Review of the Well Founded Coalgebras Programme

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The categorical roots of this programme were in the work of Gerhard Osius on *Categorical Set Theory* (1974) and Christian Mikkelsen on recursion in toposes (1976).

The set-theoretic roots were in John von Neumann's theorem (1923&8) that induction for predicates entails recursion for functions. These papers were also part of the introduction of the axiom-scheme of replacement, completing ZFC, the system that most pure mathematicians claim to use. However, it seems that in 100 years set theorists have failed to explain the recursion theorem, replacement or the essential role of **im**predicativity to the wider mathematical community.

I wanted an intuitionistic categorical account of transfinite recursion for my book *Practical Foundations of Mathematics* (1999). I ended up writing *Intuitionistic Sets and Ordinals* (1996), which introduced plump ordinals, but failed to prove the fixed point theorem intuitionistically. In 1997, Dito Pataraia did so in a *much simpler* way that I should have spotted myself. This shattered my morale.

My book included the definition of a well founded coalgebra and the proof that it enjoys recursion in the form of coalgebra-to-algebra homomorphisms. It required the functor to preserve inverse images, but I was challenged to weaken this assumption to just monos, which brought me back to the subject in 2019.

I knew that I would have to use Pataraia's result in the generalisation and the proof was not so difficult. What has kept me down rabbit holes was that Pataraia's result needed fundamental re-thinking and that the whole community is stubbornly ignorant about the mathematics and the history. I was bullied off MathOverflow for asking about these things.

I reluctantly began work on a history paper, *Old and new proofs of the order-theoretic fixed point theorem*, about the period between 1904 and 1923 when Set Theory was temporarily the vehicle for advances in the foundations of mathematics. For example, as someone who has done constructions in new systems, I came to identify with Friedrich Hartogs (1914) as the first user of Ernst Zermelo's "type theory". It is also clear that "Zorn's Lemma" (1935), "Tarski's Theorem" (1955) and the "Bourbaki–Witt Theorem (1950) were all clearly stated by Kazimierz Kuratowski (1922) in *Une Méthode d'élimination des Nombres Transfinis des Raisonnements Mathématiques.*

Back in the category theory, part of the re-thinking of Pataraia was its generalisation to categories. In place of inflationary monotone endofunctions we have well pointed endofunctors. These were publicised by Max Kelly in 1980, but instead of studying them as a (2-)categorist, he wrote a far from *unified treatment of transfinite constructions...* Instead, I found *one specific* directed colimit that replaces ordinals here as well as in the poset case.

My central paper, *Well founded coalgebras and recursion*, has been with referees since November 2021. It not only weakens the assumption to preserving monos but replaces them with general factorisation systems. Using these for the structure map of the coalgebra yields a very general notion of extensionality that has many of the strange features of set theory.

The plan was to apply these results to other categories and factorisations to give an account of the various intuitionistic ordinals. Indeed it does so for the plump ones (which are what categorists would have invented if their minds hadn't been clouded by set theory) but is very unsatisfactory for the more familiar notion ("thin" in my terminology).

I am currently using this idea to give *A categorical replacement for replacement*: that well founded coalgebras admit an extensional reflection for *any* factorisation system. I *interpreted* this in ZF in a recent ItaCa lecture and intend to show how to construct transfinite iteration of functors at CT26 (remotely). *Recovering* ZF involves other questions about its meaning that are less important for a categorist.

Regarding ordinals, it is now clear that they contribute *nothing* to induction, recursion or replacement. Given that their use anyway depends on **im**predicativity, this raises serious questions for those who propose a *predicative* version of them. In fact they have a different use to measure complexity of proofs, but maybe the "arithmetic" for this should be re-considered afresh.