that $(a,b) \in w' \Longrightarrow (a,b) \in w$. Conversely, by roundedness of w, we have

$$\begin{aligned}
(a,b) \in w \quad &\Rightarrow \quad \exists a'b'.\,(a \prec\!\!\prec a') \wedge (b \prec\!\!\prec b') \wedge (a',b') \in w \\
&\equiv \quad \exists a'b'.\,(K_a \subset U_{a'}) \wedge (L_b \subset V_{b'}) \wedge (a',b') \in w \\
&\Rightarrow \quad \exists a'b'.\,(K_a \times L_b \subset U_{a'} \times V_{b'}) \wedge (a',b') \in w \\
&\Rightarrow \quad K_a \times L_b \subset \bigcup \{U_{a'} \times V_{b'} \mid (a',b') \in w\} \\
&\equiv \quad (a,b) \in w'.
\end{aligned}$$

□

Theorem 8.9 Let X and Y be locally compact sober topological spaces that have bases (U_a, K_a) and (V_b, K_b) using non-empty compact subspaces with single interpolation. Then the product $X \times Y$ has a basis $(U_a \times V_b, K_a \times V_b)$ of the same kind, indexed by the product preorder. □

Remark 8.10 The way-below relation for $X \times Y$ is given by

$$(a,b) \prec\!\!\prec h \equiv \exists k\ell.\,(a \prec\!\!\prec k) \wedge (b \prec\!\!\prec \ell) \wedge (k \times \ell \prec\!\!\prec' h)$$

where $$h' \prec\!\!\prec' h \equiv \forall (a',b') \in h'.\,\exists (a,b) \in h.\,(a' \prec\!\!\prec a) \wedge (b' \prec\!\!\prec b).$$

However, as you might imagine given the difficulty of the foregoing proof in Point–Set Topology, showing directly that this satisfies the axioms for an abstract basis and that it provides the product in the category of abstract bases and matrices is well beyond the scope of this paper, see [work in progress].

These formulae say that any cover $h \subset A \times B$ has a refinement $k \times \ell$ consisting of a regular array of rectangles. Allowing more general patterns of covers than these adds to the complexity of the proofs but not to the generality of locally compact spaces that we can consider.

Really, we would like a new axiomatisation of a relation that *generates* the full way-below relation.

Naturally, the same issue arises in Formal Topology, although it is worse because the covering sets are infinite.

However, we should not allow this residual complication distract us from the achievement of obtaining a basis for the product that is simply the product of the bases. If we had required bases to have finite meets and joins we would have had to generate lattices.

Remark 8.11 The natural analogue of Proposition 8.3 in Locale Theory is a tensor product of complete join-semilattices [Joh82, §§II 2.12–14]. In general, however, this differs from the product in Point–Set Topology, but they do agree for locally compact spaces.

Defining products of formal covers predicatively apparently requires them to be inductively generated in the sense of Proposition 7.22.

In contrast to these difficulties with the product, it is easy to describe the abstract basis for the exponential Σ^X. We therefore take this as our starting point and define a space ΣX in a similar way to Proposition 5.14. Then we justify the superscript by proving that it has the universal property of the exponential, using matrices.

Afterwards we take advantage of our earlier work in this paper to characterise Σ^X as the lattice of open subspaces of X, equipped with the Scott topology (Proposition 2.11). This approach gives the result uniformly across all of the formulations of topology, whereas the numerous accounts in domain theory are only valid in the setting of Point–Set Topology.

Note carefully that the following definitions reverse the order relations.

Lemma 8.12 Let $(A, \sqsubseteq_X, \prec\!\!\prec_X)$ be an abstract basis satisfying the primary rules. For $k, \ell \in \mathsf{Fin}\,(A)$ and $L \in \mathsf{Fin}\,(\mathsf{Fin}\,(A))$, define

$$(k \prec\!\!\prec_{\Sigma X} L) \equiv \exists \ell \in L.\,(k \prec\!\!\prec_{\Sigma X} \ell) \equiv \exists \ell \in L.\,(\ell \prec\!\!\prec_X k) \equiv \exists \ell \in L.\,\forall a \in \ell.\,(a \prec\!\!\prec_X k),$$

$$(k \sqsubseteq_{\Sigma X} \ell) \equiv (\ell \sqsubseteq_X k) \equiv \forall a \in \ell. \exists b \in k. (a \sqsubseteq_X b),$$

$$\bullet_{\Sigma X} \equiv \circ_X \qquad \text{and} \qquad k \sqcap_{\Sigma X} \ell \equiv k \sqcup_X \ell.$$

Then $(\mathsf{Fin}\,(A), \sqsubseteq_{\Sigma X}, \prec\!\!\!\prec_{\Sigma X})$ is a prime stable basis with single interpolation, boundedness above and strong intersection.

Proof Prime means that L is essentially redundant and stable that it has finite meets with respect to $\sqsubseteq_{\Sigma X}$, given by unions of finite subsets or concatenations of lists, as shown. These meets satisfy the strong intersection rule:

$$\begin{array}{lrcl} & (k \prec\!\!\!\prec_{\Sigma X} L_1) \wedge (k \prec\!\!\!\prec_{\Sigma X} L_2) & \equiv & \exists \ell_1 \in L_1. \exists \ell_2 \in L_2. (\ell_1 \prec\!\!\!\prec_X k) \wedge (\ell_2 \prec\!\!\!\prec_X k) \\ & & \Leftrightarrow & k \prec\!\!\!\prec_{\Sigma X} (L_1 \sqcap_{\Sigma X} L_2), \\ \text{where} & (L_1 \sqcap_{\Sigma^X} L_2) & \equiv & \{\ell_1 \sqcup_X \ell_2 \mid \ell_1 \in L_1,\ \ell_2 \in L_2\}. \end{array}$$

Transitivity of $\prec\!\!\!\prec_{\Sigma X}$ is that of $\prec\!\!\!\prec_X$ and single interpolation is

$$\begin{array}{rcl} (k \prec\!\!\!\prec_{\Sigma^X} L) & \equiv & \exists \ell \in L. (\ell \prec\!\!\!\prec_X k) \\ & \Rightarrow & \exists h. \exists \ell \in L. (\ell \prec\!\!\!\prec_X h \prec\!\!\!\prec_X^1 k) \\ & \Rightarrow & \exists h. (k \prec\!\!\!\prec_{\Sigma X} h \prec\!\!\!\prec_{\Sigma X} L), \end{array}$$

by the Wilker rule for X. We deduce the one for ΣX, using $H \equiv \{h\} \prec\!\!\!\prec_{\Sigma X}^1 \{\ell\} \subset L$. □

What about the rounded union property, which is needed in Lemma 4.3 and so in Theorem 8.14?

Lemma 8.13 Let X have abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ satisfying single interpolation. Then the basis for ΣX is bounded below and satisfies the rounded union rule iff that for X is bounded above and satisfies the strong intersection rule.

Proof For the boundedness properties this is just $k \prec\!\!\!\prec_{\Sigma X} \ell \iff \ell \prec\!\!\!\prec_X k$.

Similarly, the rounded union rule for ΣX,

$$k_1 \prec\!\!\!\prec_{\Sigma X} \ell \succ\!\!\!\succ_{\Sigma X} k_2 \implies \exists k.\ k \prec\!\!\!\prec_{\Sigma X} \ell \quad \wedge \quad k_1 \prec\!\!\!\prec_{\Sigma X} k \succ\!\!\!\succ_{\Sigma X} k_2,$$

is

$$k_1 \succ\!\!\!\succ_X \ell \prec\!\!\!\prec_X k_2 \implies \exists k.\ k \succ\!\!\!\succ_X \ell \quad \wedge \quad k_1 \succ\!\!\!\succ_X k \prec\!\!\!\prec_X k_2.$$

This is the rounded intersection rule, but generalised from a single element to a list ℓ: if each $b \in \ell$ requires k_b, the list needs the union $k \equiv \bigcup\{k_b \mid b \in \ell\}$. Lemma 3.9 showed that the strong and rounded intersection rules are equivalent, using single interpolation. □

Continuing with the abstract approach, here is the universal property. The simplicity of this result is our reward for the messy argument in Lemma 8.8.

Theorem 8.14 The object ΣX is the exponential Σ^X because there is a bijection between continuous maps

$$\sigma : \Gamma \times X \to \Sigma \qquad \text{and} \qquad \phi : \Gamma \to \Sigma X$$

that is natural with respect to pre-composition with $f : \Delta \to \Gamma$.

Proof Let Γ and X have bases using non-empty compact subspaces indexed by A and B respectively. By Proposition 8.2 and Lemma 8.8, the map σ corresponds to an open subspace $W \subset \Gamma \times X$ and hence to a subset $w \subset A \times B$ that is lower, rounded and saturated in each component.

By Theorem 4.21, the map $\phi : \Gamma \to \Sigma X$ corresponds to a matrix $\langle\, a \,|\, \phi \,|\, \ell \,\rangle$ with $\ell \in \mathsf{Fin}\,(B)$ that

(a) has the partition property;

(b) is covariant, uniformly bounded and filtered in ℓ;

(c) is contravariant (lower), rounded and saturated in a; and

(d) is rounded in ℓ with respect to $\prec\!\!\!\prec_{\Sigma X}$.

The first of these is redundant because $\prec\!\!\!\prec_{\Sigma X}$ is prime. The second says that

$$\langle\, a \,|\, \phi \,|\, \ell \,\rangle \iff \forall b \in \ell. \langle\, a \,|\, \phi \,|\, b \,\rangle,$$

so we have a bijection if we define $\quad (a, b) \in w \iff \langle\, a \,|\, \phi \,|\, b \,\rangle$.

The third condition on the matrix is half of that on the subset w, whilst the fourth is the other half because

$$\begin{aligned}(a,b) \in w \;\equiv\; \langle\, a \,|\, \phi \,|\, b \,\rangle \;\;&\Leftrightarrow\;\; \exists \ell.\, \langle\, a \,|\, \phi \,|\, \ell \,\rangle \;\wedge\; (\ell \prec\!\!\!\prec_{\Sigma X} b) \\ &\equiv\;\; \exists \ell.\, \forall b' \in \ell.\, (a, b') \in w \;\wedge\; (b \prec\!\!\!\prec_X \ell).\end{aligned}$$

For naturality, by Theorem 4.21, the composite of $f : \Delta \to \Gamma$ with either σ or ϕ corresponds to the saturated pre-composition with the matrix for f. □

Now we link this back to more familiar presentations by identifying the basis for Σ^X. The parentheses are a mnemonic for the fact that we are using the reverse order on $\mathsf{Fin}\,(A)$, *cf.* Remark 2.7.

Proposition 8.15 Let X have concrete basis (U_a, K_a) using compact subspaces. Then the exponential Σ^X is (isomorphic to) the lattice of open subspaces of X with the Scott topology (Proposition 2.11). This has a concrete basis using compact subspaces given by

$$\mathbf{V}_{(\ell)} \;\equiv\; \mathcal{K}_\ell \qquad \text{and} \qquad \mathbf{L}_{(\ell)} \;\equiv\; \{U_\ell\} \quad \text{or} \quad \{V \mid U_\ell \subset V\},$$

where

$$U_\ell \;\equiv\; \bigcup \{U_a \mid a \in \ell\} \quad \text{and} \quad \mathcal{K}_\ell \;\equiv\; \{U \mid \forall a \in \ell.\, K_a \subset U\}.$$

Proof The universal property in the case $\Gamma \equiv \mathbf{1}$ puts the points of Σ^X in bijection with the open subspaces of X. Then a formal point $p \subset \mathsf{Fin}\,(A)$ in the sense of Definition 5.1 corresponds to an open subspace $U \subset X$ by

$$U \;=\; \bigvee^{\uparrow} \{U_\ell \mid \ell \in p\} \;\equiv\; U_p \qquad \text{and} \qquad p \;\equiv\; \{\ell \mid \mathcal{K}_\ell \ni U\},$$

which is the directed basis expansion. This join is preserved by membership in any Scott-open family $\mathbf{V} \subset \Sigma^X$:

$$\begin{aligned}\mathbf{V} \ni U \quad &\Longleftrightarrow \quad \mathbf{V} \ni \bigvee^{\uparrow} \{U_\ell \mid \mathcal{K}_\ell \ni U\} \\ &\Longleftrightarrow \quad \exists \ell.\, \mathbf{V} \ni U_\ell \;\wedge\; \mathcal{K}_\ell \ni U \quad \equiv \quad \exists \ell.\, U \in \mathcal{K}_\ell \;\wedge\; \{U_\ell\} \subset \mathbf{V}.\end{aligned}$$

Hence we have a concrete basis expansion for $\mathbf{V} \subset \Sigma^X$ if we take $\mathbf{V}_{(\ell)} \equiv \mathcal{K}_\ell$ and either $\mathbf{L}_{(\ell)} \equiv \{U_\ell\}$ or its saturation $\{U \mid U_\ell \subset U\}$. The way-below relation is

$$\{U_k\} \;\equiv\; \mathbf{L}_{(k)} \;\subset\; \mathbf{V}_{(L)} \;\equiv\; \bigcup \{\mathcal{K}_\ell \mid \ell \in L\},$$

which is, as required,

$$\exists \ell \in L.\;\, (U_k \in \mathcal{K}_\ell) \;\Longleftrightarrow\; \exists \ell \in L.\, (\ell \prec\!\!\!\prec_X k) \;\equiv\; (k \prec\!\!\!\prec_{\Sigma X} L).$$

The filter property is

$$\mathbf{V}_{(k)} \ni U \in \mathbf{V}_{(\ell)} \;\Longleftrightarrow\; U \in \mathbf{V}_{(k \sqcup_X \ell)}.$$

We would like to compare this choice of basis with the formulae in Section 5. Relative to the isomorphism above, Definition 5.4 gave

$$\mathbf{V}_{(\ell)} \;\equiv\; \{p \mid \ell \in p\} \;\cong\; \{U \mid \mathcal{K}_\ell \ni U\} \;\equiv\; \mathcal{K}_\ell,$$

and we have shown that every Scott-open family $\mathbf{V}$ is a union of these. Similarly, the formula for the basic compact subspace in Theorem 5.12 was

$$\begin{aligned}\mathbf{L}_{(\ell)} \;\;&\equiv\;\; \{p \mid \forall L.\, \ell \prec\!\!\!\prec_{\Sigma X} L \Rightarrow L \between p\} \\ &=\;\; \{p \mid \forall k.\, k \prec\!\!\!\prec_X \ell \Rightarrow k \in p\} \\ &=\;\; \{p \mid \ell \lhd_X p\} \quad \cong \quad \{U \mid U_\ell \subset U\},\end{aligned}$$

using roundedness of p, Definition 7.4 and Lemma 6.8. □

Corollary 8.16 Every Scott-open filter is expressible in the manner of Lemma 3.12, as

$$\mathcal{K} \;=\; \bigcup \{\mathcal{K}_\ell \mid \ell \in s\} \subset \Omega, \qquad \text{where} \qquad s \equiv \{\ell \mid \mathcal{K} \ni U_\ell\} \subset \mathsf{Fin}\,(A)$$

is a $\prec\!\!\prec$-filter in the directed basis (Proposition 3.1).

Proof The correspondence is the basis expansion of $\mathcal{K} \subset \Sigma^X$. Since $\mathcal{K}$ is inhabited, so is s. For roundedness, we use the directed basis expansion of $U_\ell \subset X$:

$$\ell \in s \;\equiv\; \mathcal{K} \ni U_\ell = \bigvee\!\!{}^{\uparrow} \{U_h \mid h \prec\!\!\prec_X \ell\} \implies \exists h.\, \mathcal{K} \ni U_h \;\wedge\; h \prec\!\!\prec_X \ell.$$

It is a $\sqsubseteq$-filter because

$$k \in s \ni \ell \;\equiv\; U_k \in \mathcal{K} \ni U_\ell \iff \mathcal{K} \ni U_{k \sqcup \ell} \;\equiv\; k \sqcup_X \ell \in s$$

and then roundedness makes it a $\prec\!\!\prec$-filter. □

The basis expansion for the simplest exponentials provides two of the axioms in the abstract description of the category in Section 10. We consider any set N (maybe, but not necessarily, $\mathbb{N}$) as the discrete locally compact space $\mathbf{Filt}(N, =)$. Its exponential Σ^N is $\mathbf{Filt}(\mathsf{Fin}\,(N), \supset)$, which is classically the powerset $\mathcal{P}(N)$ with the Scott topology (Proposition 5.14). This topology is the free frame on N.

Proposition 8.17 For $\xi \in \Sigma^N$ and $F \in \Sigma^{\Sigma^N}$,

$$F\xi \iff \exists \ell \in \mathsf{Fin}\,(N).\ (\forall n \in \ell.\, \xi n) \;\wedge\; F\,\{n \in N \mid n \in \ell\},$$

which we call the ***Scott principle***. For $N \equiv \mathbf{1}$, this is the ***Phoa principle***[1]

$$F\sigma \iff F\bot \;\vee\; \sigma \;\wedge\; F\top.$$

Proof From the universal property of the exponentials and the Sierpiński space (Theorem 8.14 and Proposition 8.2), the elements $\xi \in \Sigma^N$ and $F \in \Sigma^{\Sigma^N}$ correspond to continuous functions $\xi : N \to \Sigma$ and $F : \Sigma^N \to \Sigma$ and so to open subspaces $U \subset N$ and $\mathbf{V} \subset \Sigma^N$.

