Transfinite Iteration of Functors
(A Categorical Replacement for Replacement, Part II)

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Abstract

A categorical principle is proposed that extends the logic of an elementary topos to have similar strength to the axiom-scheme of Replacement. This is that any well founded coalgebra have an extensional reflection and it gets its force from using the "mono" class of *general* categories and factorisation systems in the definition of extensionality. The principle is invoked here to construct transfinite iteration of functors, of which the leading example is the von Neumann hierarchy in set theory.

This note provides the details of the construction of transfinite iterates of functors (in particular the Mirimanoff–von Neumann cumulative hierarchy) from the **Proposal** that all well founded coalgebras have extensional reflections. I lectured on this remotely at CT26 on 13 July 2026. It is an *incomplete draft*, so please get a new copy if you'd like to discuss it with me.

1 Introduction

After elementary toposes were introduced [Law70], there was a need to show that they could do (more or less) anything that set theory could do. So there were several accounts of how the "type theory" of mainstream mathematics (products, exponentials, powersets, predicate calculus and recursion) could be interpreted using the structure of a topos.

The one author who took the ($\in$) relation seriously in a categorical setting was Gerhard Osius [Osi74]. Any relation ($\prec$) on a set A may be expressed as a *coalgebra* for the covariant powerset functor

$$A \longrightarrow \mathcal{P}(A) \qquad \text{by} \qquad a \mapsto \{b \mid b \prec a\}$$

and it obeys the *extensionality* rule

$$\forall a, b{:}A. \quad \frac{\forall c.\, c \prec a \iff c \prec b}{a = b}$$

exactly when the function $A \to \mathcal{P}(A)$ is 1–1. Osius saw that *homomorphisms* of extensional coalgebras

$$\begin{tikzcd} \mathcal{P}A \arrow[r, "\triangleright"] & \mathcal{P}B \\ A \arrow[u, hook] \arrow[r, "\triangleright"] & B \arrow[u, hook] \end{tikzcd}$$

characterise *subset inclusion* when ($\prec$) is taken as membership.

To model set theory like this, we must also say that ($\in$) is *well founded*, which Osius did using a notion of *recursive* coalgebra. However, the intuitionistic *induction scheme*

$$\forall \phi. \quad \frac{\forall b. \quad \big(\forall c.\, c \prec b \Rightarrow \phi c\big) \Longrightarrow \phi b}{\forall a.\, \phi a}$$

may also be expressed in terms of coalgebras:

Definition 1.1 A coalgebra $\alpha : A \to TA$ for a functor T that preserves monos is ***well founded*** if, for all monos $i : U \rightarrowtail A$, in the diagram

$$\begin{tikzcd}
TU \arrow[rr, tail, "Ti"] & & TA \\
H \arrow[u] \arrow[ur, phantom, very near start, "\urcorner"] \arrow[r, dotted] & U \arrow[r, tail, "i"] & A \arrow[u, "\alpha"']
\end{tikzcd}$$

if the pullback $H \to A$ factors through $U \rightarrowtail A$ *then* in fact $i : U \cong A$.

We will call this diagram the ***broken pullback*** [Tay96, Tay99, Tay21].

Example 1.2 In the leading case $T \equiv \mathcal{P}$, the subobject U encodes the predicate ϕ as usual. Then the pullback is

$$\begin{aligned}
H &\equiv \{(b, V) \mid \alpha(b) = V \subset U \subset A\} \\
&\cong \big\{b \bigm| \{c \mid c \prec b\} \subset \{c \mid \phi c\}\big\} \\
&= \{b \mid \forall c.\, c \prec b \Rightarrow \phi c\} \subset A,
\end{aligned}$$

which is the *induction hypothesis*. Therefore the inclusion $H \rightarrowtail U$ is the upper line of the induction scheme and $i : U \cong A$ is the conclusion. □

Osius used these ideas to re-construct Zermelo set theory, but we are going to make a substantial generalisation of them, for a different purpose.

The idea from set theory that inspires this work is associated with Andrzej Mostowski [Mos49], but first appeared in Johann von Neumann's formulation of the ordinals [vN23]:

Any *well founded* relation $(A, \prec)$ can be made *extensional*.

As originally and usually stated, the newly created extensional relation has to be $(\in)$ (membership), for which the construction requires the axiom-scheme of Replacement.

However, if instead we allow the new relation to be expressed as a coalgebra structure, Replacement is no longer required: it can be constructed as the quotient by a co-recursively defined equivalence relation, within the logic of a topos [Tay96, Thm. 2.11].

Replacement is the one major feature of set theory that is not captured by the logic of a topos. Whilst it is not actually needed for the content of Mostowski's theorem and there are many other invocations of it that miss the point (such as saying that anything bijective with a set is also a set), there are in fact genuine mathematical applications.

The principal uses seem to involve transfinite iterates of functors. The leading example of this is the ***cumulative hierarchy*** called O_α by Dimitry Mirimanoff [Mir17, §9] and then Ernst Zermelo but since called after von Neumann. (Mirimanoff, who mainly studied number theory rather than set theory, also identified well-foundedness as the solution to the "antinomies" of set theory and gave the order-theoretic characterisation of the subset relation.)

It is tempting to interpret Replacement in terms of Universes or the image of functions from Small objects to Large ones. However, the cumulative hierarchy shows that it is interesting *in the absence of such assumptions* because it *constructs skyscrapers from plans on the ground.*

We are going to use *category theory* and not set theory to do this. The first generalisation that any categorist makes when taking an idea from another subject is to allow *any* category and *any* endofunctor of it in place of **Set** and $\mathcal{P}$. But when we move from **Set** to other categories, the concept of injective function (as used in the "predicates" testing well-foundedness and in the coalgebra formulation of "extensionality") comes up for negotiation: Other categories have many different notions of "subspace", and dually of the "quotient" that is associated with Mostowski.