Example 5.2 gave the singleton basis for N and Proposition 3.1 the directed one, with

$$U_\ell \;\equiv\; K_\ell \;\equiv\; \ell \subset N \quad \text{and so} \quad \mathcal{K}_\ell \;\equiv\; \{\xi \in \Sigma^N \mid \forall n \in \ell.\, \xi n\}.$$

Then, by Proposition 8.15, the basis for the $X \equiv \Sigma^N$ has

$$(\xi \in \mathbf{V}_{(\ell)}) \;\equiv\; (\xi \in \mathcal{K}_\ell) \;\equiv\; (\forall n \in \ell.\, \xi n)$$

and

$$(\mathbf{L}_{(\ell)} \subset \mathbf{V}) \;\equiv\; (U_\ell \in \mathbf{V}) \;\equiv\; F\,\{n \in N \mid n \in \ell\}.$$

Therefore the basis expansion,

$$\xi \in \mathbf{V} \iff \exists \ell.\ (\xi \in \mathbf{V}_{(\ell)}) \;\wedge\; (\mathbf{L}_{(\ell)} \subset \mathbf{V}),$$

is the same as the Scott principle.

For $N \equiv \mathbf{1}$, any $\ell \in \mathsf{Fin}\,(N)$ is either $\ell \equiv \circ$ or $\ell \equiv \bullet$, so the existential quantification reduces to a binary disjunction, in the cases of which the universal quantifier ranges over the empty set and the singleton. □

9 Σ-split subspaces

This section needs to be re-considered to make a smoother transition from the other subjects to ASD. Probably the nucleus $\mathcal{E}$ and the class $\mathcal{M}$ could be integrated into the earlier development.

We take advantage of exponentials to provide a fourth way of seeing bases using Scott-open families.

We follow the analogy of Theorem 6.12. Parts (a) and (b) were about concrete and abstract bases, which we have already shown to be equivalent. In (e), a space with a basis indexed by $(A, \sqsubseteq)$ is a subspace of $\mathbf{Filt}(A, \sqsubseteq)$, so we begin by modifying the notion of subspace to obtain a

[1] After Wesley Phoa [Pho90, Hyl91], whose name is of southeast Asian origin and is pronounced a little like French *poire*.

corresponding result for bases using Scott-open filters. From this we will derive a notion of nucleus corresponding to part (c) and then how to split it to obtain the subspace. Finally, we sketch how this provides an “algebraic theory” analogous to that of frames for part (d).

Whilst it may be questionable to use the same name (*nucleus*) for two things (j and $\mathcal{E}$) that satisfy different equations, we will see that they play the same role in their respective subjects, namely to define subspaces.

By a simple categorical argument using this presentation, we also deduce that local compactness is *necessary* for exponentiability. For the long history of the investigation of function-spaces in topology, see [Isb86].

Recall that, for an inclusion $i : X \hookrightarrow Y$ of topological spaces, Y has the ***subspace topology*** if each open subspace $U \subset X$ is the restriction $V \cap X$ of some open $V \subset Y$. We consider the situation when there is an operation $U \mapsto V$ that provides this and is Scott-continuous:

Theorem 9.1 For any locally compact sober space X with a basis $(U_a, \mathcal{K}_a)$ using Scott-open families indexed by $(A, \sqsubseteq)$ there are continuous maps

$$i : X \longrightarrow Y \equiv \mathbf{Filt}(A, \sqsubseteq) \quad \text{and} \quad I : \Sigma^X \longrightarrow \Sigma^Y \quad \text{such that} \quad ix \in IU \iff x \in U$$

that are defined by

$$ix \;\equiv\; \{a \mid x \in U_a\} \quad \text{and} \quad IU \;\equiv\; \bigcup \{V_a \mid \mathcal{K}_a \ni U\} \;\equiv\; \{p \mid \exists a.\, (\mathcal{K}_a \ni U) \;\wedge\; (a \in p)\}.$$

Conversely, any such pair (i, I) defines a basis on X by

$$U_a \;\equiv\; i^{-1}V_a \;\equiv\; \{x : X \mid a \in ix\} \qquad \text{and} \qquad \mathcal{K}_a \;\equiv\; \{U \mid \{b \mid a \sqsubseteq b\} \in IU\}.$$

Moreover, these translations are inverse.

Proof The filteredness conditions for a concrete basis using Scott-open families (Definition 2.6(b,c)) give those for ix in Proposition, so this is a point of $\mathbf{Filt}(A, \sqsubseteq)$. The subspace IU is a union of basic open subspaces. Then

$$ix \in IU \;\equiv\; \exists a.\, x \in U_a \;\wedge\; \mathcal{K}_a \ni U \iff x \in U$$

by the basis expansion for X.

Conversely, U_a is an inverse image of an open subspace, whilst $\mathcal{K}_a$ is Scott-open because I is Scott-continuous. The basis expansion for X follows from that for Y and the equation for (i, I) because

$$\begin{aligned} x \in U \;&\Leftrightarrow\; ix \in IU \iff \exists a.\, ix \in V_a \;\wedge\; L_a \subset IU \\ &\Leftrightarrow\; \exists a.\; a \in ix \;\wedge\; \{b \mid a \sqsubseteq b\} \in IU \\ &\Leftrightarrow\; \exists a.\; x \in U_a \;\wedge\; \exists c.\, \mathcal{K}_c \ni U \;\wedge\; c \in \{b \mid a \sqsubseteq b\} \\ &\Leftrightarrow\; \exists a.\, x \in U_a \;\wedge\; \mathcal{K}_a \ni U. \end{aligned}$$

Finally, the definitions are inverse because

$$ix \ni a \iff x \in U_a \qquad \text{and} \qquad \uparrow a \in IU \iff \mathcal{K}_a \ni U. \qquad \square$$

Example 9.2 Any set N with the singleton basis (Example 5.2) is a Σ-split subspace of Σ^N by

$$in \;\equiv\; \lambda m.\, (n = m) \qquad \text{and} \qquad I\phi \;\equiv\; \lambda\psi.\, \exists m.\, \phi m \wedge \psi m. \qquad \square$$

This version for locally compact locales and formal covers could go in Section 7:

Lemma 9.3 Let $\prec\!\!\!\ll$ and $\lhd$ be related as in Lemma 7.2 and define

$$ju \;\equiv\; \{a \mid a \lhd u\} \qquad \text{and} \qquad \mathcal{E}u \;\equiv\; \{a \mid \exists \ell.\, a \prec\!\!\!\ll \ell \subset u\}.$$

Then there are isomorphic quotients of frames

$$\begin{array}{ccc} & \mathcal{P}(A) & \\ \swarrow & & \searrow \\ \{u \mid u = ju\} & \underset{j}{\overset{\mathcal{E}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} & \{u \mid u = \mathcal{E}u\} \end{array}$$

Proof The axioms for a nucleus and the translations of three of the conditions in Lemma 7.2 using j and $\mathcal{E}$ are

$$u \subset ju = j(ju), \qquad u \subset j(\mathcal{E}u),$$

$$\mathcal{E}u \subset ju \qquad \text{and} \qquad \mathcal{E}(ju) \subset \mathcal{E}u.$$

From these we deduce $j(\mathcal{E}u) = ju$ and $\mathcal{E}(ju) = \mathcal{E}u$. These are the equations for an isomorphic between the splittings of the idempotents j and $\mathcal{E}$ on $\mathcal{P}(A)$. Note that $j(\mathcal{E}u)$ is the basis expansion of u and *cf.* the remarks following Theorem 6.10. $\square$

Theorem 9.4 Let Ω be a frame with concrete basis using Scott-open families $(U_a, \mathcal{K}_a)$ indexed by $(A, \sqsubseteq)$. Then there are maps $i^* : \mathcal{D}(A, \sqsubseteq) \twoheadrightarrow \Omega$ and $i_*, I : \Omega \rightarrowtail \mathcal{D}(A, \sqsubseteq)$, where i^* is a frame homomorphism, i_* preserves arbitrary meets and I is Scott continuous. These satisfy the equations

$$i^* \cdot i_* = i^* \cdot I = \mathsf{id}_\Omega \qquad \text{and} \qquad i_* \cdot i^* = j$$

where j and $\mathcal{E} \equiv I \cdot i^*$ are given by

$$ju \equiv \{a \mid a \lhd u\} \qquad \text{and} \qquad \mathcal{E}u \equiv \{a \mid \exists \ell.\, a \prec\!\!\prec \ell \subset u\}.$$

Proof We already know all of this structure apart from I and $\mathcal{E}$. In particular, i^* is the inverse image operation for the inclusion $i : X \to \mathbf{Filt}(A, \sqsubseteq)$ in Theorem 9.1, which also defined, for $u \in \Omega$ (so $u = ju$),

$$Iu \equiv \{a \mid \mathcal{K}_a \ni u\} \equiv \{a \mid \exists k.\, a \prec\!\!\prec k \subset u\},$$

by Lemma 7.2. By the first part of Theorem 7.10, if $u = ju$ then $Iu \subset u$, so $i^* Iu \subset i^* u = ju = u$.

Conversely, if $a \in u = ju$ then, using Lemma 7.2,

$$a \lhd \{b \mid b \prec\!\!\prec a\} \equiv Ij\{a\} \subset Iju \equiv Iu,$$

so $u \lhd Iu$ and then $i^* \cdot I = \mathsf{id}_\Omega$. The maps I and $\mathcal{E} \equiv I \cdot i^*$ are Scott-continuous because of their definition using the Scott-open families $\mathcal{K}_a$. $\square$

Corollary 9.5 Any continuous frame Ω is related in the same way to $\mathcal{D}(\Omega)$ by

$$i^* u \equiv \bigvee u, \qquad i_* a \equiv \downarrow a \equiv \{b \mid b \leq a\} \qquad \text{and} \qquad Ia \equiv \downarrow\!\!\downarrow a \equiv \{b \mid b \ll a\}. \qquad \square$$

We now describe this categorical structure more formally, because it offers a way of constructing a general locally compact space from data on $\mathbf{Filt}(A, \sqsubseteq)$.

Definition 9.6 For locally compact sober spaces X and Y, a ***Σ-split inclusion*** is a continuous map $i : X \rightarrowtail Y$ together with a Scott-continuous map $I : \Sigma^X \to \Sigma^Y$ such that

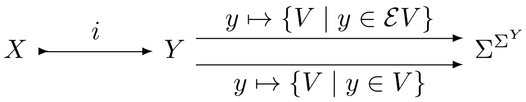
$$ix \in IU \iff x \in U \qquad \text{or} \qquad \Sigma^i \cdot I = \mathsf{id}_{\Sigma^X}.$$

The other composite, $\mathcal{E} \equiv I \cdot \Sigma^i : \Sigma^Y \to \Sigma^Y$, is called a ***nucleus*** and satisfies

$$\mathcal{E}(U \wedge V) = \mathcal{E}(\mathcal{E}U \wedge \mathcal{E}V) \qquad \text{and} \qquad \mathcal{E}(U \vee V) = \mathcal{E}(\mathcal{E}U \vee \mathcal{E}V).$$

If nuclei $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy $\mathcal{E}_1 \cdot \mathcal{E}_2 = \mathcal{E}_2 = \mathcal{E}_2 \cdot \mathcal{E}_1$ then the subspace defined by $\mathcal{E}_2$ is a Σ-split subspace of that defined by $\mathcal{E}_1$ and then we write $\mathcal{E}_2 \subset \mathcal{E}_1$.

Lemma 9.7 The following diagram is an equaliser in the category of locally compact sober spaces:

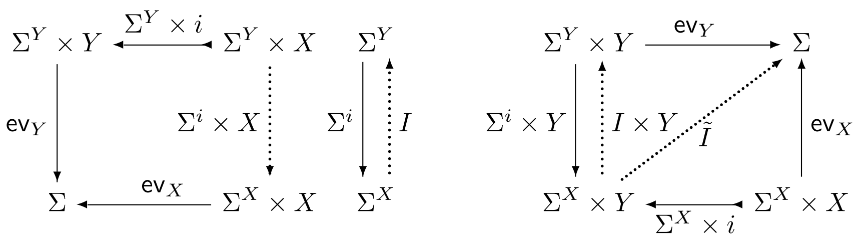


so the points of X are those $y : Y$ that are ***admissible***, $\forall V \in \Sigma^Y.\ y \in \mathcal{E}V \iff y \in V$.

Proof For any $y \in Y$ that satisfies $y \in V \iff y \in \mathcal{E}V \equiv I\Sigma^i V$ for all open $V \subset Y$, let $\mathcal{P} \equiv \{U \subset X \mid y \in IU\}$. Then
(a) $y \in IX \equiv IX \equiv I(\Sigma^i Y) \equiv \mathcal{E}Y \iff y \in Y$, which is true;
(b) dually $y \notin I\emptyset$ since $y \notin \emptyset \subset Y$;
(c) $y \in IU \wedge y \in IV \iff y \in IU \cap IV \iff y \in I\Sigma^i(IU \cap IV) \equiv I(\Sigma^i IU \cap \Sigma^i IV) \equiv I(U \cap V)$;
(d) dually $y \in IU \vee y \in IV \iff y \in IU \cup IV \iff y \in I(U \cup V)$; and
(e) the family $\mathcal{P} \equiv \{U \mid y \in IU\} \subset \Sigma^X$ is Scott-open because I is Scott-continuous.
Hence $\mathcal{P}$ is a formal point (Definition 3.13) of X, so by sobriety of X there is a unique point $x \in X$ with $x \in U \iff y \in IU$, but $x \in U \iff ix \in IU$ so $y = ix$. Then $ix \in V \iff x \in U \equiv i^*V \iff u \in IU \equiv I(i^*V) \equiv \mathcal{E}V \iff y \in V$, so $y = ix$ by sobriety of Y. □

The notion of a Σ-splitting can be used to prove a famous result about locally compact spaces in a uniform way across all three settings. We showed in the previous section that they admit exponentials, but in fact they are the *only* spaces that do so. The following argument was inspired by Peter Johnstone's observation that $(-)^X$, if it exists, preserves injectivity [Joh82, Lemma VII 4.10], and Dana Scott's characterisation of injective spaces as continuous lattices with his topology [**?**].

Lemma 9.8 Let $\mathcal{C}$ be a category with finite products, Σ an object of C and $\mathcal{M} \subset \mathcal{C}$ a subcategory (*i.e.* it is closed under composition) that is closed under product with objects of $\mathcal{C}$. Also let $i : X \to Y$ be a map in $\mathcal{M}$ such that the exponentials Σ^X and Σ^Y exist. Then i is Σ-split, so maps Σ^i and I exist with $\Sigma^i \cdot I = \mathsf{id}_{\Sigma^X}$.



Proof We spell out the universal properties of ev_X and ev_Y to make it clear that we are not using any other exponentials besides these. By that of ev_X there is a unique map Σ^i that makes the square on the left commute:

$$\mathsf{ev}_X \cdot (\Sigma^i \times X) \ = \ \mathsf{ev}_Y \cdot (\Sigma^Y \times i).$$

By injectivity of Σ with respect to $\Sigma^X \times i$, there is some map $\tilde{I}$ making the lower right triangle commute and then by the universal property of ev_Y there is a unique map I making the upper triangle commute:

$$\mathsf{ev}_X \ = \ \tilde{I} \cdot (\Sigma^X \times i) \qquad \text{and} \qquad \tilde{I} \ = \ \mathsf{ev}_Y \cdot (I \times Y).$$

Then

$$\begin{aligned} \mathsf{ev}_X \cdot (\Sigma^i \times X) \cdot (I \times X) &= \mathsf{ev}_Y \cdot (\Sigma^Y \times i) \cdot (I \times X) \ = \ \mathsf{ev}_Y \cdot (I \times i) \\ &= \mathsf{ev}_Y \cdot (I \times Y) \cdot (\Sigma^X \times i) \\ &= \tilde{I} \cdot (\Sigma^X \times i) \ = \ \mathsf{ev}_X, \end{aligned}$$

whence $\Sigma^i \cdot I = \mathsf{id}_{\Sigma^X}$ by uniqueness in the universal property of ev_X. $\square$

Theorem 9.9 Let X be any topological space, locale or inductively presented formal cover for which the exponential Σ^X exists in that category. Then X is locally compact.

The class $\mathcal{M}$ needs more careful consideration.

Proof These three categories have products by Remark **??**. Since the space $Y \equiv \Sigma^A$ is locally compact, it has an exponential Σ^Y. For the class $\mathcal{M}$ we take
(a) inclusions with the subspace topology in Point–Set Topology, so $\mathcal{M}$ is closed under products with other spaces by the construction of the Tychonov product topology;
(b) sublocale inclusions in Locale Theory, which are the regular monomorphisms, so $\mathcal{M}$ is closed under products by simple category theory;
(c) cover extensions in Formal Topology, *i.e.* $(A, \sqsubseteq, \lhd_X) \hookrightarrow (A, \sqsubseteq, \lhd_Y)$ where $a \lhd_Y u \Longrightarrow a \lhd_X u$, so $\mathcal{M}$ is closed under products because ...
Then by the Lemma, $i : X \hookrightarrow Y$ is Σ-split, so by Theorem 9.1 and its analogues, X has a basis using Scott-open families indexed by $(A, \sqsubseteq)$, making it locally compact. $\square$

Theorem 9.10 Let $\mathcal{E}$ be a Scott-continuous endofunction of a continuous frame Ω (for a locally compact locale Y) such that

$$\mathcal{E}(U \wedge V) \;=\; \mathcal{E}(\mathcal{E}U \wedge \mathcal{E}V) \qquad \text{and} \qquad \mathcal{E}(U \vee V) \;=\; \mathcal{E}(\mathcal{E}U \vee \mathcal{E}V).$$

Then there is a Σ-split sublocale $i : X \rightarrowtail Y$ with X locally compact and $\mathcal{E} = I \cdot \Sigma^i$. If $Y \equiv \mathbf{Filt}(A, \sqsubseteq)$ then X is given by the formal cover defined by

$$a \lhd u \quad \equiv \quad \mathcal{E}B_a \leqslant \mathcal{E}B_u$$

and has a concrete basis. These are unique up to unique isomorphism.

It would be better to do this by defining $\prec\!\!\!\lessdot$ from $\mathcal{E}$ and taking advantage of the constructions earlier in this paper, instead of invoking external results about continuous lattices. Also deduce the result for locally compact sober spaces using choice.

Proof From either equation, $\mathcal{E}$ is an idempotent on Ω. Splitting it, we write i^* for the epi part because the equations make this a frame homomorphism, with a right adjoint $i^* \dashv i_*$. Then $i^* \cdot i_*$ is also the identity on the smaller lattice, whilst the composite $j \equiv i_* \cdot i^*$ is a localic nucleus, so the splitting is a frame that defines a sublocale $i : X \rightarrowtail Y$. Neither i_* nor j need be Scott continuous, but $\mathcal{E}$ and hence I are, so the smaller frame is a continuous lattice and X is locally compact. The cover relation is

$$\begin{aligned} a \lhd u \quad &\equiv \quad a \in ju \;\equiv\; \{a\} \subset i_*(i^* u) \\ &\Leftrightarrow \quad i^*\{a\} \subset i^* u \;\Longleftrightarrow\; I(i^*\{a\}) \subset I(i^* u) \\ &\equiv \quad \mathcal{E}B_a \leqslant \mathcal{E}B_u. \end{aligned}$$

$\square$

Notice in this proof that we pass irreversibly from using I to i_*. This is where we lose the track of the chosen Scott-open family $\mathcal{K}_a$ and are just left with $\Uparrow U_a$ defined by the order on the frame, *cf.* Proposition 7.15.

The one remaining part of Theorem 6.12 is (d), that bases correspond to quotients, reflecting the way in which frames are algebras. The analogue of this for local compactness requires us to generalise what we understand by Algebra. The structure that we have discussed in this is section is *intrinsic* because the topologies are now algebras *whose carriers are themselves spaces* instead of sets.

We can discuss algebras over general categories using the notion of a *monad*, along with its *Eilenberg–Moore category*, which was characterised by Jon Beck. Although Beck himself never published his eponymous result, several category theory textbooks have accounts of this topic, such as [Tay99, Section 7.5].

Theorem 9.11 The contravariant self-adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$ on the category of locally compact locales is monadic. The same holds for locally compact sober spaces, assuming the Axiom of Choice.

Proof Adapting Beck's theorem to our situation, $\Sigma^{(-)}$ must reflect invertibility and create $\Sigma^{(-)}$-split equalisers. The former is sobriety and the latter is essentially our notion of Σ-split subspace. The equation for a nucleus was first expressed using the λ-calculus [B], but [G] showed using bases that this is equivalent to our lattice-theoretic form. In Section 11 we will show that nuclei are interdefinable with abstract bases. □

Monadicity offers a notion of "completeness" for a categorical situation. The idea of Abstract Stone Duality that gave it its name was to consider monadicity of this adjunction in any category for which it is meaningful as a defining axiom and develop a symbolic calculus from that.

On this completeness principle, we are keen to accept all of the spaces that it offers as "locally compact". However, the abstract bases that we obtain in this way only satisfy the primary axioms. On the other hand, we needed the secondary ones to put the category of abstract bases and matrices into a manageable form. The distinction in terminology arises from this mis-match, so whichever choice we make, it would be necessary to employ arguments like those in Section 3 so show that this abstract category (is equivalent to one that) satisfies the monadicity property.

10 Abstract Stone Duality

The first three accounts of general topology that we considered relied on either the set of points or the algebra of open subspaces of a space. Our final approach is a formal language that is tailored to the intrinsic structure of the category of locally compact spaces as we set it out in the previous two sections. There are more extensive introductions to this calculus elsewhere: the one in Section 4 of [I] is the closest to the setting here, whilst the related paper [J] applies this to real analysis; for mathematical foundations and an overview of the motivations of ASD see [O].

Those who are attuned to the strength of the logic that a mathematical argument is using will already have noticed how little is needed to manipulate abstract bases. Formal Topologists shun impredicative universal quantification (*e.g.* in Lemma 7.18), but they still need it over infinite subsets, whereas it is finitary for abstract bases. In place of possibly nested implications, we just use *coherent sequents*, which are entailments between existentially quantified conjunctions.

The cost of working in a very weak system is that the proofs are much more laborious, so the construction of a model of our axioms using abstract bases and matrices alone will occupy an entire paper, but the reward is that we will be a step closer towards a link with interval computation. Here, therefore, we are just describing a *notation* for the structure that we have considered, which is valid in Point–Set Topology and Locale Theory because of the previous parts of this paper.

The specific objective of this paper for the development of Abstract Stone Duality is actually to show that the nuclei (Definition 9.6) that had been used in previous work to define (sub)types may be replaced by abstract bases. The results of the next section will therefore be valid in the calculus that we set out in this one.

This calculus speaks directly about points and functions, unlike Locale Theory and Formal Topology, whilst being much more concise than the set-theoretic notation that is used in Point–Set Topology.

Presenting syntax and its equivalence with other mathematical structures takes a lot of space, so we do this rather tersely, which unfortunately leaves us with just a lot of bullet points. If you have never seen such a calculus (in particular the λ-notation for functions) before, you should study one of the numerous treatments of the *simply typed* λ*-calculus* and its *denotational semantics* that now exist for masters' students in theoretical computer science. Beware, however, that our λ-calculus is *restricted* in that only exponentials of the form Σ^X are allowed.

We only introduce sets such as $\mathbb{N}$ to seed the generation of types, not as the ontology of their points or open subspaces. Here is all that we require of them:

Axiom 10.1 ***Sets*** form an ***arithmetic universe***, which is a category with
(a) finite limits ($\mathbf{1}$, $A \times B$ and equalisers);

(b) stable disjoint coproducts (disjoint unions);

(c) stable effective quotients of equivalence relations; and

(d) stable free monoids $\mathsf{Fin}\,(A)$.

We do not actually need quotients to define abstract bases, but if we use the properties in Axiom 10.8 to identify the sets from amongst all types, quotients turn out to be definable anyway [C]; for free monoids see [E].

Axiom 10.2 The ***types*** of ASD are formed as follows:

(a) any *set* (as just defined) is a type;

(b) if X and Y are types then so is their *product* $X \times Y$;

(c) if X is a type then so is its *exponential* Σ^X; and

(d) any Σ-*split subtype* (Axiom 10.11 or Remark 10.16) of a type is another type.

The ***interpretation***, ***denotation*** or ***semantics*** of a type is a locally compact sober topological space, locale or Formal Topology.

Axiom 10.3 As is customary, we write

$$x_1 : X_1,\ \ldots,\ x_n : X_k \quad \vdash \quad t : Y$$

for a ***term*** t of type Y, possibly containing (at most) free variables $x_1, \ldots, x_n$ of types $X_1, \ldots, X_n$ respectively. The ***interpretation*** of t is a continuous function

$$[\![\, t \,]\!] : [\![X_1]\!] \times \cdots \times [\![X_k]\!] \longrightarrow [\![Y]\!],$$

where $[\![X_1]\!]$, …, $[\![X_k]\!]$, $[\![Y]\!]$ are locally compact spaces that have been chosen as the denotations of the types $X_1, \ldots, X_n, Y$. We shall not use the brackets because we do not really need to distinguish between (syntactic) terms and their (topological) denotations in this paper.

The steps of a proof are *equations* between terms,

$$x_1 : X_1,\ \ldots,\ x_n : X_k \quad \vdash \quad t_1 = t_2 : Y,$$

except that if $Y \equiv \Sigma$ we write $\Leftrightarrow$ instead of $=$, whilst since Σ and Σ^X are lattices we can define $\Rightarrow$ or $\leqslant$ in terms of $\Leftrightarrow$ or $=$ and $\wedge$ or $\vee$. These equations between terms are interpreted as equations between continuous functions. Since equations arise as the results of deductions, we must allow them to occur as hypotheses, especially in Axiom 10.11 and for induction.

The list of type variables and equational hypotheses is called the ***context*** and is usually (partially) abbreviated to the letter Γ,

$$\Gamma \ \vdash\ t : Y \qquad \text{or} \qquad \Gamma \ \vdash\ t_1 = t_2 : Y,$$

or even omitted altogether if it is clear.

Axiom 10.4 There are terms $\langle s, t\rangle$, $\pi_0 p$, $\pi_1 p$, $\lambda x.\, \phi$ and ϕt that are associated with the product and exponential types in the usual way:

$$\frac{\Gamma \ \vdash\ s \equiv \pi_0 p : X \qquad \Gamma \ \vdash\ t \equiv \pi_0 p : Y}{\Gamma \ \vdash\ p \equiv \langle s, t\rangle : X \times Y} \qquad\qquad \frac{\Gamma,\ x : X \ \vdash\ \sigma \equiv \phi x : \Sigma}{\Gamma \ \vdash\ \phi \equiv \lambda x.\, \sigma : \Sigma^X}$$

Topologically, we are using Proposition 8.2 and Theorems 8.9 and 8.14 to write λ-terms of type Σ^X instead of open subspaces of X. However, there are some additional conditions below to make this work correctly.

Axiom 10.5 The types $\Sigma \equiv \Sigma^{\mathbf{1}}$ and Σ^X are distributive lattices and we may use $\top$, $\bot$, $\wedge$ and $\vee$ (but not $\neg$ or $\Rightarrow$) on their terms, because of Proposition 8.2.

Combining these operations with recursion over a list or (Kuratowski-) finite subset of a set A, we have membership and both forms of quantification,

$$a \in \ell, \qquad \forall a \in \ell.\, \phi a \qquad \text{and} \qquad \exists a \in \ell.\, \phi a$$

as terms of type Σ, if $\phi : \Sigma^A$.

Remark 10.6 To give the topology on X we need more than that Σ^X be a lattice. The key to classifying open and closed subspaces is the ***Phoa principle*** (Proposition 8.17),

$$F\sigma \iff F\bot \lor \sigma \land F\top.$$

This is rather more important than its simple form suggests. We deduce that
(a) if $\sigma \Rightarrow \tau$ then $F\sigma \Rightarrow F\tau$;
(b) more generally, any $F : \Sigma^Y \to \Sigma^X$ preserves the lattice order, which we therefore call ***intrinsic*** and write as $\leqslant$;
(c) the symbols $\neg$, $\Rightarrow$ and $\Leftrightarrow$ are therefore not allowed *within* terms of type Σ, but we use $\Rightarrow$ and $\Leftrightarrow$ instead of $\leqslant$ and $=$ for the order and equality *between* such terms;
(d) if $F\top \Rightarrow G\top$ then $\sigma \land F\sigma \Leftrightarrow \sigma \land (F\bot \lor F\top) \Rightarrow \sigma \land F\top \Rightarrow \sigma \land G\top \Rightarrow G\sigma$; and
(e) similarly if $F\bot \Rightarrow G\bot$ then $F\sigma \Rightarrow G\sigma \lor \sigma$.

The last two observations may be formulated as the following two fundamental rules for topological reasoning:

Axiom 10.7 Let $\alpha, \beta : \Sigma$ be terms that may depend on $\sigma : \Sigma$ (so $\alpha \equiv F\sigma$ and $\beta \equiv G\sigma$) and the variables in Γ. Then

$$\frac{\Gamma,\ \sigma \Leftrightarrow \top \quad\vdash\quad \alpha \Longrightarrow \beta}{\Gamma \quad\vdash\quad \sigma \land \alpha \Longrightarrow \beta} \qquad\text{and}\qquad \frac{\Gamma,\ \sigma \Leftrightarrow \bot \quad\vdash\quad \alpha \Longrightarrow \beta}{\Gamma \quad\vdash\quad \alpha \Longrightarrow \beta \lor \sigma}$$

The top lines say that $\alpha \Rightarrow \beta$ holds in the subspace U or C of Γ on which $\sigma \Leftrightarrow \top$ or $\bot$. Then the rules allow us to deduce the more complex implications in the *whole* space. We call these principles after Gerhard ***Gentzen*** because of the loose resemblance to his rules for implication and negation in the sequent calculus [Gen35, Section III]. The (*positive*) rule on the left is used very commonly and is easily overlooked, so for illustration we spell out its use in the proof of Lemma 11.1.

The (*negative*) one on the right, on the other hand, may be surprising to an intuitionistic *set* theorist, but it is a theorem of intuitionistic *locale* theory. For example, Japie Vermeulen [Ver94] stated it in the form of the dual Frobenius law for proper maps, *cf.* Definition 10.12(f) below.

Axiom 10.8 Any set N (Axiom 10.1) has
(a) *equality* $n, m : N \vdash (n = m) : \Sigma$, as a term in itself, not just an equation between terms,

$$\frac{\Gamma \ \vdash\ n = m : N}{\Gamma \ \vdash\ (n = m) \Leftrightarrow \top : \Sigma}$$

making the set *discrete*;
(b) *existential quantification*, $\phi : \Sigma^N \vdash \exists n.\, \phi n : \Sigma$, *cf.* Section 13;

$$\frac{\Gamma,\ n : N \ \vdash\ \phi n \Longrightarrow \sigma}{\Gamma \ \vdash\ \exists n.\, \phi n \Longrightarrow \sigma}$$

(c) and *definition by description* (Example 5.2),

$$\frac{\Gamma \ \vdash\ \exists n.\, \phi n \Leftrightarrow \top \qquad\qquad \Gamma,\ n, m : N \ \vdash\ \phi n \ \land\ \phi m \Longrightarrow (n = m)}{\Gamma,\ m : N \ \vdash\ \phi m \iff (m = \mathsf{the}\ n.\, \phi n)}$$

making it *sober*.

In fact these are the three conditions that make $i : X \hookrightarrow \Sigma^X$ a Σ-split inclusion, (Example 9.2) since $(n = m) \Leftrightarrow inm$ and $(\exists n.\, \phi n) \Leftrightarrow I\top\phi$.

We further require that any function $f : M \to N$ between sets give rise to a term $m : M \vdash fm : N$ whose denotation is f and that semantic equality between such functions be stated as an equation in the syntax.

Axiom 10.9 The ***Scott principle*** (Proposition 8.17) is that, for any set N,

$$F : \Sigma^{\Sigma^N},\ \xi : \Sigma^N \quad\vdash\quad F\xi \iff \exists \ell.\, (\forall n \in \ell.\, \xi n) \ \land\ F(\lambda n. n \in \ell).$$

This fully captures the infinitary aspects of general topology; in particular, we deduce that all terms preserve directed joins in the following sense:

Lemma 10.10 Let $G : \Sigma^{\Sigma^X}$, $\phi_\ell : \Sigma^X$ and $\alpha_\ell : \Sigma$ for $\ell \in \mathsf{Fin}\,(N)$ be such that

$$\phi_k \Longrightarrow \phi_{k\sqcup\ell} \Longleftarrow \phi_\ell, \quad \alpha_\circ \iff \top \quad \text{and} \quad \alpha_{k\sqcup\ell} \iff \alpha_k \wedge \alpha_\ell.$$

Then
$$G\left(\exists\ell.\,\phi_\ell \;\wedge\; \alpha_\ell\right) \iff \exists\ell.\,(G\phi_\ell) \;\wedge\; \alpha_\ell.$$

Proof Let $\xi \equiv \lambda n.\,\alpha_{\{n\}}$, so $\alpha_\ell \iff \forall n \in \ell.\,\xi n$, and

$$F \;\equiv\; \lambda\zeta.\,G\left(\exists\ell.\,\phi_\ell \;\wedge\; \forall n \in \ell.\,\zeta n\right).$$

Then
$$F(\lambda n.\,n \in k) \iff G\left(\exists\ell.\,\phi_\ell \;\wedge\; (\ell \subset k)\right) \iff G\phi_k,$$
so
$$\exists\ell.\,(\forall n \in \ell.\,\xi n) \;\wedge\; F(\lambda n.\,n \in \ell) \iff \exists\ell.\,\alpha_\ell \;\wedge\; G\phi_\ell,$$
which is equal by Axiom 10.9 to $F\xi \iff G\left(\exists\ell.\,\phi_\ell \;\wedge\; \alpha_\ell\right)$, as required. □

Finally we come to the characteristic feature of Abstract Stone Duality that encapsulates the study of locally compact spaces in this paper:

Axiom 10.11 Let $\mathcal{E}$ be a *nucleus* on a type Y (Definition 9.6). Then
(a) we *form* the subtype $X \equiv \{Y \mid \mathcal{E}\} \hookrightarrow Y$;
(b) if $\Gamma \vdash t : Y$ is a term of type Y that is *admissible* with respect to $\mathcal{E}$ in the sense of Lemma 9.7,

$$\Gamma,\ \psi : \Sigma^Y \;\vdash\; \psi t \iff \mathcal{E}\psi t,$$

then we *introduce* the term $\Gamma \;\vdash\; \mathsf{admit}\,t : X$ of type X;
(c) we *eliminate* $\Gamma \vdash s : X$ to give $\Gamma \vdash is : Y$;
(d) if $\Gamma \vdash \psi : \Sigma^Y$ then we *introduce* $\Gamma \vdash \Sigma^i\psi \equiv \lambda x.\,\psi(ix) : \Sigma^X$;
(e) we *eliminate* $\Gamma \vdash \phi : \Sigma^X$ to give $\Gamma \vdash I\phi : \Sigma^Y$; and
(f) the β- and η-rules (for admissible t) are

$$\mathsf{admit}(is) \;=\; s, \qquad i(\mathsf{admit}\,t) \;=\; t,$$

$$\phi s \;=\; (I\phi)(is), \qquad \text{and} \qquad I(\Sigma^i\psi) \;=\; \mathcal{E}\psi.$$

The motivation and details of this calculus were given in [B].