The abstraction of "subspaces" and "quotients" that we will exploit has roots in Emmy Noether's Moderne Algebra and was formalised in category theory by Peter Freyd and Max Kelly [FK72]:

Definition 1.3 Two maps $e : X \twoheadrightarrow Q$ and $m : V \hookrightarrow Y$ in any category are ***orthogonal***, written $e \perp m$, if, for any two maps f and g such that the square commutes, there is a unique morphism $h : Q \to V$ that makes the two triangles commute:

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Q \arrow[d, "g"] \arrow[dl, dotted, "h"'] \\ V \arrow[r, hook, "m"] & Y \end{tikzcd}$$

Then a ***factorisation system*** is a pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$ such that

(a) the classes $\mathcal{E}$ and $\mathcal{M}$ each contain all isomorphisms;

(b) they are each closed under composition;

(c) $e \perp m$ for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$ and

(d) every morphism $f : X \to Y$ can be expressed as $f = e\,; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

We will introduce several factorisation systems in this paper and use the $\mathcal{M}$-class in each of them to define "predicates" and "extensionality". (In principle one could use two *different* classes of "monos" in these roles, but it turns out for all of the examples that we consider here that well-foundedness in our categorical sense is equivalent to the standard intuitionistic induction scheme in the proof rule above.)

Definition 1.4 A coalgebra $\alpha : A \to TA$ is ***extensional*** if its structure map α belongs to the class $\mathcal{M}$. Extensional well founded coalgebras are called ***ensembles*** because they have very similar properties to set theory. We write **CoAlg**, **WfCoAlg** and **Ens** for the categories of coalgebras, well founded ones and ensembles, with coalgebra homomorphisms.

This is enough to state our proposed *categorical replacement for Replacement*:

Proposal 1.5 Every well founded coalgebra has an extensional reflection. $\square$

Example 1.6 In the case where $\mathcal{C} \equiv \mathbf{Set}$, $T \equiv \mathcal{P}$, $\mathcal{M}$ consists of 1–1 functions and $\mathcal{E}$ of surjective ones, the extensional reflection is the one associated with Mostowski and can be constructed in topos logic, without the need for an additional axiom. $\square$

In the next section we will list the exact requirements on $\mathcal{C}$, T, $\mathcal{E}$ and $\mathcal{M}$ for this Proposal to take effect, along with the main results from [Tay21] on which we rely.

In fact, for the applications in this paper, the Proposal is a Theorem of intuitionistic Zermelo–Fraenkel set theory, *via* a more general Adjoint Functor Theorem that is discussed in [Tay26].

However, we do not consider the Proposal to be a *theorem* but a new *axiom* (or *definition*), just as the notions of elementary topos, cartesian closed category *etc.* are treated as definitions in categorical logic and not theorems of set theory. Category theory is a mature discipline that has found numerous applications in and beyond pure mathematics: it does not require set theory as a foundation.

Categorical logic works in parallel with type theory. There are precise translations between adjunctions and systems of introduction and elimination rules. We build more powerful logics by adding further such systems or equivalently asserting the existence of adjoints of previously defined functors. Replacing Replacement in the native language of category theory is similarly to be done by adding new adjunctions.

The only reason for *proving* the Proposal in IZF is that, after a century, no-one has so far shown that IZF is inconsistent. So our Proposal has as firm a basis as IZF does and Kurt Gödel's Incompleteness Theorem teaches us that we cannot ask for any more than that. However, conversely, someone may use the Proposal as a better way of understanding whatever holes there may be in IZF and thereby sink it.

Historically, Set Theorists (including Ernst Zermelo, who was skeptical of the re-formulations of his system) adopted Replacement because they felt that their system was aesthetically incomplete without it. The first mathematical application was von Neumann's representation of the ordinals [vN23]. Whilst we, as categorical logicians, do not share the tastes of set theorists, this Proposal is likewise put forward as something that *ought* to be true in our subject.

The category **Set** doesn't have any interesting factorisation systems, but **Pos** has at least two of them that are suitable. In Section 3 we show how these provide two constructive notions of ordinals. One of them is sufficiently similar to the classical one to show that our later constructions capture what we expect of transfinite recursion and iteration. The other is the first application of the Proposal, to ordinals themselves.

Section 4 develops a small modification to the category of posets into a construction of transfinite recursion, *i.e.* iteration of an endofunction $s : E \to E$ of a complete semilattice.

The remaining sections work towards transfinite iteration of an endofunctor $S : \mathbf{Set} \to \mathbf{Set}$ that could be the powerset. We give an introduction to the simple fibred category theory that is required for this:

Section 5 shows how to turn ordered indexations into discrete opfibrations and described the fibre and base factorisations in the fibred category **DFPos**.

Section 6 introduces the functor to be iterated and the fibred notions of well founded and extensional coalgebras for it.

Section 7 recalls how composition of opfibrations provide fibred coproducts and applies them to define fibred joins of ensembles.

Section 8 puts these things together to construct transfinite iterates of endofunctors.

Remark 1.7 A better metaphor than skyscrapers is stellar parallax:

Galileo Galilei was frustrated that he couldn't use parallax to prove the heliocentric theory. He could only measure about one minute of arc ($\frac{1}{60}$ degree, where the apparent diameter of the full Moon is about $\frac{1}{2}$ degree), whereas the parallax of the nearest stars is a fraction of a second ($\frac{1}{60}$ minute). Friedrich William Bessel and Friedrich Georg Wilhelm von Struve did measure this in 1838 and 1843, both using major advances in the manufacture of lenses by Joseph Ritter von Fraunhofer. Now the Gaia space telescope can measure parallax for stars 10,000 times further away.

Parallax and orthogonality are both *inverse* relationships. Our constructions are tightly engineered, like Fraunhofer's lenses: an $\mathcal{M}$-subobject has to be closed under *very rich* structure, so there are very *few* of them. There are far *more* $\mathcal{E}$-maps as a result: they need not be "surjective" at all but can be very sparse. As a result they *generate* huge mathematical objects, just as modern telescopes can measure huge stellar distances.

The central idea in this paper was first proposed in [Tay99, Rk 9.6.16].

2 Behind the Proposal

This section recalls the assumptions and key results from the theory of well founded and extensional coalgebras [Tay21] and also from the proof of the Proposal using the axiom of Replacement in [Tay26]. However, we will not discuss set theory here and in fact there is no need to read this section in order to understand the rest of the paper.

Proposition 2.10 is the only result from this section that will be used in the rest of this paper.

For the papers just mentioned, the category $\mathcal{C}$ and the factorisation system $(\mathcal{E}, \mathcal{M})$ must have the following properties. **Set** (or any elementary topos) has them and they are easily inherited by the other categories that we will consider.

Assumptions 2.1

(a) $\mathcal{C}$ is locally small;

(b) $\mathcal{C}$ has an initial object $\emptyset$;

(c) all maps $\emptyset \to X$ are in $\mathcal{M}$, so that $\emptyset$ is the least *sub*object of any object;

(d) for any $m \in \mathcal{M}$ and $f \in \mathcal{C}$ with the same codomain, the pullback f^*m must exist in $\mathcal{C}$;

(e) $\mathcal{C}$ must be well powered, *i.e.* each object has a *set* (object of the working topos) $\mathsf{Sub}\,(X)$ of

$\mathcal{M}$-subobjects $U \rightarrowtail X$;

(f) $\mathcal{M} \to \mathcal{C}$ must create directed colimits, which means that any directed diagram of $\mathcal{M}$-maps must have a colimiting cocone of $\mathcal{M}$-maps and any other cocone of $\mathcal{M}$-maps must have a mediator that is also in $\mathcal{M}$;

(g) there must be a functor $\mathcal{C} \to \mathbf{Set}$ that also creates these colimits [Tay26].

To understand (c), note that it *fails* in the category of rings because $\mathbb{Z} \to \mathbb{Z}/(n)$ is not mono.

We use the following easily proved properties of factorisation systems frequently:

Lemma 2.2 Any $m \in \mathcal{M}$ and $f \in \mathcal{C}$ satisfy $f^*m \in \mathcal{M}$ and $f \in \mathcal{M} \iff f\,;m \in \mathcal{M}$. $\square$

Two further properties will also be required, but will need more attention:

Assumptions 2.3

(a) Every $\mathcal{M}$-map m must satisfy the cancellation property of ***plain monos***, namely

$$f\,;m \;=\; g\,;m \Longrightarrow f = g.$$

(b) The functor T that is used to define coalgebras must preserve $\mathcal{M}$-maps:

$$m \in \mathcal{M} \Longrightarrow Tm \in \mathcal{M}.$$

The first is needed because the isomorphism $i : U \cong A$ in Definition 1.1 is often obtained by finding $p : A \to U$ with $p\,;i = \mathsf{id}_A$ but not necessarily proving directly that $i\,;p = \mathsf{id}_U$. Our leading example of the covariant powerset satisfies (b), but we will carry through our development for *any* endofunctor of **Set** that does so.

The assumptions have some easy consequences, although the proofs are given in [Tay21].

Lemma 2.4 If $\alpha : A \to TA$ is well founded or extensional then so is $T\alpha : TA \to TTA$. $\square$

Lemma 2.5 The forgetful functors $\mathbf{Ens} \to \mathbf{WfCoAlg} \to \mathbf{CoAlg} \to \mathcal{C}$ create colimits of diagrams that are directed posets. $\square$

Proposition 2.6 Transfinite iteration of a functor T starting at $\emptyset$ stays within the preorder of extensional well founded T-coalgebras.

Proof The initial case is $\emptyset \hookrightarrow T\emptyset$, the successor is that T preserves $\mathcal{M}$-maps and the infinitary case (however interpreted) involves colimits of $\mathcal{M}$-maps. All of these stay within **Ens**. $\square$

The inductive and recursive properties of well founded coalgebras do not come directly from their definition but from the following construction, whose roots are in [vN23].

Construction 2.7 Let $i : (B, \beta) \hookrightarrow (A, \alpha)$ be a subcoalgebra. Then its ***relative successor*** $k : (C, \gamma) \hookrightarrow (A, \alpha)$ is given by pullback of α and Ti in $\mathcal{C}$. Since C lies between B and TB, we call it a ***sandwich*** of B.

$$\begin{tikzcd}
& A \arrow[r, "\alpha"] & TA & \\
& C \arrow[u, hook, "k"] \arrow[r, "c"] \arrow[ur, phantom, "\urcorner", very near start] \arrow[drr, "\gamma"'] & TB \arrow[u, hook, "Ti"'] \arrow[dr, hook, "Tj"] & \\
B \arrow[uur, hook, "i"] \arrow[ur, "j"] \arrow[urr, "\beta"'] & & & TC \arrow[uul, hook, "Tk"']
\end{tikzcd}$$

The pullback mediator $j : B \to C$ makes $(B, \beta) \hookrightarrow (C, \gamma) \hookrightarrow (A, \alpha)$ as subcoalgebras (initial segments) when we define $\gamma \equiv c;Tj$. We will write sB for the relative successor (C); the operation s is inflationary because $j : B \hookrightarrow sB$. $\square$