However, it has been very difficult to define nuclei for topologically interesting spaces, for example two sections of [I] were devoted to introducing the nucleus for the Dedekind reals (Example 5.3). In the following section we will show how abstract bases can be used instead.

Having stated the syntax and axioms, we turn to their topological meaning, which was inspired by that of Locale Theory (Definition 6.2).

Definition 10.12
(a) Terms $t : X$ are ***formal points*** of the type X;
(b) terms $\phi : \Sigma^X$ are ***formal open subspaces*** of X;
(c) a term $t : X$ ***lies in*** the open subspace classified by ϕ if $\phi t \Leftrightarrow \top$ and in the corresponding closed subspace if $\phi t \Leftrightarrow \bot$;
(d) terms of type Σ^{Σ^X} are interpreted as ***Scott-open families*** of open subspaces;
(e) in particular, a ***formal compact subspace*** of X is a term

$$K : \Sigma^{\Sigma^X} \quad \text{such that} \quad K\top \iff \top \quad \text{and} \quad K(\phi \wedge \psi) \iff K\phi \wedge K\psi,$$

where ϕ and ψ are terms of type Σ^X that denote open subspaces $U, V \subset X$;
(f) because of the negative Gentzen rule (Axiom 10.7), any formal compact subspace also satisfies the so-called ***dual Frobenius law***,

$$K\left(\lambda x.\,\sigma \vee \phi x\right) \iff \sigma \;\vee\; K\phi,$$

so long as σ does not depend on x;

(g) the formal compact subspace $K : \Sigma^{\Sigma^X}$ ***covers*** the open subspace $\phi : \Sigma^X$ if $K\phi \Leftrightarrow \top$; and

(h) a term $t : X$ ***lies in*** (the saturation of) the compact subspace $K : \Sigma^{\Sigma^X}$ if $K \leqslant \lambda\phi.\,\phi t$, so whenever $\phi : \Sigma^X$ satisfies $K\phi$ it also has ϕt: see Section 9 of [J].

A lot more general topology and real analysis may be expressed in this calculus, as [J] describes. For example, just as a space with a Σ-valued *equality* is *discrete* (Axiom 10.8(a)), so an *inequality* or *apartness* makes it *Hausdorff*. The term K serves as a *universal quantifier* over a formal compact subspace, although such subspaces are not necessarily representable as spaces or types in our calculus, because not all compact subspaces of a locally compact space are locally compact.

This paper, on the other hand, is concerned with how Abstract Stone Duality expresses local compactness. Accordingly, we rewrite our fundamental definition using the new notation, just as Definition 7.1 did in terms of locales. The λ-terms $\phi, \beta_a : \Sigma$ correspond to open subspaces and Scott-open families.

Definition 10.13 A ***concrete basis using λ-terms*** consists of

(a) for each $a : A$, terms $\beta_a : \Sigma^X$ and $K_a : \Sigma^{\Sigma^X}$;

(b) if $a \sqsubseteq b$ then $\beta_a \leqslant \beta_b$ and $K_a \geqslant K_b$, so $\beta_a x \Longrightarrow \beta_b x$ and $K_b\phi \Longrightarrow K_a\phi$;

(c) $\beta_a x \ \wedge\ \beta_b x \iff \exists c.\,\beta_c x \ \wedge\ (a \sqsubseteq c \sqsupseteq b)$; and

(d) $\phi x \iff \exists a.\,\beta_a x \ \wedge\ K_a\phi$.

As we have already seen in the other settings, from concrete bases we may derive abstract ones, Σ-split subspaces and nuclei:

Lemma 10.14 Any concrete basis (β_a, K_a) for X using λ-terms indexed by A defines a Σ-split inclusion $i : X \to \Sigma^A$ by

$$ix \ \equiv\ \lambda a.\,\beta_a x \qquad \text{and} \qquad I\phi \ \equiv\ \lambda\xi.\,\exists a.\,K_a\phi \ \wedge\ \xi a.$$

Conversely, given such an inclusion, the basis is

$$\beta_a \ \equiv\ \lambda x.\,ixa \qquad \text{and} \qquad K_a \ \equiv\ \lambda\phi.\,I\phi(\lambda b.\,a \sqsubseteq b)$$

and these translations are inverse.

Proof The basis gives a Σ-splitting because

$$(I\phi)(ix) \ \equiv\ \exists a.\,K_a\phi \ \wedge\ \beta_a x \iff \phi x.$$

Conversely, the Σ-splitting yields a basis because

$$\begin{aligned}\phi x \ \Leftrightarrow\ I\phi(ix) \ &\equiv\ \exists a.\,B_a(ix) \ \wedge\ \mathcal{L}_a(I\phi)\\ &\equiv\ \exists a.\,ixa \ \wedge\ I\phi(\lambda b.\,a \sqsubseteq b) \ \equiv\ \exists a.\,\beta_a x \ \wedge\ \mathcal{K}_a\phi.\end{aligned}$$

These translations are inverse because $ixa \iff \beta_a x$ and $K_a\phi \iff I\phi(\lambda b.\,a \sqsubseteq b)$ and we can recover $I\phi\xi$ from the latter. ☐

Lemma 10.15 Any concrete basis (β_a, K_a) using λ-terms gives rise to an abstract basis $(A, \sqsubseteq, \prec\!\!\!\!\prec)$, where

$$(a \prec\!\!\!\!\prec \ell) \ \equiv\ K_a\beta_\ell \ \equiv\ K_a\,(\lambda x.\,\exists b \in \ell.\,\beta_b x).$$

If the K_a preserve meets, so they are formal compact subspaces, then $\prec\!\!\!\!\prec$ obeys the strong intersection rule.

Proof It would be instructive to examine how he following arguments correspond to those in Section 2. Co- and contravariance of $\prec\!\!\!\!\prec$ follow from that of β_ℓ and A_a respectively. For the Wilker rule we use the basis expansion of β_c, switch to a directed basis and then apply K_a:

$$\begin{aligned}\beta_k \ \equiv\ \exists c \in k.\,\beta_c \ &=\ \exists c \in k.\,\exists b.\,\beta_b \ \wedge\ A_b\beta_c\\ &=\ \exists b.\,\beta_b \ \wedge\ \exists c \in k.\,A_b\beta_c\\ &=\ \exists \ell.\,\beta_\ell \ \wedge\ \forall b \in \ell.\,\exists c \in k.\,A_b\beta_c.\\ \text{Hence}\qquad a \prec\!\!\!\!\prec k \ \equiv\ A_a\beta_k \ &\Leftrightarrow\ \exists \ell.\,A_a\beta_\ell \ \wedge\ \forall b \in \ell.\,\exists c \in k.\,A_b\beta_c\\ &\equiv\ \exists \ell.\,a \prec\!\!\!\!\prec \ell \prec\!\!\!\!\prec^1 k\end{aligned}$$

by Lemma 10.10. For the weak intersection rule, the directed basis expansion of β_ℓ gives

$$\beta_\ell \;=\; \exists b.\, \beta_b \,\wedge\, A_b\beta_\ell \;=\; \exists k.\, \beta_k \,\wedge\, \forall b \in k.\, A_b\beta_\ell \;\geqslant\; \beta_k \,\wedge\, (k \prec\!\!\!\prec \ell).$$

Hence, using $\beta_c \wedge \beta_d = \exists e.\, \beta_e \wedge (c \sqsupseteq e \sqsubseteq d)$ in the equality,

$$\beta_k \,\wedge\, (k \prec\!\!\!\prec \ell_1) \,\wedge\, (k \prec\!\!\!\prec \ell_2) \;\leqslant\; \beta_{\ell_1} \,\wedge\, \beta_{\ell_2} \;=\; \exists h.\, \beta_h \,\wedge\, h \sqsubseteq \ell_1 \sqcap \ell_2$$

and therefore, by Lemma 10.10 again,

$$K_a\beta_k \,\wedge\, (k \prec\!\!\!\prec \ell_1) \,\wedge\, (k \prec\!\!\!\prec \ell_2) \implies \exists h.\, K_a\beta_h \,\wedge\, h \sqsubseteq \ell_1 \sqcap \ell_2.$$

The strong case is similar but simpler. □

Remark 10.16 In the next section we will show conversely that any abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ for which A is a set in the sense of Axiom 10.1 defines a nucleus $\mathcal{E}$. From this we obtain a Σ-split subtype $X \hookrightarrow \Sigma^A$ equipped with a concrete basis (β_a, K_a) using λ-terms. Moreover any term $\xi : \Sigma^A$ that is a formal point (rounded bounded located filter) for the abstract basis is admissible for $\mathcal{E}$ and therefore provides a term of X.

Hence we may replace Axiom 10.11 with the following rules:

(a) *formation* of the type $X \equiv \mathsf{Spec}\,(A, \sqsubseteq, \prec\!\!\!\prec)$;

(b) *introduction* of a term $\Gamma \vdash \mathsf{admit}(\xi) : X$ whenever $\Gamma \vdash \xi : \Sigma^A$ is a formal point;

(c) *elimination* of $x : X$ to get $\beta_a x : \Sigma$ for each element $a : A$ of the basis;

(d) *introduction* of $\lambda x.\, \Psi(\lambda a.\, \beta_a x) : \Sigma^X$ given $\Psi : \Sigma^{\Sigma^A}$; and

(e) *elimination* of $\phi : \Sigma^X$ to get $K_a\phi : \Sigma$ for each $a : A$; where

(f) the β- and η-rules are the main themes of the paper,

$$\mathsf{admit}(\lambda a.\, \beta_a x) \;=\; x, \qquad \beta_a(\mathsf{admit}\,\xi) \iff \xi a,$$

$$\phi x \iff \exists a.\, \beta_a x \wedge K_a\phi \qquad \text{and} \qquad K_a\phi_\ell \iff (a \prec\!\!\!\prec \ell).$$

Then Lemma 10.14 provides the maps i and I that we need to recover Axiom 10.11. □

We may also translate the results of Sections 3 and 4 to upgrade bases using λ-terms to obey the secondary axioms and to use matrices defined by

$$\langle\, a \,|\, f \,|\, b \,\rangle \;\equiv\; K_a\big(\lambda x.\, \gamma_b(fx)\big),$$

to characterise continuous functions (terms, morphisms) $fX \to Y$ where X and Y have bases (β, K_a) and (γ_b, L_b) respectively.

11 Abstract bases and nuclei

This section was the core calculation on which the paper was built. The plan and details of the proofs need to be checked.

In this section we prove the correspondence between an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ satisfying the primary axioms and an ASD nucleus $\mathcal{E}$ (Definition 9.6), entirely within the calculus that we set out in the previous section. This justifies the replacing the subtype-formation rule in Axiom 10.11 with that in Remark 10.16. The following account is a much simplified version of the one in [G].

Theorem 9.1 gave the plan for the construction. We must first introduce $\mathbf{Filt}(A, \sqsubseteq)$ as an object in ASD, as we did in Point–Set Topology in Proposition 5.14 and Locale Theory in Lemma 6.11. We do this by defining a nucleus $\mathcal{E}^0$ on Σ^A and identifying the admissible terms.

Lemma 11.1 The term $\mathcal{E}^0$ defined by $\mathcal{E}^0\Phi\xi \;\equiv\; \exists a.\, \xi a \,\wedge\, \Phi(\lambda b.\, a \sqsubseteq b)$ is a nucleus.

Proof We spell out this simple argument in detail because it illustrates the (positive) Gentzen rule (Axiom 10.7), whilst any $\Phi : \Sigma^A \to \Sigma$ preserves the intrinsic order (Remark 10.6(b)).

$$\begin{array}{lcll}
a \sqsubseteq b,\ b \sqsubseteq c & \vdash & a \sqsubseteq c & \text{transitivity} \\
a \sqsubseteq b & \vdash & b \sqsubseteq c \implies a \sqsubseteq c & \text{Gentzen} \\
a \sqsubseteq b & \vdash & \lambda c.\, b \sqsubseteq c \ \leqslant\ \lambda c.\, a \sqsubseteq c & \lambda\text{-abstraction} \\
a \sqsubseteq b & \vdash & \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi(\lambda c.\, a \sqsubseteq c) & \text{intrinsic monotonicity} \\
\ldots & \vdash & (a \sqsubseteq b) \ \wedge\ \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi(\lambda c.\, a \sqsubseteq c) & \text{Gentzen} \\
\ldots & \vdash & \exists b.\, (\lambda b.\, a \sqsubseteq b) b \ \wedge\ \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi(\lambda c.\, a \sqsubseteq c), & \exists
\end{array}$$

where the last line is in fact $\Leftrightarrow$ because we may put $b \equiv a$. Hence $\mathcal{E}^0\Phi(\lambda b.\, a \sqsubseteq b) \Leftrightarrow \Phi(\lambda b.\, a \sqsubseteq b)$. Then, with either $\wedge$ or $\vee$,

$$\begin{aligned}
\mathcal{E}^0(\mathcal{E}^0\Phi \overset{\vee}{\underset{\wedge}{}} \mathcal{E}^0\Psi)\xi \ &\equiv\ \exists a.\, \xi a \ \wedge\ (\mathcal{E}^0\Phi \overset{\vee}{\underset{\wedge}{}} \mathcal{E}^0\Psi)(\lambda b.\, a \sqsubseteq b) \\
&\Leftrightarrow\ \exists a.\, \xi a \ \wedge\ (\Phi \overset{\vee}{\underset{\wedge}{}} \Psi)(\lambda b.\, a \sqsubseteq b) \ \equiv\ \mathcal{E}^0(\Phi \overset{\vee}{\underset{\wedge}{}} \Psi)\xi. \qquad \square
\end{aligned}$$

Next we verify that $\mathcal{E}^0$ defines the object that we want by proving that a term $\xi : \Sigma^A$ is a filter iff it is *admissible* for $\mathcal{E}^0$ in the sense of Lemma 9.7, satisfying $\mathcal{E}^0\Phi\xi = \Phi\xi$ for all Φ. Note that such a term ξ may have parameters, so these points are "generalised" ones in the sense of sheaf theory; they are test maps to an equaliser from a general object.

Lemma 11.2 If $\xi : \Sigma^A$ is admissible for $\mathcal{E}^0$ then it is covariant, bounded and filtered.

Proof We use admissibility with respect to various Φ. For covariance, let $\Phi \equiv \lambda\zeta.\, \zeta a$, so

$$\xi a \ \equiv\ \Phi\xi \iff \mathcal{E}^0\Phi\xi \ \equiv\ \exists b.\, \xi b \ \wedge\ (b \sqsubseteq a).$$

Then, for filteredness, let $\Phi \equiv \lambda\zeta.\, \zeta b \wedge \zeta c$, so

$$\xi b \wedge \xi c \ \equiv\ \Phi\xi \iff \mathcal{E}^0\Phi\xi \iff \exists a.\, \xi a \ \wedge\ (b \sqsupseteq a \sqsubseteq c).$$

Finally, $\Phi \equiv \lambda\zeta.\, \top$ gives boundedness: $\top \equiv \Phi\xi \Leftrightarrow \mathcal{E}^0\Phi\xi \Leftrightarrow \exists a.\, \xi a$. $\square$

Lemma 11.3 If ξ is covariant then $\mathcal{E}^0\Phi\xi \implies \Phi\xi$.

Proof As in Lemma 11.1, we may write covariance as $\xi b \vdash (\lambda c.\, b \sqsubseteq c) \leqslant \xi$.
Since any Φ preserves the intrinsic order $\leqslant$, we have $\xi b \vdash \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi\xi$.
Using the Gentzen rule we deduce that $\xi b \ \wedge\ \Phi(\lambda c.\, b \sqsubseteq c) \implies \Phi\xi$. $\square$

Lemma 11.4 If ξ is bounded and filtered then $\Phi\xi \implies \mathcal{E}^0\Phi\xi$.

Proof By the Scott principle (Axiom 10.9), $\Phi\xi \iff \exists\ell.\, (\forall b \in \ell.\, \xi b) \ \wedge\ \Phi(\lambda b.\, b \in \ell)$.

By induction on ℓ, we claim that ξ satisfies

$$\exists c.\, \xi c \ \wedge\ \forall b \in \ell.\, (c \sqsubseteq b).$$

In the base case $\ell \equiv \circ$, this is boundedness of ξ, whilst filteredness of ξ gives the induction step. Then $(\lambda b.\, b \in \ell) \leqslant (\lambda b.\, c \sqsubseteq b)$, so $\Phi(\lambda b.\, b \in \ell) \implies \Phi(\lambda b.\, c \sqsubseteq b)$ since Φ preserves the intrinsic order. Hence $\exists c.\, \xi c \ \wedge\ \Phi(\lambda b.\, c \sqsubseteq b)$, which is $\mathcal{E}^0\Phi\xi$. $\square$

Lemma 11.5 The object $\mathbf{Filt}(A, \sqsubseteq)$ that is defined by the nucleus $\mathcal{E}^0$ on Σ^A has a basis using λ-terms with

$$B_a \ \equiv\ \lambda\xi.\, \xi a \qquad \text{and} \qquad \mathcal{L}_a \ \equiv\ \lambda\Phi.\, \Phi(\lambda b.\, a \sqsubseteq b),$$

where the general open subspaces are those $\Phi : \Sigma^{\Sigma^A}$ such that $\Phi = \mathcal{E}^0\Phi$ and the basis expansion is

$$\Phi\xi \iff \mathcal{E}^0\Phi\xi \ \equiv\ \exists a.\, B_a\xi \ \wedge\ \mathcal{L}_a\Phi \ \equiv\ \exists a.\, \xi a \ \wedge\ \Phi(\lambda b.\, a \sqsubseteq b). \qquad \square$$

Lemma 10.14 actually embeds a space with a basis indexed by the preorder $(A, \sqsubseteq)$ into $\mathbf{Filt}(A, \sqsubseteq)$ rather than Σ^A.

Lemma 11.6 Any concrete basis using λ-terms gives rise to a nucleus on $\mathbf{Filt}(A, \sqsubseteq)$ with

$$\mathcal{E}\Phi\xi \;\equiv\; \exists a\ell.\, \xi a \;\wedge\; (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\, \Phi(\lambda c.\, b \sqsubseteq c).$$

Proof Let Φ be an open subspace of $\mathbf{Filt}(A, \sqsubseteq)$, so $\Phi = \mathcal{E}^0\Phi$, then

$$\begin{array}{rcll}
\Sigma^i\Phi & \equiv & \lambda x.\, \Phi(ix) \;\equiv\; \lambda x.\, \Phi(\lambda b.\, \beta_b x) & \\
& = & \lambda x.\, \mathcal{E}^0\Phi(\lambda b.\, \beta_b x) & \\
& \equiv & \lambda x.\, \exists b.\, (\lambda b'.\, \beta_{b'} x) b \;\wedge\; \Phi(\lambda c.\, b \sqsubseteq c) & \text{Lemma 11.1} \\
& = & \exists b.\, \beta_b \;\wedge\; \Phi(\lambda c.\, b \sqsubseteq c) & \\
& = & \exists \ell.\, \beta_\ell \;\wedge\; \forall b \in \ell.\, \Phi(\lambda c.\, b \sqsubseteq c) & \\
K_a(\Sigma^i\Phi) & \Leftrightarrow & \exists \ell.\, K_a\beta_\ell \;\wedge\; \forall b \in \ell.\, \Phi(\lambda c.\, b \sqsubseteq c) & \text{Lemma 10.10} \\
\mathcal{E}\Phi\xi & \equiv & I(\Sigma^i\Phi)\xi \;\equiv\; \exists a.\, \xi a \;\wedge\; K_a(\Sigma^i\Phi) & \text{Lemma 10.14} \\
& \Leftrightarrow & \exists a\ell.\, \xi a \;\wedge\; a \prec\!\!\!\prec \ell \;\wedge\; \forall b \in \ell.\, \Phi(\lambda c.\, b \sqsubseteq c). &
\end{array}$$

The equations for a nucleus follow from the fact that $\mathcal{E} = I \cdot \Sigma^i$. ☐

Now we show that any *abstract* basis defines a nucleus too.

Lemma 11.7 Given any co- and contravariant relation $\prec\!\!\!\prec$, let

$$\mathcal{K}_a\Phi \;\equiv\; \exists \ell.\, (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\, \Phi(\lambda c.\, b \sqsubseteq c)$$

and $$\mathcal{E}\Phi\xi \quad\equiv\quad \exists a.\, B_a\xi \;\wedge\; \mathcal{K}_a\Phi \quad\equiv\quad \exists a\ell.\, \xi a \;\wedge\; (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\, \Phi(\lambda c.\, b \sqsubseteq c).$$

Then we recover

$$\mathcal{K}_a\Phi \iff \mathcal{E}\Phi(\lambda b.\, a \sqsubseteq b) \qquad \text{and} \qquad (a \prec\!\!\!\prec \ell) \iff \mathcal{K}_a B_\ell \iff \mathcal{E}B_\ell(\lambda b.\, a \sqsubseteq b).$$

Also, $\mathcal{E}$ satisfies $\mathcal{E} = \mathcal{E}^0 \cdot \mathcal{E} = \mathcal{E} \cdot \mathcal{E}^0$ and is recovered from $\prec\!\!\!\prec$.

Proof By covariance of $a \prec\!\!\!\prec \ell$ in ℓ,

$$\mathcal{K}_a B_\ell \;\equiv\; \exists k.\, (a \prec\!\!\!\prec k) \;\wedge\; \forall b \in k.\, \exists c \in \ell.\, b \sqsubseteq c \iff (a \prec\!\!\!\prec \ell).$$

Contravariance of $a \prec\!\!\!\prec \ell$ in a transfers to $\mathcal{K}_a$; using this, $\mathcal{K}_a$ is recovered from $\mathcal{E}$. We leave the last part to the reader since we will not use it. ☐

Now we must use the properties of an abstract basis to prove that $\mathcal{E}$ satisfies the two equations in Definition 9.6. However, since any term preserves the intrinsic order, we already have

$$\mathcal{E}(\Phi \wedge \Psi) \;\leqslant\; (\mathcal{E}\Phi) \wedge (\mathcal{E}\Psi) \qquad \text{and} \qquad (\mathcal{E}\Phi) \vee (\mathcal{E}\Psi) \;\leqslant\; \mathcal{E}(\Phi \vee \Psi),$$

so we only need to prove the reverse inequalities. The weak intersection rule gives the first and the Wilker rule the second.

Lemma 11.8 If $\prec\!\!\!\prec$ satisfies the weak intersection rule

$$(a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\, (b \prec\!\!\!\prec k_1 \wedge b \prec\!\!\!\prec k_2) \implies a \prec\!\!\!\prec k_1 \sqcap k_2,$$

then $\mathcal{K}_a(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \implies \mathcal{K}_a(\Phi \wedge \Psi)$ and so $\mathcal{E}(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \;\leqslant\; \mathcal{E}(\Phi \wedge \Psi)$.

Proof Using the formulae for $\mathcal{K}_a$ in Lemma 11.7 three times,

$$\begin{array}{cl}
& \mathcal{K}_a(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \\
\Leftrightarrow & \exists \ell.\, (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\, \mathcal{E}\Phi(\lambda c.\, b \sqsubseteq c) \;\wedge\; \mathcal{E}\Psi(\lambda c.\, b \sqsubseteq c) \\
\Leftrightarrow & \exists \ell.\, (a \prec\!\!\!\prec \ell) \;\wedge\; \forall b \in \ell.\, \left\{ \begin{array}{cl} & \exists k_1.\, (b \prec\!\!\!\prec k_1) \;\wedge\; \forall c_1 \in k_1.\, \Phi(\lambda d.\, c_1 \sqsubseteq d) \\ \wedge & \exists k_2.\, (b \prec\!\!\!\prec k_2) \;\wedge\; \forall c_2 \in k_2.\, \Psi(\lambda d.\, c_2 \sqsubseteq d). \end{array} \right.
\end{array}$$

Taking the unions of the k-lists for all $b \in \ell$ and using covariance of $\prec\!\!\!\leqslant$ with respect to k, this implies

$$\exists k_1 k_2. \begin{cases} \quad \exists \ell.\, (a \prec\!\!\!\leqslant \ell) \;\wedge\; (\forall b \in \ell.\, b \prec\!\!\!\leqslant k_1 \wedge b \prec\!\!\!\leqslant k_2) \\ \wedge \quad \forall c_1 \in k_1.\, \Phi(\lambda d.\, c_1 \sqsubseteq d) \\ \wedge \quad \forall c_2 \in k_2.\, \Psi(\lambda d.\, c_2 \sqsubseteq d). \end{cases}$$

By the weak intersection rule, the top line implies $a \prec\!\!\!\leqslant k_1 \sqcap k_2$, which is

$$\exists \ell'.\, (a \prec\!\!\!\leqslant \ell') \;\wedge\; \forall b \in \ell'.\, (\exists c_1 \in k_1.\, b \sqsubseteq c_1) \;\wedge\; (\exists c_2 \in k_2.\, b \sqsubseteq c_2),$$

possibly with a different list ℓ'. Then we match $\exists c$ with $\forall c$ and use

$$(b \sqsubseteq c) \;\wedge\; \Phi(\lambda d.\, c \sqsubseteq d) \implies \Phi(\lambda d.\, b \sqsubseteq d)$$

from Lemma 11.1, the fact that Φ preserves the intrinsic order, the Gentzen rule (Axiom 10.7) and Lemma 11.7 to obtain

$$\exists \ell'.\, (a \prec\!\!\!\leqslant \ell') \;\wedge\; \forall b \in \ell'.\, \Phi(\lambda d.\, b \sqsubseteq d) \;\wedge\; \Psi(\lambda d.\, b \sqsubseteq d) \;\equiv\; \mathcal{K}_a(\Phi \wedge \Psi).$$

Hence we have shown that $\mathcal{K}_a(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \implies \mathcal{K}_a(\Phi \wedge \Psi)$. Then by Lemma 11.7,

$$\begin{aligned} \mathcal{E}(\Phi \wedge \Psi)\xi \;&\equiv\; \exists a.\, \xi a \;\wedge\; \mathcal{K}_a(\Phi \wedge \Psi) \\ &\Rightarrow\; \exists a.\, \xi a \;\wedge\; \mathcal{K}_a(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \;\equiv\; \mathcal{E}(\mathcal{E}\Phi \wedge \mathcal{E}\Psi). \end{aligned}$$ □

In the Wilker rule it is convenient to consider existential quantification instead of binary disjunction:

Lemma 11.9 If $\prec\!\!\!\leqslant$ satisfies the Wilker rule

$$a \prec\!\!\!\leqslant k \implies \exists \ell.\, (a \prec\!\!\!\leqslant \ell) \;\wedge\; \forall b \in \ell.\, \exists c \in k.\, (b \prec\!\!\!\leqslant c),$$

then $\qquad \mathcal{K}_a(\exists i.\, \Phi_i) \implies \mathcal{K}_a(\exists i.\, \mathcal{E}\Phi_i) \qquad$ and so $\qquad \mathcal{E}(\exists i.\, \Phi_i) \leqslant \mathcal{E}(\exists i.\, \mathcal{E}\Phi_i)$

and in particular $\mathcal{E} \leqslant \mathcal{E} \cdot \mathcal{E}$.

Proof Using Lemma 11.7 several times, the Wilker rule in the second line and $h \equiv \{c\}$ half-way down,

$$\begin{aligned} \mathcal{K}_a(\exists i.\, \Phi_i) \;&\equiv\; \exists k.\, (a \prec\!\!\!\leqslant k) \;\wedge\; \forall c \in k.\, \exists i.\, \Phi_i(\lambda d.\, c \sqsubseteq d) \\ &\Rightarrow\; \exists k\ell.\, (a \prec\!\!\!\leqslant \ell) \;\wedge\; \big(\forall b \in \ell.\, \exists c \in k.\, b \prec\!\!\!\leqslant c\big) \\ &\qquad\qquad \wedge \big(\forall c \in k.\, \exists i.\, \Phi_i(\lambda d.\, c \sqsubseteq d)\big) \\ &\Rightarrow\; \exists \ell.\, (a \prec\!\!\!\leqslant \ell) \;\wedge\; \forall b \in \ell.\, \exists ci.\, (b \prec\!\!\!\leqslant c) \;\wedge\; \Phi_i(\lambda d.\, c \sqsubseteq d) \\ &\Rightarrow\; \exists \ell.\, (a \prec\!\!\!\leqslant \ell) \;\wedge\; \forall b \in \ell.\, \exists i. \\ &\qquad\qquad \exists h.\, (b \prec\!\!\!\leqslant h) \;\wedge\; \forall c \in h.\, \Phi_i(\lambda d.\, c \sqsubseteq d) \\ &\Rightarrow\; \exists \ell.\, (a \prec\!\!\!\leqslant \ell) \;\wedge\; \forall b \in \ell.\, \exists i.\, \mathcal{E}\Phi_i(\lambda c.\, b \sqsubseteq c) \\ &\equiv\; \mathcal{K}_a(\exists i.\, \mathcal{E}\Phi_i). \end{aligned}$$

Then $\quad \mathcal{E}(\exists i.\, \Phi_i)\xi \;\equiv\; \exists a.\, \xi a \wedge \mathcal{K}_a(\exists i.\, \Phi_i) \implies \exists a.\, \xi a \wedge \mathcal{K}_a(\exists i.\, \mathcal{E}\Phi_i) \;\equiv\; \mathcal{E}(\exists i.\, \mathcal{E}\Phi_i)\xi.$ □

We leave the following similar but simpler results to the reader:

Lemma 11.10

(a) If $\mathcal{E}$ satisfies $\mathcal{E}(\mathcal{E}\Phi \wedge \mathcal{E}\Psi) \leqslant \mathcal{E}(\Phi \wedge \Psi)$ then $\prec\!\!\!\leqslant$ obeys the weak intersection rule;

(b) if $\mathcal{E}$ satisfies $\mathcal{E}(\exists i.\, \Phi_i) \leqslant \mathcal{E}(\exists i.\, \mathcal{E}\Phi_i)$ then $\prec\!\!\!\leqslant$ obeys the Wilker rule;

(c) $\mathcal{E} \cdot \mathcal{E} \leqslant \mathcal{E}$ iff $\prec\!\!\!\leqslant$ is transitive;

(d) $\mathcal{E}\top = \top$ iff $\prec\!\!\!\leqslant$ is bounded above;

(e) $\mathcal{E}\bot = \bot$ iff no $a \in A$ has $a \prec\!\!\!\leqslant \circ$ (Section 13);

(f) $\mathcal{E}$ preserves binary meets iff $\prec\!\!\!\leqslant$ satisfies the strong intersection rule;

(g) $\mathcal{E} \leqslant \mathcal{E}^1 \cdot \mathcal{E}$ iff $\prec\!\!\!\leqslant$ satisfies single interpolation, where $\mathcal{E}^1$ is defined from $\prec\!\!\!\leqslant^1$ in the same way that $\mathcal{E}$ was defined from $\prec\!\!\!\leqslant$:

$$\mathcal{E}^1\Phi\xi \;\equiv\; \exists ab.\, \xi a \;\wedge\; (a \prec\!\!\!\leqslant b) \;\wedge\; \Phi(\lambda c.\, b \sqsubseteq c).$$ □

This completes the proof that $\mathcal{E}$ is a nucleus, so we can use it in Axiom 10.11 to form a type:

Theorem 11.11 Every abstract basis obeying the primary rules presents a locally compact object in Abstract Stone Duality. Hence the new subtype formation rule in Remark 10.16 is justified. □

Now that we have established the equivalence between abstract bases and nuclei we turn to that between their respective notions of formal point, in Definition 5.1 and Lemma 9.7.

Lemma 11.12 If ξ is admissible then it is ***covariant***, $\xi a \ \wedge\ (a \sqsubseteq b) \Longrightarrow \xi b$.

Proof We need to be careful because the verbatim proof of Lemma 11.2 gave roundedness instead. For $a \in A$, let $\Phi_a \equiv \lambda\zeta.\,\zeta a$, so by Lemma 11.7,

$$\mathcal{E}\Phi_a\xi \iff \exists c\ell.\,\xi c \ \wedge\ (c \prec\!\!\!\prec \ell) \ \wedge\ \forall d \in \ell.\, d \sqsubseteq a.$$

Then, for admissible ξ, since $\sqsubseteq$ is transitive we have

$$a \sqsubseteq b \quad \vdash \quad \xi a \iff \Phi_a\xi \iff \mathcal{E}\Phi_a\xi \Longrightarrow \mathcal{E}\Phi_b\xi \iff \Phi_b\xi \iff \xi b$$

and the stated result follows from the Gentzen rule. □

Lemma 11.13 If ξ is admissible then it is ***rounded***, $\xi c \iff \exists a.\,\xi a \wedge (a \prec\!\!\!\prec c)$.
Conversely, if ξ is rounded then $\mathcal{E}^0\Phi\xi \Longrightarrow \mathcal{E}\Phi\xi$ for any Φ.

Proof Consider $\Phi \equiv \lambda\zeta.\,\zeta c$ and use covariance for $\ell \sqsubseteq \{c\}$.

$$\begin{array}{rcll}
\xi c & \equiv & \Phi\xi \iff \mathcal{E}\Phi\xi & \text{def } \Phi \\
& \equiv & \exists a\ell.\,\xi a \ \wedge\ (a \prec\!\!\!\prec \ell) \ \wedge\ \forall b \in \ell.\,(\lambda c'.\,b \sqsubseteq c')c & \text{Lemma 11.7} \\
& \Leftrightarrow & \exists a.\,\xi a \ \wedge\ \exists \ell.\,a \prec\!\!\!\prec \ell \sqsubseteq c & \text{Notation 1.7} \\
& \Leftrightarrow & \exists a.\,\xi a \wedge (a \prec\!\!\!\prec c). & \text{covariance for } \ell \sqsubseteq \{c\} \\
\mathcal{E}^0\Phi\xi & \equiv & \exists b.\,\xi b \ \wedge\ \Phi(\lambda c.\,b \sqsubseteq c) & \text{Lemma 11.1} \\
& \Leftrightarrow & \exists ab.\,\xi a \ \wedge\ (a \prec\!\!\!\prec b) \ \wedge\ \Phi(\lambda c.\,b \sqsubseteq c) & \text{rounded} \\
& \equiv & \mathcal{E}^1\Phi\xi \Longrightarrow \mathcal{E}\Phi\xi, &
\end{array}$$

where $\mathcal{E}^1\Phi\xi \equiv \exists ab.\,\xi a \wedge (a \prec\!\!\!\prec b) \wedge \Phi(\lambda c.\,b \sqsubseteq c)$. □

Lemma 11.14 If ξ is admissible then it is ***located***,

$$\xi a \ \wedge\ (a \prec\!\!\!\prec \ell) \Longrightarrow \exists b.\,\xi b \ \wedge\ (b \in \ell).$$

Conversely, if ξ is located then $\mathcal{E}\Phi\xi \Longrightarrow \mathcal{E}^0\Phi\xi$ for any Φ.
In particular, if a is empty $(a \prec\!\!\!\prec \circ)$ then $\xi a \Leftrightarrow \bot$.