Lemma 2.8 Any well founded coalgebra (A, α) is a ***well founded object***, in the sense that
(a) it has a set (object of a topos) $\mathsf{Sub}\,(A)$ of subcoalgebras $B \rightarrowtail A$;
(b) the relative successor subcoalgebra defines an inflationary monotone function

$$s : \mathsf{Sub}\,(A) \longrightarrow \mathsf{Sub}\,(A) \qquad \text{with} \qquad B \rightarrowtail sB \rightarrowtail A;$$

(c) A (considered as its largest subcoalgebra $A \hookrightarrow A$) is the unique fixed point:

$$B = sB \hookrightarrow A \iff B = A. \qquad \square$$

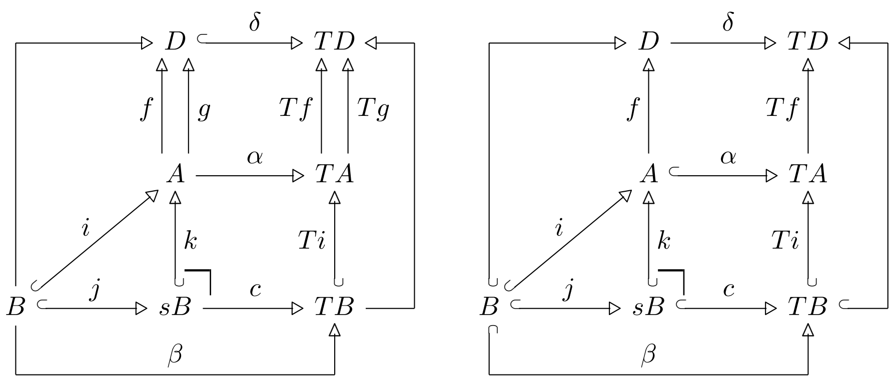
Proposition 2.9 Let D be a poset with $\bot$, $\top$ and directed joins, $s : D \to D$ an inflationary monotone endofunction whose only fixed point is $\top$ and ϕ a predicate on D such that $\phi(\bot)$, $\phi(d) \Rightarrow \Phi(sd)$ and ϕ is closed under directed unions. Then $\phi(\top)$. $\square$

This is used to prove the one non-trivial result from [Tay21, Lemma 7.3] that we will need in this paper.

Proposition 2.10 Let $\alpha : A \longrightarrow TA$ and $\delta : D \longrightarrow TD$ be coalgebras, where A is well founded and δ is plain mono. Then
(a) there is at most one homomorphism $A \longrightarrow D$; and
(b) if A and D are $\mathcal{M}$-extensional, *i.e.* $\alpha, \delta \in \mathcal{M}$, then any such homomorphism is also in $\mathcal{M}$.

Hence the category **Ens** of $\mathcal{M}$-extensional well founded coalgebras is a preorder and its morphisms are $\mathcal{M}$-maps.

Proof Assumptions 2.1(c,f) ensure that the properties hold for $B \equiv \emptyset$ and are preserved by directed unions. Here are the induction steps:



(a) Suppose that $i\,;f = i\,;g : B \longrightarrow D$. Then $Ti\,;Tf = Ti\,;Tg$ and

$$k\,;f\,;\delta \;=\; k\,;\alpha\,;Tf \;=\; c\,;Ti\,;Tf \;=\; c\,;Ti\,;Tg \;=\; k\,;\alpha\,;Tg \;=\; k\,;g\,;\delta,$$

so the diagram commutes from sB to TD. Hence $k\,;f = k\,;g : sB \longrightarrow D$ because δ is plain mono.
(b) Suppose that $B \hookrightarrow A \longrightarrow D$ is in $\mathcal{M}$. We use Assumption 2.3(b) and Lemma 2.2. Since the class $\mathcal{M}$ is closed under T and composition and the whole diagram commutes, $sB \hookrightarrow TD$ is also in $\mathcal{M}$. Finally $sB \hookrightarrow D$ is in $\mathcal{M}$ too by the cancellation property.

Hence these properties hold for the unique fixed point $B \equiv A$. $\square$

The category **Ens** also has binary meets, but we apparently don't need them here.

The proof of the Proposal (that every well founded coalgebra has an extensional reflection) from IZF uses the same notion of well founded object and induction proof. Here is the induction step:

Lemma 2.11 If $A \twoheadrightarrow E$ is an extensional reflection and B is a sandwich of A then B also has an extensional reflection F.

$$\begin{tikzcd}
 & TB \arrow[rr] & & TF \arrow[dr, hook] & \\
TA \arrow[ur] \arrow[rr] & & TE \arrow[ur, hook] \arrow[rr, hook] & & TG \\
 & B \arrow[ul] \arrow[uu] \arrow[rr, two heads] \arrow[drrr] & & F \arrow[ul] \arrow[uu, hook] \arrow[dr, hook, dotted] & \\
A \arrow[uu] \arrow[ur] \arrow[rr, two heads] & & E \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, hook] & & G \arrow[uu, hook]
\end{tikzcd}$$

Proof We use Assumption 2.3(b) and Lemma 2.2 again. Factorise $B \longrightarrow TA \longrightarrow TE$ as $B \twoheadrightarrow F \hookrightarrow TE$. Since E is extensional, so are TE and F, whilst $E \hookrightarrow F$ by Proposition 2.10(b). Now if also $B \longrightarrow G \hookrightarrow TG$ then orthogonality of $B \twoheadrightarrow F$ and $G \hookrightarrow TG$ gives $F \hookrightarrow G$. $\square$

There is another proof in [Tay21, §8A], *via* the characterisation of the extensional reflection as *any* $\mathcal{E}$-homomorphism to an $\mathcal{M}$-extensional coalgebra but this relies on a lot of theory.