Proof Consider $\Phi \equiv \lambda\zeta.\,\exists b \in \ell.\,\zeta b$. Then with $k \equiv \ell$ and $b \equiv d$,

$$\begin{array}{rcll}
\xi a \ \wedge\ (a \prec\!\!\!\prec \ell) & \Rightarrow & \exists ak.\,\xi a \ \wedge\ (a \prec\!\!\!\prec k) \ \wedge\ \forall d \in k.\,\exists b \in \ell.\,d \sqsubseteq b & \\
& \equiv & \mathcal{E}\Phi\xi \iff \Phi\xi \ \equiv\ \exists b \in \ell.\,\xi b. & \text{def } \mathcal{E}, \Phi \\
\mathcal{E}\Phi\xi & \equiv & \exists a\ell.\,\xi a \wedge (a \prec\!\!\!\prec \ell) \wedge \forall b \in \ell.\,\Phi(\lambda c.\,b \sqsubseteq c) & \text{def } \mathcal{E} \\
& \Rightarrow & \exists b\ell.\,\xi b \wedge (b \in \ell) \wedge \forall b' \in \ell.\,\Phi(\lambda c.\,b' \sqsubseteq c) & \text{located} \\
& \Rightarrow & \exists b.\,\xi b \ \wedge\ \Phi(\lambda c.\,b \sqsubseteq c) \ \equiv\ \mathcal{E}^0\Phi\xi. & \text{Lemma 11.1}
\end{array}$$

□

Lemma 11.15 If ξ is admissible then it is ***bounded*** and ***filtered***,

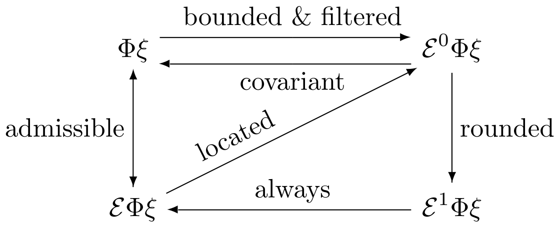
$$\exists a.\,\xi a \qquad \text{and} \qquad \xi b \ \wedge\ \xi c \Longrightarrow \exists a.\,\xi a \ \wedge\ (b \sqsupseteq a \sqsubseteq c).$$

Proof The proof of boundedness is the same as in Lemma 11.2, but that for filteredness uses the roundedness property above. With $\Phi \equiv \lambda\zeta.\,\zeta b \wedge \zeta c$ as before,

$$\begin{array}{rcl}
\xi b \ \wedge\ \xi c & \equiv & \Phi\xi \iff \mathcal{E}\Phi\xi \\
& \Rightarrow & \mathcal{E}^0\Phi\xi \ \equiv\ \exists a.\,\xi a \ \wedge\ \Phi(\lambda d.\,a \sqsubseteq d) \ \equiv\ \exists a.\,\xi a \ \wedge\ (b \sqsupseteq a \sqsubseteq c).
\end{array}$$

□

Proposition 11.16 A term $\xi : \Sigma^A$ is admissible for $\mathcal{E}$ iff it is rounded, bounded, covariant, filtered and located for $\prec\!\!\prec$. Hence the introduction and elimination rules for terms of type X and the introduction rule for Σ^X in Remark 10.16 are justified.



Proof The preceding lemmas deduce the other properties from admissibility. Conversely, if ξ is rounded and located then $\mathcal{E}^0\Phi\xi \Leftrightarrow \mathcal{E}\Phi\xi$ by Lemmas 11.13 and 11.14, whilst if it is bounded, covariant and filtered then $\mathcal{E}^0\Phi\xi \Leftrightarrow \Phi\xi$ by Lemmas 11.3 and 11.4. ☐

This completes the proof of the soundness and completeness of the axioms for a abstract basis as an account of concrete bases for locally compact sober spaces, locales, formal topologies and objects of ASD.

12 Bases using compact subspaces

We began with a natural definition of basis that uses compact subspaces, but in most of our discussion we have used Scott-open families instead, at the cost of the weak rule for intersections. We would nevertheless like to restore the strong intersection rule because of Proposition 1.13 and Lemma 8.13. In this section we use Lawson's Lemma 3.11 and so the axiom of Dependent Choice to convert the weaker forms to the stronger ones. We consider the result for abstract bases in detail first and the concrete one afterwards.

Notation 12.1 Let $(A, \sqsubseteq, \prec\!\!\prec)$ be an abstract basis that satisfies the single interpolation rule. A ***Lawson sequence*** $\vec{a}$ is one of the form

$$a_\infty \prec\!\!\prec \cdots \prec\!\!\prec a_2 \prec\!\!\prec a_1 \prec\!\!\prec a_0, \qquad \text{that is,} \qquad \forall i < \infty.\ a_\infty \prec\!\!\prec a_{i+1} \prec\!\!\prec a_i,$$

and we let $\vec{A}$ be the set of such sequences. We write $\vec{\ell}$ for a list or finite subset of $\vec{A}$ (not a sequence of unrelated lists) and ℓ_∞ for the list $\{b_\infty \mid \vec{b} \in \vec{\ell}\}$. Then we define

$$a \prec\!\!\prec \vec{\ell} \;\equiv\; a \prec\!\!\prec \ell_\infty \qquad\qquad \vec{b} \prec\!\!\prec k \;\equiv\; \exists i < \infty.\ b_i \prec\!\!\prec k$$

and $$\vec{a} \prec\!\!\prec \vec{\ell} \;\equiv\; \exists i < \infty.\, a_i \prec\!\!\prec \ell_\infty \;\equiv\; \exists i < \infty.\, a_i \prec\!\!\prec \vec{\ell} \;\equiv\; \vec{a} \prec\!\!\prec \ell_\infty.$$

As in Lemma 3.7, $\vec{a} \sqsubseteq \vec{b}$ is defined by $(\vec{a} \prec\!\!\prec \vec{b}) \vee (\vec{a} = \vec{b})$.

Lemma 12.2 Using Dependent Choice, Lawson sequences may be interpolated between individual basis elements and between lists of them:

$$a \prec\!\!\prec k \implies \exists \vec{b}.\, a \prec\!\!\prec \vec{b} \prec\!\!\prec k \qquad \text{and} \qquad k \prec\!\!\prec^1 \ell \implies \exists \vec{h}.\, k \prec\!\!\prec^1 \vec{h} \prec\!\!\prec^1 \ell.$$

Proof By repeated single interpolation, as in Lemma 3.11, given $a \prec\!\!\prec k$ there are

$$a \prec\!\!\prec b_\infty \prec\!\!\prec \cdots \prec\!\!\prec b_2 \prec\!\!\prec b_1 \prec\!\!\prec b_0 \prec\!\!\prec k,$$

so $a \prec\!\!\prec b_\infty$ and $\exists i < \infty.\, b_i \prec\!\!\prec k$. Conversely, if these hold then $a \prec\!\!\prec k$. Since $(k \prec\!\!\prec^1 \ell) \equiv \forall b \in k.\, \exists c \in \ell.\, b \prec\!\!\prec c$, the second part iterates the first over the list k. ☐

Lemma 12.3 Transitivity, single interpolation and boundedness above hold:

$$\vec{a} \prec\!\!\prec \vec{k} \prec\!\!\prec \vec{\ell} \implies \vec{a} \prec\!\!\prec \vec{\ell} \implies \exists \vec{b}.\, \vec{a} \prec\!\!\prec \vec{b} \prec\!\!\prec \vec{\ell} \qquad \text{and} \qquad \exists \vec{b}.\, \vec{a} \prec\!\!\prec \vec{b}.$$

Proof By interpolation, using the previous result,

$$\begin{array}{rcl}\vec{a} \prec\!\!\!\prec \vec{k} \prec\!\!\!\prec \vec{\ell} & \equiv & \exists i.\, a_i \prec\!\!\!\prec k_\infty \;\wedge\; \forall \vec{b} \in \vec{\ell}.\, \exists j.\; b_\infty \prec\!\!\!\prec b_j \prec\!\!\!\prec \ell_\infty \\ & \Rightarrow & \exists i.\, a_i \prec\!\!\!\prec k_\infty \prec\!\!\!\prec \ell_\infty \implies \vec{a} \prec\!\!\!\prec \vec{\ell} \\ \vec{a} \prec\!\!\!\prec \vec{\ell} & \equiv & \exists i.\; a_{i+1} \prec\!\!\!\prec a_i \prec\!\!\!\prec \ell_\infty \\ & \Rightarrow & \exists \vec{b} i.\; a_{i+1} \prec\!\!\!\prec \vec{b} \prec\!\!\!\prec a_i \prec\!\!\!\prec \ell_\infty \implies \exists \vec{b}.\; \vec{a} \prec\!\!\!\prec \vec{b} \prec\!\!\!\prec \vec{\ell}\end{array}$$

and for any $\vec{a}$, interpolate $a_2 \prec\!\!\!\prec a_1 \prec\!\!\!\prec \vec{b} \prec\!\!\!\prec a_0$, so that $\vec{a} \prec\!\!\!\prec \vec{b}$. □

Lemma 12.4 The Wilker and strong intersection rules hold in the form

$$(\vec{a} \prec\!\!\!\prec \vec{k}) \;\wedge\; (\vec{a} \prec\!\!\!\prec \vec{\ell}) \implies \exists \vec{h}.\, (\vec{a} \prec\!\!\!\prec \vec{h} \prec\!\!\!\prec^1 \vec{k}) \;\wedge\; (\vec{h} \prec\!\!\!\prec^1 \vec{\ell}).$$

Proof We use the greater of i and j, the weak intersection and Wilker rules in A (*cf.* Lemma 3.7) and the list form of Lawson interpolation:

$$\begin{array}{rcl}(\vec{a} \prec\!\!\!\prec \vec{k}) \;\wedge\; (\vec{a} \prec\!\!\!\prec \vec{\ell}) & \equiv & (\exists i < \infty.\, a_i \prec\!\!\!\prec k_\infty) \;\wedge\; (\exists j < \infty.\, a_j \prec\!\!\!\prec \ell_\infty) \\ & \Rightarrow & \exists i < \infty.\; a_{i+1} \prec\!\!\!\prec a_i \prec\!\!\!\prec k_\infty,\, \ell_\infty \\ & \Rightarrow & \exists i < \infty.\, \exists h'h''.\; a_{i+1} \prec\!\!\!\prec h'' \prec\!\!\!\prec^1 h' \sqsubseteq k_\infty,\, \ell_\infty \\ & \Rightarrow & \exists h'h''\vec{h}.\; \vec{a} \prec\!\!\!\prec h'' \prec\!\!\!\prec^1 \vec{h} \prec\!\!\!\prec^1 h' \sqsubseteq k_\infty,\, \ell_\infty \\ & \Rightarrow & \exists \vec{h}.\; \vec{a} \prec\!\!\!\prec \vec{h} \prec\!\!\!\prec^1 \vec{k} \;\wedge\; \vec{h} \prec\!\!\!\prec^1 \vec{\ell}.\end{array}$$

□

Lemma 12.5 If the given basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ is bounded below, has rounded unions, is positive or prime then $\vec{A}$ has the same property.

Proof Given $\vec{a}$, we have $c \prec\!\!\!\prec a_\infty$ since A is bounded below and then $c \prec\!\!\!\prec \vec{b} \prec\!\!\!\prec a_\infty$, so $\vec{b} \prec\!\!\!\prec \vec{a}$.

Similarly, if $\vec{b}, \vec{c} \prec\!\!\!\prec \vec{a}$ then $b_i \prec\!\!\!\prec a_\infty$ and $c_j \prec\!\!\!\prec a_\infty$, so there are d and $\vec{e}$ with $b_i \prec\!\!\!\prec d \prec\!\!\!\prec \vec{e} \prec\!\!\!\prec a_\infty$ and $c_j \prec\!\!\!\prec b$. Hence $\vec{b}, \vec{c} \prec\!\!\!\prec \vec{e} \prec\!\!\!\prec \vec{a}$.

For positivity, $\vec{a} \prec\!\!\!\prec \vec{\circ} \iff \exists i.\, a_i \prec\!\!\!\prec \circ$.

For primality, $\vec{a} \prec\!\!\!\prec \vec{\ell} \equiv \exists i.\, a_i \prec\!\!\!\prec \ell_\infty \implies \exists ib.\, a_i \prec\!\!\!\prec b \in \ell_\infty \implies \exists \vec{b}.\, \vec{a} \prec\!\!\!\prec \vec{b} \in \vec{\ell}$, where $\vec{b} \in \vec{\ell}$ is the member for which $b_\infty = b \in \ell_\infty$. □

Theorem 12.6 Any abstract basis that satisfies the single interpolation rule is isomorphic (in the sense of Remark 4.22) to one that also satisfies boundedness above and the strong intersection rule, by the matrices

$$\langle\, a \,|\, f \,|\, \vec{b} \,\rangle \;\equiv\; a \prec\!\!\!\prec \vec{b} \qquad \text{and} \qquad \langle\, \vec{b} \,|\, g \,|\, a \,\rangle \;\equiv\; \vec{b} \prec\!\!\!\prec a.$$

Proof We have verified all of the hypotheses of Lemma 3.7, so it provides $\sqsubseteq$ for the new basis. We may show that these matrices have the required properties and are inverse by similar methods. In particular, they are both uniformly bounded, $\langle\, a \,|\, f \,|\, \vec{b} \,\rangle$ is uniformly weakly filtered and $\langle\, \vec{b} \,|\, g \,|\, a \,\rangle$ is strongly but non-uniformly filtered. □

Notation 12.7 Now let $(U_a, \mathcal{K}_a)$ be a concrete basis for a space X using Scott-open families that is indexed by $(A, \sqsubseteq, \prec\!\!\!\prec)$. We define an $\vec{A}$-indexed basis for the same space by

$$U_{\vec{a}} \;\equiv\; U_{a_\infty} \qquad \text{and} \qquad \mathcal{K}_{\vec{a}} \;\equiv\; \bigcup\!\!\!\uparrow \{\mathcal{K}_{a_i} \mid i < \infty\}$$

or $\mathcal{K}_{\vec{a}}\phi \;\equiv\; \exists i.\, \mathcal{K}_{a_i}\phi$ in ASD notation.

Lemma 12.8 These have the variance properties and agree with the way-below relation.

Proof By Lemma 3.7, if $\vec{a} \prec\!\!\!\prec \vec{b} \equiv \exists i.\, a_\infty \prec\!\!\!\prec a_i \prec\!\!\!\prec b_\infty$ then $U_{\vec{a}} = U_{a_\infty} \subset U_{b_\infty} = U_{\vec{b}}$ and

$$\vec{a} \prec\!\!\!\prec \vec{b} \implies \exists i.\, \forall j.\, a_i \prec\!\!\!\prec b_\infty \prec\!\!\!\prec b_j \implies \forall j < \infty.\, \exists i < \infty.\, \mathcal{K}_{a_i} \supset \mathcal{K}_{b_j}$$

so $$\mathcal{K}_{\vec{a}} \equiv \bigcup \{\mathcal{K}_{a_i} \mid i < \infty\} \;\supset\; \bigcup \{\mathcal{K}_{b_j} \mid j < \infty\} \equiv \mathcal{K}_{\vec{b}}\,.$$

Also $$\mathcal{K}_{\vec{a}} \ni U_{\vec{\ell}} \iff \exists i < \infty.\, \mathcal{K}_{a_i} \ni U_{\ell_\infty} \iff \exists i < \infty.\, a_i \prec\!\!\!\prec \ell_\infty \;\equiv\; \vec{a} \prec\!\!\!\prec \vec{\ell}.$$ □

Lemma 12.9 The filter property for concrete bases is satisfied.

Proof If $x \in U_{\vec{a}} \equiv U_{a_\infty}$ and $x \in U_{\vec{b}} \equiv U_{b_\infty}$ then there is some c with $x \in U_c$ and $a_\infty \sqsupseteq c \sqsubseteq b_\infty$. By the basis expansion of U_c and Lemma 12.2 there are $e \prec\!\!\prec \vec{d} \prec\!\!\prec c$ with $x \in U_e \subset U_{d_\infty} = U_{\vec{d}} \subset U_c$, so $\vec{a} \sqsupseteq \vec{d} \sqsubseteq \vec{b}$. □

Lemma 12.10 The basis expansion is satisfied.

Proof We use the basis expansion in A twice and then interpolate $b \prec\!\!\prec \vec{a} \prec\!\!\prec c$:

$$\begin{aligned}
x \in U \quad &\Rightarrow \quad \exists b.\, x \in U_b \;\wedge\; \mathcal{K}_b \ni U \\
&\Rightarrow \quad \exists bc.\, x \in U_b \;\wedge\; \mathcal{K}_b \ni U_c \;\wedge\; \mathcal{K}_c \ni U \\
&\Rightarrow \quad \exists \vec{a}.\, x \in U_{a_\infty} \;\wedge\; \mathcal{K}_{a_\infty} \supset \mathcal{K}_{\vec{a}} \equiv \bigcup_{i<\infty} \mathcal{K}_{a_i} \supset \mathcal{K}_{a_0} \ni U \\
&\Rightarrow \quad \exists \vec{a}.\, x \in U_{\vec{a}} \;\wedge\; \mathcal{K}_{\vec{a}} \ni U \\
&\Rightarrow \quad \exists \vec{a}.\, x \in U_{a_\infty} \;\wedge\; \mathcal{K}_{a_\infty} \ni U \\
&\Rightarrow \quad \exists b.\, x \in U_b \;\wedge\; \mathcal{K}_b \ni U \implies x \in U.
\end{aligned}$$ □

Theorem 12.11 Every sober topological space that has a basis using Scott-open families (Definition 2.6) also has one using compact subspaces.

Proof We have constructed an abstract basis that satisfies the strong intersection rule and a concrete one whose Scott-open families are filters by Lemma 3.12. Hence these are the neighbourhood filters of compact subspaces by Proposition 3.15. □

Remark 12.12 *Is there a counterexample in a locale in a topos without Dependent Choice?*

Remark 12.13 *Presumably this is also valid in Martin-Löf Type Theory.*

Remark 12.14 The other parts of the theory of abstract bases *per se* can be carried out in an ***arithmetic universe*** (Axiom 10.1). This has no notion of sequence, so how can we accommodate Lawson's lemma into this view?

In fact, we do not need sequences in *general*, just the ability to *interpolate* something that can *generate* a sequence given its endpoints $\langle a_\infty \prec\!\!\prec a_0\rangle$, as in Lemma 12.2.

In a foundational setting that can encode infinite objects (functions $\mathbb{N} \to A$), we can use the interpolation property and Dependent Choice *once* to pick a sequence and then "remember" it for further use. If, however, we cannot represent the whole sequence, we can achieve the same thing, so long as whenever we repeat the process of selecting its terms, we are guaranteed to obtain the same result as before. In other words, the choice needs to be made *deterministically*.

In the *free* arithmetic universe, the subobjects are recursively enumerable. Therefore, by imposing a fixed way of scheduling parallel computations, we have a deterministic way of selecting an element of any inhabited subobject. In traditional recursion theory this is based on Stephen Kleene's theorem [Kle43], but unfortunately the literature in arithmetic universes has not yet been developed adequately to provide an idiomatic analogue.

Thus, instead of working with a *actually* infinite sequence, we encode it by its endpoints $\langle a_\infty \prec\!\!\prec a_0\rangle$ and use an interpolation *operator* μ that takes $\langle a_\infty \prec\!\!\prec a_i\rangle$ to $\langle a_\infty \prec\!\!\prec a_{i+1}\rangle$, so that the *potentially* infinite sequence consists of as many iterates as we actually require. Then we define

$$a \prec\!\!\prec \langle b_\infty \prec\!\!\prec b_0\rangle \;\equiv\; a \prec\!\!\prec b_\infty \qquad \text{and} \qquad \langle b_\infty \prec\!\!\prec b_0\rangle \prec\!\!\prec a \;\equiv\; \exists i.\, \mu^i\langle b_\infty \prec\!\!\prec b_0\rangle \prec\!\!\prec a,$$

the latter being understood as $c \prec\!\!\prec a$ where $\mu^i\langle b_\infty \prec\!\!\prec b_0\rangle = \langle b_\infty \prec\!\!\prec c\rangle$. After Lemma 12.2 above, the sequence $\vec{a}$ can be replaced throughout by $\langle a_\infty \prec\!\!\prec a_0\rangle$ because we need no further analysis of it.

Therefore Theorem 12.6 is valid in any arithmetic universe that has such a deterministic choice operation on inhabited subobjects, in particular in the free one. □

The reason for using the formulation of abstract bases without $\sqsubseteq$ (Lemma 3.7) is that we cannot expect μ to respect another $\sqsubseteq$. Ideally, given $a_\infty \prec\!\!\prec a_i$ and $b_\infty \prec\!\!\prec b_j$ with $a_\infty \sqsubseteq b_\infty$ and $a_i \prec\!\!\prec b_j$, we would like to chose interpolants such that $a_{i+1} \prec\!\!\prec b_{j+1}$ too, but this does not seem to be possible.

This completes the proof, which we began in Section 3, that abstract bases may be taken to satisfy all of the additional "convenient" properties in Definitions 1.10 and 1.11 as well as the primary ones in Definition 1.8.

13 Overt spaces

The notion of overtness has arisen independently under various names in several constructive disciplines: located subspaces in Constructive Analysis, open locales, positivity in Formal Topology and liveness in Process Algebra. It is often said to be invisible classically, but deeper investigation makes new use of some old ideas: The arguments that we employed in Section 5 when we tried to construct a traditional topological space directly from an abstract basis will turn up again here, whilst the properties of overt subspaces of metric spaces look very like the Newton–Raphson method for solving equations.

It is easiest to give the initial definition of this concept using Abstract Stone Duality, but we then characterise it using abstract bases and formal covers and work with these. Finally we prove a Theorem that links Topology to Computability.

Topologically, overtness is the lattice dual of compactness, the latter being related to the universal quantifier. For example, whereas a compact subspace of a Hausdorff space is closed, so an overt subspace of a discrete space is open. Similarly, an open subspace or direct image of an overt subspace is again overt. These ideas are explored in the context of real analysis in [J].

The word "overt" was introduced in [C], but in English it means "open", not in the simple sense, but that of being *explicit*. We shall see that this is appropriate because the concept is related to having computational *evidence*.

Definition 13.1 A locally compact space X is ***overt*** if it has a term $\exists_X : \Sigma^X \to \Sigma$ that obeys the rules for ***existential quantification***:

$$\frac{\frac{\ldots,\ x : X \quad \vdash \quad \phi x \Longrightarrow \sigma}{}}{\ldots \quad \vdash \quad \exists x.\, \phi x \Longrightarrow \sigma}$$

By our Axiom 10.8(b), any set (Axiom 10.1) is an overt space.

In classical topology, where the Sierpiński space Σ just has the two points $\top$ and $\bot$, we just have $\exists_X U \ \equiv\ (U \neq \emptyset)$ in any space. However, it is actually the points rather than Excluded Middle that make overtness trivial (Remark 13.5).

Overt locales were first studied by André Joyal, Miles Tierney [JT84] and Peter Johnstone [Joh84], who called them *open* because for such locales $!_X : X \to \mathbf{1}$ is an open map, *i.e.* there is a left adjoint $\exists_X \dashv !_X^*$ satisfying the Frobenius law below. The name needed to be changed because overt *sub*spaces are often *closed* [J].

A formal cover is overt if it has an additional structure called a positivity (Theorem 13.10).

Lemma 13.2 A space X is overt iff there is a term $\lozenge : \Sigma^X \to \Sigma$ that satisfies

$$\lozenge \bot \iff \bot \qquad \text{and} \qquad x : X,\ \phi : \Sigma^X \ \vdash\ \phi x \Longrightarrow \lozenge\, \phi.$$

Then $\lozenge \equiv \exists_X$ and this also preserves joins and satisfies the ***Frobenius law***

$$\sigma : \Sigma,\ \phi : \Sigma^X \quad \vdash \quad \lozenge\,(\sigma \land \phi) \iff \sigma \ \land\ \lozenge\,\phi,$$

cf. Definition 10.12(f).

Proof These are consequences of the adjunction $\exists_X \dashv !_X^*$ and respectively the negative and positive Gentzen rules (Axiom 10.7). See [J] for further discussion. □

Definition 13.3 More generally,

(a) we define an ***overt subspace*** of a (not necessarily overt) space X to be *any* operator $\lozenge : \Sigma^X \to \Sigma$ that preserves joins;

(b) it is ***inhabited*** if $\lozenge \top \Leftrightarrow \top$; and

(c) a point $x : X$ is an ***accumulation point*** of $\lozenge$ if $(\lambda\phi.\, \phi x) \leqslant \lozenge$, so $\phi x \Longrightarrow \lozenge\,\phi$ for all $\phi : \Sigma^X$.

The last is the dual of the condition $\mathcal{K} \subset \mathcal{P}$ for a (formal) point to lie in a (saturated) formal compact subspace (Definitions 3.16, 6.2(g) and 10.12(h)).

Therefore $\lozenge \equiv \exists_X$ makes any overt space X into an overt subspace of itself for which every $x : X$ is an accumulation point.

For subspaces, we may regard the accumulation points as providing the extent of $\Diamond$. However, we can only understand overtness if we regard these points as just by-products of the operator (or of its equivalent positivity). Sometimes there is an open or closed subspace that has the same points, as explained in [J], but in general the extent of an overt subspace need not be locally compact.

Beware that being inhabited does not mean *ā priori* that the subspace has an accumulation point: we have a Theorem to prove about this.

Example 13.4 For any sequence $f : \mathbb{N} \to X$, the operator $\Diamond U \equiv \exists n.\, fn \in U$ defines an overt subspace. In this, the limit of any convergent subsequence is an accumulation point (hence the name). □

Remark 13.5 In fact, we may replace $\mathbb{N}$ here with anything that we consider to be a "set" in whichever logical foundations we are using. Hence any space that has enough points (*cf.* Warning 6.17) is overt. However, this also means that overtness depends on the strength of our chosen foundations.

Indeed, for any $\Diamond$ operator in Point–Set Topology with Excluded Middle,

$$\begin{array}{rrcl} & V & \equiv & \{x \mid \exists U.\, x \in U \wedge \neg \Diamond U\} \;=\; \bigcup \{U \mid \neg \Diamond U\} \\ \text{and} & C & \equiv & \{x \mid \forall U.\, x \in U \Rightarrow \Diamond U\} \end{array}$$

are complementary open and closed subspaces such that $\Diamond U \iff U \between C$. □

Leaving the uninteresting classical case behind, the preservation of joins invites characterisation in terms of the basis:

Proposition 13.6 Overt subspaces correspond bijectively to ***positivities***. These are subsets $r \subset A$ of the basis that are rounded and located, or equivalently upper and positive (*cf.* Lemma 5.8 and Definition 6.14),

$$r \ni b \Longrightarrow \exists a.\, r \ni a \prec\!\!\prec b \qquad \text{and} \qquad r \ni a \prec\!\!\prec \ell \Longrightarrow r \between \ell \;\equiv\; \exists b.\, r \ni b \in \ell$$

$$\text{or} \qquad r \ni a \sqsubseteq b \Longrightarrow r \ni b \qquad \text{and} \qquad r \ni a \lhd u \Longrightarrow r \between u \;\equiv\; \exists b.\, r \ni b \in u,$$

$$\text{where} \qquad r \;\equiv\; \{a \mid \Diamond U_a\} \qquad \text{and} \qquad \Diamond U \iff \exists a.\ (a \in r) \;\wedge\; \mathcal{K}_a \ni U.$$

Then a formal point $p \subset A$ is an accumulation point of $\Diamond$ iff $p \subset r \subset A$.

Proof Proposition 4.14 and (the proof of) Lemma 7.11 characterised the subset r. We recover $\Diamond$ from r by the basis expansion and r from $\Diamond$ by roundedness. The containment $p \subset r$ is the restriction of the definition of an accumulation point to the basis and this is recovered for the same reason. □

Now we turn to the characterisation of overt spaces using $\lhd$ and $\prec\!\!\prec$.

Lemma 13.7 If a space has a positive basis (with no $a \prec\!\!\prec \circ$) then it is overt.

Proof Let $\Diamond U \equiv \exists a.\, (\mathcal{K}_a \ni U)$, so by hypothesis

$$\Diamond \bot \;\equiv\; \Diamond U_\circ \;\equiv\; \exists a.\, (\mathcal{K}_a \ni U_\circ) \;\equiv\; \exists a.\, (a \prec\!\!\prec \circ) \;\iff\; \bot.$$

Then $\Diamond$ is $\exists_X$ by Lemma 13.2 because, by the basis expansion,

$$x \in U \;\iff\; \exists a.\, (x \in U_a) \;\wedge\; (\mathcal{K}_a \ni U) \;\Longrightarrow\; \exists a.\, (\mathcal{K}_a \ni U) \;\equiv\; \Diamond U.$$

By the same argument as in Lemma 3.3, the positivity is $r \equiv \underline{A} \equiv \{b \mid \exists a.\, a \prec\!\!\prec b\}$. □

However, we cannot obtain a positive basis for an overt space "negatively" by just omitting the a with $a \prec\!\!\prec \circ$, *cf.* Lemma 3.6.

Notation 13.8 For any overt space X with concrete basis using Scott-open families $(U_a, \mathcal{K}_a)$ indexed by $(A, \sqsubseteq, \prec\!\!\prec)$, let

$$A^+ \;\equiv\; \{a \mid \exists x.\, x \in U_a\} \subset A.$$

This is the positivity that corresponds to $\lozenge \equiv \exists_X$ by Proposition 13.6. Since it is located, we never have $A^+ \ni a \prec\!\!\!\prec \circ$.

The key result is (our version of) a lemma[2] from Peter Johnstone's investigation of Locale Theory *without* Excluded Middle [Joh84, Lemma 2.5]. He stated it as $a \ll (b+c) \Longrightarrow (a \ll b) \vee (c \in A^+)$, which in our notation is $a \prec\!\!\!\prec \ell \sqcup \{c\} \Longrightarrow (a \prec\!\!\!\prec \ell) \vee (c \in A^+)$.

Whilst it may appear that we are simply cutting ℓ down to its intersection with A^+, doing that need not yield a constructively finite subset [Kur20]. However, Scott openness of $\mathcal{K}_a$ does provide some suitable finite $k \subset \ell \cap A^+$.

Lemma 13.9 $\quad a \prec\!\!\!\prec \ell \iff \exists k.\, a \prec\!\!\!\prec k \sqsubseteq \ell \ \wedge\ k \subset A^+$.

Proof $\quad \mathcal{K}_a \ni U_\ell \iff \mathcal{K}_a \ni \bigcup\{U_b \mid b \in \ell \cap A^+\} \iff \exists k.\, \mathcal{K}_a \ni U_k \ \wedge\ k \subset \ell \cap A^+$. □

In particular, $a \prec\!\!\!\prec c \Longrightarrow (a \prec\!\!\!\prec \circ) \vee (c \in A^+)$. We use the form above to eliminate empty basic subspaces from the interpolants that are provided by the Wilker and intersection rules for the given basis:

Theorem 13.10 A space is overt iff it has a positive abstract basis.

Proof We already have the reverse direction. Forwards, we may restrict the basis expansion to A^+ because

$$x \in U \iff \exists a.\, x \in U_a \ \wedge\ \mathcal{K}_a \ni U \iff \exists a.\, x \in U_a \ \wedge\ a \in A^+ \ \wedge\ \mathcal{K}_a \ni U.$$

It still obeys the filter property because in the statement

$$x \in U_a \ \wedge\ x \in U_b \Longrightarrow \exists c.\, x \in U_c \ \wedge\ (a \sqsupseteq c \sqsubseteq b),$$

we have $a, b, c \in A^+$. Hence the concrete basis $(U_a, \mathcal{K}_a)$ may be cut down to A^+.

Now we prove the Wilker and weak intersection rules that make $(A^+, \sqsubseteq, \prec\!\!\!\prec)$ an abstract basis. First we apply them for the given basis A and then we use Johnstone's lemma to reduce the interpolant:

$$\begin{array}{llll}
a \prec\!\!\!\prec \ell & \Rightarrow & \exists k.\, a \prec\!\!\!\prec k \prec\!\!\!\prec^1 \ell & \text{Wilker} \\
& \Rightarrow & \exists kh.\, a \prec\!\!\!\prec h \sqsubseteq k \prec\!\!\!\prec^1 \ell \ \wedge\ (h \subset A^+) & \text{Lemma 13.9} \\
& \Rightarrow & \exists h \subset A^+.\, a \prec\!\!\!\prec h \prec\!\!\!\prec^1 \ell & \\
a \prec\!\!\!\prec k \prec\!\!\!\prec \ell_1 \ \wedge\ k \prec\!\!\!\prec \ell_2 & \Rightarrow & \exists \ell'.\, a \prec\!\!\!\prec \ell' \sqsubseteq \ell_1 \sqcap \ell_2 & \text{intersection} \\
& \Rightarrow & \exists \ell' h.\, a \prec\!\!\!\prec h \sqsubseteq \ell' \sqsubseteq \ell_1 \sqcap \ell_2 \ \wedge\ (h \subset A^+) & \text{Lemma 13.9} \\
& \Rightarrow & \exists h \subset A^+.\, a \prec\!\!\!\prec h \sqsubseteq \ell_1 \sqcap \ell_2. &
\end{array}$$

Finally, since A^+ is located, $A^+ \ni a \prec\!\!\!\prec \ell \Longrightarrow \exists b.\, b \in \ell \cap A^+$, so A^+ is a positive basis. □

In Formal Topology, the usual definition of overtness is this:

Theorem 13.11 A space is overt iff there is a positivity $r \subset A$ such that $b \triangleleft b^+ \equiv \{b\} \cap r$.