The limit case is based on Assumption 2.1(g) and the axiom of Replacement, but that needs some explanation from Set Theory that we don't want to give here. It is discussed in [Tay26].

The point of view of this paper is that *the Proposal should stand on its own*, potentially justified in other foundational settings.

3 Ordinals as coalgebras in Pos

Which of the facts in this section are not obvious and so should be proved here?

In order to talk about "transfinite" iteration we need some ordinals. Of course we are not going to treat them as "doing something infinitely often, and then some more", but even the traditional rigorous accounts unfortunately use excluded middle literally at every step. On the other hand, there are *multiple* constructive notions, some of which have some of the old properties and others obey a different collection. There is a study of them in [Tay23], which contains the proofs of results in this section.

However, our purpose is to demonstrate the generalised use of well founded and extensional coalgebras, so we focus on the "ordinals" defined by two factorisation systems in the category **Pos** of posets and order-preserving functions. With these, we use the following analogue of the covariant powerset functor:

Lemma 3.1 The category **Pos** of partially ordered sets and monotone (order-preserving) functions satisfies Assumptions 2.1 with either of the factorisation systems below. $\square$

Definition 3.2 The ***lower sets functor*** $\mathcal{D} : \mathbf{Pos} \to \mathbf{Pos}$ is defined by

$$\begin{array}{lcl}
\mathcal{D}(A, \leq) & \equiv & \{U \mid U \subset_\downarrow A\} \equiv \{U \subset A \mid \forall a, b \in A.\, a \leq b \in U \Rightarrow a \in U\} \\
\mathcal{D}fU & \equiv & \{b \in B \mid \exists a \in U.\, b \leq_B f(a)\} \qquad \text{for } f : (A, \leq) \to (B, \leq) \text{ and } U \subset_\downarrow A, \\
U \subset_\downarrow A & \equiv & \forall a, u{:}A.\ a \leq u \in U \Longrightarrow a \in U.
\end{array}$$

Lemma 3.3 A $\mathcal{D}$-coalgebra is given by a poset $(A, \leq)$ and another binary relation, $(\prec)$, that satisfies

$$c \leq b \prec a \Longrightarrow c \prec a \qquad \text{and} \qquad c \prec b \leq a \Longrightarrow c \prec a.$$

Moreover, a $\mathcal{D}$-coalgebra is well founded in our categorical sense, using either $\mathcal{R}$ or $\mathcal{L}$ below for the predicates, iff $(\prec)$ is a well founded relation in the traditional intuitionistic sense (Example 1.2) [Tay23].. $\square$

Slim ($\mathcal{R}$) ordinals

Proposition 3.4 $({}^\perp\mathcal{R}, \mathcal{R})$ is a factorisation system for **Pos**, where
(a) $\mathcal{R}$ consists of the inclusions with the restricted order, which are the regular monos; and
(b) ${}^\perp\mathcal{R}$ consists of the monotone functions that are surjective on points, which are the plain epis.
Moreover, $\mathcal{R}$ and $\mathcal{D}$ satisfy Assumptions 2.3 [Tay23]. $\square$

Lemma 3.5 A $\mathcal{D}$-coalgebra is $\mathcal{R}$-extensional iff $(\prec)$ is extensional in the traditional sense and ***meta-transitive***, *i.e.*

$$\forall a, b, c. \quad (\forall d.\, d \prec c \Rightarrow d \prec b) \;\wedge\; (b \prec a) \Longrightarrow (c \prec a). \qquad \square$$

The most commonly cited constructive definition of a well ordered relation $(\prec)$ is one that is well founded, extensional and transitive in the *ordinary* sense. We call those ***thin ordinals***, but they do not fit easily into our theory. ***Slim*** or $\mathcal{R}$***-ordinals*** obey this stronger property, which serves as a useful compromise with the classical situation:

Lemma 3.6 If $(\prec)$ is a well founded trichotomous relation on a set A, *i.e.*

$$\forall a, b. \quad a \prec b \;\vee\; a = b \;\vee\; b \prec a,$$

then
$$(\forall c.\, c \prec b \Rightarrow c \prec a) \iff (b \prec a \vee b = a) \;\equiv\; (b \preceq a)$$

and $(\prec)$ is meta-transitive, so $(A, \preceq, \prec)$ is an $\mathcal{R}$-extensional $\mathcal{D}$-coalgebra. $\square$

Lemma 3.7 The ***one-point successor*** of a coalgebra $(A, \leq, \prec)$ is $A^+ \equiv A + \{\infty\}$ with
(a) $(\leq)$, α and $(\prec)$ on the A-part: the same as for A;
(b) $\alpha(\infty) = A$, so $a \prec \infty$ and $a \leq \infty$ for all $a \in A$;
(c) $\infty \leq \infty$ but $\infty \not\prec \infty$; and
(d) $\infty \not\leq a$ and $\infty \not\prec a$ for $a \in A$.
If A is an $\mathcal{R}$-ordinal then so is A^+ and there is coalgebra inclusion $A \hookrightarrow A + \{\infty\}$. However, the operation $(-)^+$ is not functorial with respect to such inclusions. $\square$

Example 3.8 We shall test our theory in particular for $\omega \equiv (\mathbb{N}, \leq, <)$ and its one-point successor ω^+, which are trichotomous, $\mathcal{R}$-extensional and well founded. $\square$

Example 3.9 More generally, the order on a Cantor normal forms below ϵ_0 can be expressed lexicographically and so is trichotomous. Hence they define $\mathcal{R}$-extensional well founded $\mathcal{D}$-coalgebras.

Warning 3.10 There are *many* more $\mathcal{R}$-extensional ordinals than this and in general they are *far* from being trichotomous. Their binary joins also have some rather pathological properties [Tay23].

The significance for us is that "transfinite recursion" for these ordinals does not split into the familiar zero, successor and limit cases. Indeed, such recursion does not lead to a clear definition of *colimit* diagrams in *categories.*

Instead we rely on the fact that iteration of the functor stays within **Ens**, which is preorder, by Propositions 2.6 and 2.10. Then all colimits are simply *joins.*

Proposition 3.11 Any well founded relation has an $\mathcal{R}$-rank that is definable in topos logic: it does not require the Proposal, because ${}^\perp\mathcal{R}$ is co-well-powered. Classically, this coincides with the traditional notion of rank, which was first defined by Mirimanoff. $\square$

Plump ($\mathcal{L}$) ordinals

There is another factorisation system on **Pos** that may be used with our theory. It is an example of requiring *richer* structure for the $\mathcal{M}$-maps and so permitting much *sparser* $\mathcal{E}$-maps (*cf.* Remark 1.7), for which we then require the Proposal.

Proposition 3.12 $({}^{\perp}\mathcal{L}, \mathcal{L})$ is a factorisation system for **Pos**, where
(a) $\mathcal{L}$ consists of lower inclusions $U \subset_{\downarrow} A$, and
(b) ${}^{\perp}\mathcal{L}$ consists of cofinal monotone functions, $e : A \longrightarrow\!\!\!\!\triangleright\!\!\triangleright B$ with $\forall b \in B.\, \exists a \in A.\, b \leq e(a)$.
Moreover, $\mathcal{L}$ and $\mathcal{D}$ satisfy Assumptions 2.3 [Tay23]. □

Definition 3.13 A subset $U \subset A$ of a coalgebra is called ***representable*** if $U = \alpha(a)$ for some $a \in A$, which is unique by plain extensionality. By definition any representable subset is $(\prec)$-bounded above and by Lemma 3.3 it must be $(\leq)$-lower:

$$\exists a{:}A.\ \forall u{:}A.\ \ u \in U \Longrightarrow u \prec a \qquad \text{and} \qquad \forall c, u{:}A.\ \ c \leq u \in U \Longrightarrow c \in U.$$

Lemma 3.14 A $\mathcal{D}$-coalgebra (A, α) or $(A, \leq, \prec)$ is $\mathcal{L}$-extensional iff every $(\prec)$-bounded $(\leq)$-lower subset $U \subset_{\downarrow} \alpha(a) \subset_{\downarrow} A$ is represented as $U = \alpha(b)$ for some unique $b \in A$, with

$$b \leq a \qquad \text{and} \qquad \forall u{:}A.\ \ u \in U \iff u \prec b.$$

These are called ***plump ordinals***.

Proof The conditions make $U \in \mathcal{D}(A)$, so the property re-states that $\alpha : A \to \mathcal{D}(A)$ is a lower inclusion. □

Remark 3.15 The operator $\mathcal{D}$ is called the ***plump successor*** and is inflationary and functorial. Robin Grayson called it the *full successor* [Gra78, §IX.2.6] but didn't develop it — that was done in [JM95, Tay96].

Plump ordinals are in many ways far better behaved categorically than other notions, but they look nothing like the combinatorial definitions such as the Cantor Normal Form. Moreover, their growth in even the simplest non-classical topos $\mathbf{Set}^{\to}$ is just as fast as the powerset, so constructing plump $\omega \cdot 2$ requires either Replacement or our Proposal:

Proposition 3.16 Under the Proposal, any well founded relation has a plump rank or $\mathcal{L}$-extensional reflection. □

The discussion in the remainder of this paper will be equally valid whether we take the base factorisation system $(\mathcal{E}, \mathcal{M})$ on **Pos** to be the slim one $({}^{\perp}\mathcal{R}, \mathcal{R})$ or the plump one $({}^{\perp}\mathcal{L}, \mathcal{L})$. For the sake of familiarity, you may prefer to imagine it to be the slim one, since that gives the familiar results for ω, ω^{+} and the Cantor normal forms.

4 Transfinite recursion

In this section we show how to use the Proposal to deduce transfinite recursion over any $\mathcal{R}$- or $\mathcal{L}$-ordinal. That is, given any such ordinal $(A, \leq, \prec)$ and a complete $\bigvee$-semilattice E with a monotone endofunction $s : E \to E$, there is a unique monotone function $f : A \to E$ such that, for all $a \in A$,

$$fa \ = \ \bigvee \{s(fb) \mid b \prec a\}.$$

So this is transfinite iteration of a funct*ion* s, which we will later adapt to funct*ors*.

A solution to this equation is not just the value of f at some *particular* ordinal but the *entire system* of values up to it.

In the first instance, we only assume that E is a poset with binary meets and a least element $\bot$; we delay introducing s and $\bigvee$ until Lemma 4.11.

Definition 4.1 The ***lax slice category*** **Pos** $/\!\!/ E$ has
(a) as a typical object: a poset $(A, \leq)$ and a monotone function $a : A \to E$ and
(b) as a typical morphism, a monotone function $f : A \to B$ such that $a \leq f\,;b$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "a"'] & \arrow[d, phantom, "\leq"] & B \arrow[dl, "b"] \\
& E &
\end{tikzcd}$$

We write $\bot : A \to E$ for the function $a \mapsto \bot$ and $\mathcal{M} /\!\!/ E$ for the class of morphisms for which $f \in \mathcal{M} \subset$ **Pos** and the inequality is equality. The double slash seems to be a common but not standard notation for the lax slice category.