Proof Suppose that there is such a positivity and let $\lozenge$ be the corresponding operator by Proposition 13.6, so $\lozenge U \equiv \exists a \in r.\, \mathcal{K}_a \ni U$. Then

$$\lozenge \bot \ \equiv\ \exists a \in r.\, \mathcal{K}_a \ni U_\circ \ \equiv\ \exists a \in r.\, a \prec\!\!\!\prec \circ \iff \bot.$$

If $x \in U_a$ then $a \in r$ since $a \triangleleft \{a\} \cap r$, so

$$\begin{array}{rll}
x \in U & \Leftrightarrow & \exists a.\, x \in U_a \ \wedge\ \mathcal{K}_a \ni U \\
& \Rightarrow & \exists a.\, a \in r \ \wedge\ \mathcal{K}_a \ni U \ \equiv\ \lozenge U.
\end{array}$$

Hence $\lozenge$ is $\exists_X$ by Lemma 13.2.

[2] He discovered it on a ferry journey and named it after the operating company, but apparently did not receive any payment for this celebrity endorsement.

Conversely, if X is overt then A^+ (Notation 13.8) is a positivity. Also, from Section 7,

$$\begin{aligned} b \lhd b^+ &\equiv (\forall a.\, a \prec\!\!\!\prec b \Longrightarrow \exists \ell.\, a \prec\!\!\!\prec \ell \subset b^+) \\ &\Leftrightarrow \big(\forall a.\, a \prec\!\!\!\prec b \Longrightarrow (a \prec\!\!\!\prec \circ) \ \vee\ (b \in A^+)\big) \end{aligned}$$

since any Kuratowski-finite $\ell \subset b^+$ must be $\circ$ or $\{b\}$. This property is true by Johnstone's Lemma 13.9. □

Finally we characterise overt subspaces by re-cycling our classical Lemma 5.9:

Definition 13.12 A space X with an abstract basis $(A, \sqsubseteq, \prec\!\!\!\prec)$ is called ***recursively enumerable*** if there is some bijection $A \cong R \subset \mathbb{N}$ where R and the image of $(\prec\!\!\!\prec) \subset A \times \mathsf{Fin}\,(A)$ are recursively enumerable.

Every object that is definable in Abstract Stone Duality is recursively enumerable, although much further work is required to develop computability theory idiomatically in our setting.

We also call an overt subspace $\lozenge$ recursively enumerable if the corresponding positivity $r \equiv \{a \mid \lozenge\, U_a\} \subset A \cong R \subset \mathbb{N}$ is recursively enumerable. Again this happens whenever $\lozenge$ is definable in ASD or, we claim, naturally occurring. Although there is potentially some ambiguity in this usage, it will be resolved by the Theorem that we aim to prove.

Lemma 13.13 An abstract basis is recursively enumerable iff there is an enumeration $k_{(-)} : \mathbb{N} \to \mathsf{Fin}\,(A)$ and a decidable predicate $\mathsf{WB}(j, a, k)$ such that, for all $i \in \mathbb{N}$, $a \in A$ and $k \in \mathsf{Fin}\,(A)$,

$$a \prec\!\!\!\prec k \iff \exists j.\ i < j \ \wedge\ k = k_j \ \wedge\ \mathsf{WB}(j, a, k).$$

Proof Stephen Kleene's Theorem [Kle43, Section 4]. □

Remark 13.14 *Is there a similar result in Martin-Löf Type Theory, maybe where $\mathsf{WB}(j, a, k)$ says that j encodes a proof that $a \prec\!\!\!\prec k_j$?*

Remark 13.15 *What is the result in an elementary topos with $\mathbb{N}$, so for Locale Theory?*

Lemma 13.16 Let $\lozenge$ be a recursively enumerable overt subspace of a (not necessarily overt but) recursively enumerable space and suppose that $\lozenge\, U$ holds. Then $\lozenge$ has an accumulation point that also lies in U.

Proof It suffices to consider $U \equiv U_a$, so $a \in r$. The result is essentially Lemma 5.9: we must find a formal point p with $a \in p \subset r$, where r is rounded and located by Proposition 13.6. We use Kleene's Theorem to modify the enumeration assumption at the beginning of the proof and then the construction proceeds in the same way from $a_0 \equiv a$. That is, except that:

At the ith stage, if $\mathsf{WB}(i, a_i, k_i)$ is false (even though some later $\mathsf{WB}(j, a_i, k_i)$ and hence $a_i \prec\!\!\!\prec k_i$ may be true) then we just let $a_{i+1} \equiv a'$ for any $r \ni a' \prec\!\!\!\prec a_i$ by roundedness of r.

If $\mathsf{WB}(i, a_i, k_i)$ is true then $a' \prec\!\!\!\prec a_i \prec\!\!\!\prec k_i$ and as before $a' \prec\!\!\!\prec k' \prec\!\!\!\prec^1 a_i, k_i$ and there is some $a_{i+1} \in r \cap k'$ by locatedness of r.

Such choices can be made because the sets are recursively enumerable, as is the resulting $p \equiv \{b \mid \exists i.\, a_i \prec\!\!\!\prec b\}$. This is also a $\prec\!\!\!\prec$-filter as before.

For locatedness, if $a_i \prec\!\!\!\prec a' \prec\!\!\!\prec k$ then, by assumption on the enumeration of $\mathsf{Fin}\,(A)$, we have $k \equiv k_j$ and $\mathsf{WB}(j, a, k)$ for some j with $i < j$. This means that $a_j \prec\!\!\!\prec a_i \prec\!\!\!\prec a' \prec\!\!\!\prec k \equiv k_j$ and then $a_{j+1} \prec\!\!\!\prec b \in k_j$, so $b \in k \cap p$.

Hence we have $a \equiv a_0 \in p \subset r$ as required. □

Theorem 13.17 Every recursively enumerable overt subspace is the image of some (non-unique) sequence $f : r \to X$, where r is the corresponding positivity, as in Example 13.4.

Proof We regard the proof of the previous result as defining a *function* that takes the starting point $a \in r$ and (deterministically) yields a formal point p_a (this is justified in the same way as in Remark 12.14 and Lemma 13.13). Then

$$\big\langle\, a \,\big|\, f \,\big|\, b \,\big\rangle \ \equiv\ (b \in p_a)$$

defines a matrix for $a \in r \subset A$ and $b \in A$ because, by the Lemma,

(a) it is trivially contravariant, rounded and saturated in a because $r \subset A$ is a set with the singleton basis (Example 5.2);

(b) it has the partition property because p_a is located with respect to $\prec\!\!\prec$;

(c) it is rounded, bounded and strongly filtered in b because p_a is a $\prec\!\!\prec$-filter;

(d) $a \in r \Longrightarrow \langle\, a \,|\, f \,|\, a \,\rangle$ because $a \in p_a$; and

(e) $a \in r \wedge \langle\, a \,|\, f \,|\, b \,\rangle \Longrightarrow b \in r$ because $p_a \subset r$.

Then Theorem 4.21 defines a continuous function $f : r \to X$ and Example 13.4 gives an overt subspace $\blacklozenge$ where

$$\blacklozenge U_b \;\equiv\; \exists a \in r.\, fa \in U_b \;\equiv\; \exists a \in r.\, \langle\, a \,|\, f \,|\, b \,\rangle \iff b \in r,$$

so $\blacklozenge$ agrees with the given operator $\lozenge$ by Proposition 13.6. □

Remark 13.18 We claim that this result makes overtness the gateway between topology and computability. Any program that takes (necessarily discrete) input data and yields (approximations to) a point of a space X is of the form in Example 13.4. Conversely, by Lemma 13.16, every definable inhabited overt subspace has a computable point. Whilst the former may be trite and the latter spectacularly infeasible as they stand, they do at least establish a purely topological characterisation of what can be done computationally.

This becomes a little less far-fetched when we restrict attention to $\mathbb{R}^n$ and its usual basis with $U_{\langle x,r\rangle} \equiv B(x,r) \equiv \{y \mid |\, x-y \,| < r\}$. It turns out that $d(x) < r$ is a reasonable notation for $\lozenge\, B(x,r)$ because it says how far x is from the nearest accumulation point. This relates overtness to ***locatedness*** in Constructive Analysis [Spi10], but familiar numeral algorithms such as Newton–Raphson iteration are also very similar to this [work in progress].

Therefore we may think about problems such as solving equations *mathematically* by adding this concept to our usual topological repertoire. Then we may hand over the resulting λ-term to a *computational* proof-theorist, who may be able to discover the accumulation points in a more efficient way.

14 Conclusion

This section needs to be rewritten.

We have proved several *weak equivalences of categories.*

Definition 14.1 In the ***category of weak abstract bases and matrices***,

(a) an *object* is an abstract basis $(A, \sqsubseteq, \prec\!\!\prec)$ that satisfies the principal axioms of Definition 1.8 (co- and contravariance, Wilker and weak intersection) and the roundedness properties of Definition 1.10 (single interpolation, rounded union and boundedness above and below);

(b) a *morphism* $\langle\; |\, f \,|\; \rangle : (A, \sqsubseteq, \prec\!\!\prec) \to (B, \sqsubseteq, \prec\!\!\prec)$ is a matrix that satisfies Definition 1.15 (co- and contravariance, roundedness on both sides, partition, boundedness, weak filteredness and saturation);

(c) the *identity map* on $(A, \sqsubseteq, \prec\!\!\prec)$ is the way-below relation, $\langle\, a \,|\, \mathsf{id}_X \,|\, b \,\rangle \equiv (a \prec\!\!\prec_X b)$; and

(d) morphisms are *composed* using the saturated composition operation in Notation 4.6:

$$\langle\, a \,|\, f\,;g \,|\, c \,\rangle \;\equiv\; \exists k.\, (a \prec\!\!\prec k) \;\wedge\; \forall a' \in k.\, \exists b.\, \langle\, a' \,|\, f \,|\, b \,\rangle \;\wedge\; \langle\, b \,|\, g \,|\, c \,\rangle.$$

Definition 14.2 The ***category of strong abstract bases and matrices*** is the full subcategory of the previous one consisting of bases that also obey the strong or rounded intersection rule. By Lemma 3.9 or 4.19, the matrices are strongly filtered.

The concrete category of "locally compact spaces and continuous maps" is weakly equivalent to one or both of these abstract categories. This is the case for each of the four formulations of topology that we have considered, in the mathematical foundations that are appropriate to that subject. We begin with Formal Topology because it is the most similar to our abstract bases.

Theorem 14.3 The category of locally compact formal covers and continuous functions is weakly equivalent to the strong abstract category, in Martin-Löf type theory.

Proof Definition 6.6 and Proposition 7.18 discussed how locally compact formal covers are defined. Proposition 7.3 derived an abstract basis $\prec\!\!\prec$ from a locally compact cover $\lhd$ and Lemmas 7.5 to 7.7 did the converse.

Proposition 7.14 translated between matrices for $\prec\!\!\prec$ and $\lhd$, the latter being the definition of continuous functions between covers that the Formal Topologists use.

The results of Sections 3 and 12, regarded solely as operations on abstract bases, show how to add the extra properties to them; we may assume Dependent Choice in doing this because it is a feature of Martin-Löf Type Theory. □

Theorem 14.4 The category of locally compact locales and continuous functions is weakly equivalent to the category of weak abstract bases and matrices, in the logic of an elementary topos. If the topos satisfies the axiom of Dependent Choice then the category is also equivalent to the strong one.

Proof Definitions 6.1 and 6.2, and Proposition 7.15 explained what locally compact locales and continuous lattices are and Proposition **??** obtained an abstract basis from them.

The converse construction turns the formal cover in the previous result into a frame or locale using Lemma 6.9 and Theorem 6.10; Theorem 7.10 characterised this using $\prec\!\!\prec$. Then Lemma 7.2 provides the Scott-open family $(\mathcal{K}_a)$ such that $\mathcal{K}_a \ni U_\ell \iff a \prec\!\!\prec \ell$.

Continuous functions, which are defined as reverse frame homomorphisms, correspond to matrices by the arguments in Section 4, with $\bigcup$, $\cap$ and $K_a\subset$ replaced by $\bigvee$, $\wedge$ and $\mathcal{K}_a\ni$. Bases may be improved to obey the single interpolation and rounded union rules by a similar translation of Proposition 3.1. If Dependent Choice is available, Section 12 showed how use it to impose the strong intersection rule. □

Theorem 14.5 The category of locally compact sober topological spaces and continuous functions is weakly equivalent to the strong category of abstract bases and matrices, in a set theory with Excluded Middle and the Axiom of Choice.

Proof Sections 1, 2, 3 and 12 showed how concrete bases using compact subspaces or Scott-open families yield abstract bases and can be improved to have all of the additional properties. Conversely, Section 5 defined a locally compact sober space from any *countable* abstract basis.

For the general case, we turn the locale in the previous result into a sober topological space. Lemma 7.11 showed that formal points for the abstract basis (Definition 5.1) agree with those for the locale and formal cover (Proposition 6.15). By Proposition 7.12 there are enough of them to make the extent (Proposition 6.16) an isomorphism between the abstract frame and the lattice of open sets of formal points (Definition 5.4). Then the Scott-open families in Lemmas 5.6 and 7.2 agree and satisfy the basis expansion. We also obtain $\mathcal{K}_a \subset U_\ell \iff a \prec\!\!\prec \ell$ from Lemma 7.2 instead of Lemma 5.10 and its preceding results. The space is sober by Lemma 5.11 without the countability restriction and in the strong case Theorem 5.12 describes the basic compact subspaces.

Section 4 showed how matrices correspond bijectively to continuous functions between sober spaces and deduced the saturated composition operation. □

Remark 14.6 Our development in Point–Set Topology in Section 5 was interrupted by the need to find enough formal points to characterise the way-below relation. We eventually proved this in Theorem 7.13, once we had the *benefit* of the concept, structure and properties of the $\lhd$ relation. In particular, we now see that we needed to apply Lemma 3.14 about maximal filters, not in the concrete frame of open sets of points (*cf.* Lemma 5.11), but in the abstract one that is defined directly from the abstract basis (Proposition 7.12). Only after doing so can we deduce that these two frames are isomorphic and hence prove the Theorem.

Remark 14.7 In Abstract Stone Duality, Lemma 10.15 showed that every concrete basis using λ-terms defines an abstract one. Conversely, the results of Section 11 constructed a nucleus $\mathcal{E}$ from any abstract basis.

Our introduction to ASD relied on the equivalence with the other formulations of topology, whereas the appropriate notion of "set" for ASD is an object of an arithmetic universe (Axiom 10.1). The construction of the strong abstract category really belongs in this much weaker logic. However, the axioms of both the topology and the foundations are then *so* weak that we have a whole paper's [work in progress] worth of work to do to construct the category, its products and its exponentials, but the outcome of this is that it is a model of ASD.

Remark 14.8 The main outcome of this lengthy investigation is that *the same structure*, at least as far as its *topological* description is concerned, is equivalent to the category of locally compact spaces in all four formulations, whereas each of those accounts has its own *ad hoc* features.

This is possible because, in the four kinds of abstract basis, *the words "set" and "relation" are understood in different ways*, since we are working in different logical foundations.

Consequently, the meaning of the notion of "continuous function" varies with logical strength. Indeed, we have a precise way of saying this: a continuous function in Point–Set Topology is a matrix (a certain kind of logical predicate on sets) that is *definable* in set theory with Excluded Middle and Choice, whereas a continuous function in Formal Topology is a matrix that is definable in Martin–Löf Type Theory, *etc.*

This is an observation that is already logically relevant for familiar spaces such as $\mathbb{N}$ and $\mathbb{R}$ that have homes in all four worlds. There are, for example, faster growing continuous real-valued functions in traditional topology than in the other subjects.

Remark 14.9 Cutting Section 4 down to just Proposition 4.14, we have a weak equivalence between the categories of

(a) locally compact spaces and operators that preserve all joins (but not necessarily meets) and

(b) bases and matrices that are co- and contravariant, rounded, saturated and have the partition property (but need not be bounded or filtered).

Again, there are results for each of the four kinds of topology. There are also further generalisations to (not necessarily distributive) continuous lattices and to bases and covers without the intersection rules.

Remark 14.10 In particular, by Proposition 13.6, overt subspace operators $\lozenge$ are in bijection with positivities (certain subset of the basis) in each of the four forms of topology. It is in this application that we see the most dramatic differences amongst the four logical settings, ranging from the classical one, where overtness is useless, to ASD, where in principle it provides an algorithm for solving a problem.

Remark 14.11 This range of different logics has a bearing on what constitutes "constructive" mathematics. Unfortunately, there is a tendency amongst those mathematicians who work in one camp to claim a monopoly on this word to the exclusion of the others. In this paper we have seen three approaches to topology that live in "constructive" worlds, by which we mean not the classical one.

If we are going to forbid Excluded Middle and the Axiom of Choice, why allow impredicativity?

But if you are going to adopt that position, how do you justify the infinite subsets that are needed in Formal Topology?

Our $\prec\!\!\!\prec$ has the advantage that its theory only uses *finite* subsets and ***coherent logic***: entailments between existentially quantified formulae. Further work will show that matrices or ASD terms that are definable in our weakest logic are *computable*. According to the Church–Turing thesis and much experience since then, there is only one notion of computability, whereas the question of which axioms and arguments count as "constructive" is open to debate.

After that, we can try to do *computation* with matrices for continuous functions between locally compact spaces.

Remark 14.12 In a different direction, we may see the axiomatisation of abstract bases as the notion of local compactness stripped of the cultural baggage of the different approaches to topology. We simply have relations between sets.

They're not just sets. We have used lists or finite sets, whilst $\mathsf{Fin}\,(A)$ is the free algebra (semilattice) for a functor on sets. The categorical mind will be able to ring many changes on this idea. In fact, this is the reason for keeping the preorder $\sqsubseteq$ even though Lemma 3.7 showed that it is redundant: it is a *clue* to possibly more general structure, such as a category.

Maybe the notion of locally compact space will be even more of a *discovery* than our already diverse opening diagram suggests.

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