In **Pos** we have pullbacks, monos, factorisation, an endofunctor, coalgebras for it, well-foundedness and extensionality. We need to lift these notions to **Pos** $/\!\!/ E$. Since they are mostly defined in terms of limits, the following fact suggests that this will be straightforward:

Lemma 4.2 The forgetful functor dom : **Pos** $/\!\!/ E \to$ **Pos** that takes $a : A \to E$ to A has a left adjoint, $A \mapsto \bot$. Therefore this forgetful functor preserves limits. $\square$

Lemma 4.3 Pullback in **Pos** $/\!\!/ E$ is given by that in **Pos**, with $p \equiv \pi_0\,; a \wedge \pi_1\,; b$. If $f \in \mathcal{M} /\!\!/ E$ then $\pi_1 \in \mathcal{M} /\!\!/ E$.

$$\begin{tikzcd}
P \arrow[rr, "\pi_0"] \arrow[dd, "\pi_1"'] \arrow[dr, dotted, "p"] \arrow[dr, phantom, very near start, "\lrcorner"] & \leq & A \arrow[dl, "a"'] \arrow[dd, "f"] \\
\geq & E & \geq \\
B \arrow[ur, "b"] \arrow[rr, "g"'] & \leq & C \arrow[ul, "c"']
\end{tikzcd}$$

Proof By the previous lemma, the poset P is the pullback in **Pos**, which is in turn that in **Set**. The map $p : P \to E$ must satisfy

$$p \leq \pi_o\,; a \quad \text{and} \quad p \leq \pi_1\,; b.$$

Any other span $(A, a) \xleftarrow{h} (D, d) \xrightarrow{k} (B, b)$ commuting at C has

$$h\,; f = k\,; g \qquad d \leq h\,; a \quad \text{and} \quad d \leq k\,; b.$$

Then the mediator $m : D \to P$ is the one in **Pos** with $h = m\,; \pi_0$ and $k = m\,; \pi_1$ and

$$d \;\leq\; (h\,; a) \wedge (k\,; b) \;=\; (m\,; \pi_0\,; a) \wedge (m\,; \pi_1\,; b) \;=\; m\,; (\pi_0\,; a \wedge \pi_1\,; b) \;=\; m\,; p.$$

In the case where $a = f\,; c$,

$$\pi_1\,; b \;\leq\; \pi_1\,; g\,; c \;=\; \pi_0\,; f\,; c \;=\; \pi_0\,; a,$$

so the meet reduces to $p = \pi_1\,; b$. This means that pullback in **Pos** $/\!\!/ E$ preserves $\mathcal{M} /\!\!/ E$ since it also preserves $\mathcal{M}$ in **Pos**. $\square$

Definition 4.4 Each of the ***factorisation systems*** $(\mathcal{M}, \mathcal{E})$ in **Pos** in Propositions 3.4 and 3.12 induces one in **Pos** $/\!\!/ E$, where
(a) $\mathcal{M} /\!\!/ E$ consists of $f \in \mathcal{M}$, with $(=)$ instead of $(\leq)$;
(b) $\mathcal{E} /\!\!/ E$ consists of $f \in \mathcal{E}$, with general $(\leq)$.
Then each class contains all isomorphisms and is closed under composition.

Morphisms actually have a *three*-part factorisation: let $f = e\,;m$ by the factorisation system in **Pos** and define

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "a"'] & \arrow[d, phantom, "\leq"] & C \arrow[dl, "b"] \\
& E &
\end{tikzcd}
\quad\longmapsto\quad
\begin{tikzcd}
A \arrow[r, "\mathsf{id}"] \arrow[dd, no head, "a"'] & A \arrow[r, two heads, "e"] \arrow[d, "f"] & B \arrow[r, tail, "m"] \arrow[d, tail, "m"] & C \arrow[dd, "b"] \\
{} \arrow[r, phantom, "\leq"] & C \arrow[r, phantom, "="] \arrow[d, "b"] & C \arrow[d, "b"] \arrow[r, phantom, "="] & {} \\
E \arrow[r, equal] & E \arrow[r, equal] & E \arrow[r, equal] & E
\end{tikzcd}
$$

In fact breaking after $(\mathsf{id}\,,\leq)$ also gives a factorisation system, but the one that we are interested in breaks at B, where the $\mathcal{E} \mathbin{/\!\!/} E$-map is $(e,\leq)$ and the $\mathcal{M} \mathbin{/\!\!/} E$-map is $(m,=)$.

Lemma 4.5 The factorisation satisfies ***orthogonality***.

Proof We are given a commutative square in **Pos** with $e \in \mathcal{E}$, $m \in \mathcal{M}$ and $e\,;g = f\,;m$,

$$
\begin{tikzcd}
A \arrow[rr, two heads, "e"] \arrow[dd, "f"'] \arrow[dr, "a"] & \arrow[d, phantom, "\leq"] & B \arrow[dd, "g"] \arrow[dl, "b"'] \\
\arrow[r, phantom, "\rotatebox{90}{$\geq$}"] & E \arrow[r, phantom, "\rotatebox{90}{$\geq$}"] \arrow[d, phantom, "="] & {} \\
C \arrow[rr, tail, "m"'] \arrow[ur, "c"] & {} & D \arrow[ul, "d"']
\end{tikzcd}
$$

together with $c = m\,;d$ and $b \leq g\,;d$ making a square in $\mathbf{Pos} \mathbin{/\!\!/} E$. The map $a : A \to E$ also satisfies the inequalities $a \leq e\,;b$ and $a \leq f\,;c$, but we don't use them.

Then orthogonality in **Pos** gives a unique $h : B \to C$ such that $f = e\,;h$ and $g = h\,;m$.

$$
\begin{tikzcd}
A \arrow[rr, two heads, "e"] \arrow[dd, "f"'] \arrow[dr, "a"] & \arrow[d, phantom, "\leq"] & B \arrow[dd, no head, "h"] \arrow[dl, "b"'] \\
\arrow[r, phantom, "\rotatebox{90}{$\geq$}"] & E \arrow[dr, phantom, "\rotatebox{90}{$\geq$}?"] & {} \\
C \arrow[ur, "c"] & & {} \arrow[ll]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{} \arrow[dd, "h"'] & & B \arrow[ll, no head] \arrow[dd, "g"] \arrow[dl, "b"'] \\
{} \arrow[ur, phantom, "\rotatebox{90}{$\geq$}?"] & E \arrow[r, phantom, "\rotatebox{90}{$\geq$}"] \arrow[d, phantom, "="] & {} \\
C \arrow[rr, tail, "m"'] \arrow[ur, "c"] & {} & D \arrow[ul, "d"']
\end{tikzcd}
$$

This is a $\mathbf{Pos} \mathbin{/\!\!/} E$ map because $b \;\leq\; g\,;d \;=\; h\,;m\,;d \;=\; h\,;c$.

Lemma 4.6 $\mathbf{Pos} \mathbin{/\!\!/} E$ and its factorisation systems obey Assumptions 2.1. □

We are going to apply the theory to a *specific* endofunctor $T : \mathbf{Pos} \mathbin{/\!\!/} E \to \mathbf{Pos} \mathbin{/\!\!/} E$, but since this will introduce other complications, we first define well-foundedness and extensionality in the *generic* case. It does have still to obey some conditions.

Definition 4.7 A ***slice-endofunctor*** $T : \mathbf{Pos} \mathbin{/\!\!/} E \to \mathbf{Pos} \mathbin{/\!\!/} E$ consists of an endofunctor $T_0 : \mathbf{Pos} \to \mathbf{Pos}$ together with

$$
\begin{tikzcd}
\mathbf{Pos} \mathbin{/\!\!/} E \arrow[r, "T"] \arrow[d, "\mathsf{dom}"'] & \mathbf{Pos} \mathbin{/\!\!/} E \arrow[d, "\mathsf{dom}"] \\
\mathbf{Pos} \arrow[r, "T_0"'] & \mathbf{Pos}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "a"'] \arrow[r, phantom, ""{name=L}] & T_0 A \arrow[d, "T_1 a"] \\
E & E
\end{tikzcd}
\quad a \longmapsto T_1 a
$$

(NB $T_1 a$ is not $T(a : A \to E)$) If $a \leq f\,;b$ then $Ta \leq T_0 f\,;T_1 b$.

For our application, we also require $m \in \mathcal{M} \mathbin{/\!\!/} E \Longrightarrow Tm \in \mathcal{M} \mathbin{/\!\!/} E$.

Lemma 4.8 A T-***coalgebra*** is a **Pos** $/\!\!/$ E morphism

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "a"'] & \leq & T_0A \arrow[dl, "T_1a"] \\
& E &
\end{tikzcd}$$

and this is ***extensional*** if $\alpha \in \mathcal{M}$ and $(\leq)$ is $(=)$. $\square$

Well-foundedness (Definition 1.1) also lifts in a very simple way:

Lemma 4.9 A T-coalgebra $\alpha : (A, f) \to T(A, f) \equiv (T_0A, F)$ is well founded in **Pos** $/\!\!/$ E iff $(\alpha : A \to T_0A)$ is a well founded T_0-coalgebra in **Pos**.

$$\begin{tikzcd}
T_0U \arrow[rrr, hook, "I"] \arrow[drr, "G"] & & {=} & T_0A \arrow[dl, "F"'] \\
\leq & & E & \leq \\
K \arrow[u, "h"] \arrow[urr, hook, dotted, "k"] \arrow[r, dotted, "j"] & U \arrow[ur, hook, "g"'] \arrow[rr, hook, "i"'] & {=} & A \arrow[uu, "\alpha"'] \arrow[ul, "f"]
\end{tikzcd}$$

Proof The T-coalgebra is the triangle on the right, consisting of the T_0-coalgebra and $\geq$.

Subobjects $i : (U, g) \rightarrowtail (A, f)$ in $\mathcal{M}$ $/\!\!/$ E are in bijection with $i : U \rightarrowtail A$ in $\mathcal{M}$ because they must have $g = i\,;f$. This is the lower right triangle, with equality.

Applying the functor T to this subobject gives the upper triangle, which also consists of an $\mathcal{M}$-map and equality, since T is required to preserve the class $\mathcal{M}$ $/\!\!/$ E.

Then by Lemma 4.3, the pullback in **Pos** $/\!\!/$ E is that in **Pos** and the Corollary says that the *whole* lower triangle KEA (*i.e.* the lower side of the pullback in **Pos** $/\!\!/$ E) also has equality.

If K factors through U in **Pos** then $k = j\,; i\,; f = j\,; g$, so the slim triangle has equality too, which means that (K, k) factors though (U, g) in **Pos** $/\!\!/$ E.

Well-foundedness in either category then says that $i : U \cong A$, so the two definitions coincide. $\square$

Now we introduce the two *particular* slice endofunctors that are relevant to our study:

Definition 4.10 The endofunction $s : E \to E$ that we want to iterate induces an endofunctor $S :$ **Pos** $/\!\!/$ $E \to$ **Pos** $/\!\!/$ E.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "a"'] & \leq & B \arrow[dl, "b"] \\
& E \arrow[d, "s"'] & \\
& E &
\end{tikzcd}$$

It acts on objects by $(a : A \to E) \mapsto (a\,; s : A \to E)$ and as the identity $(f \mapsto f)$ on morphisms. It is faithful but not necessarily full. It is a slice endofunctor (Definition 4.7) with $S_0 = \mathsf{id}_{\mathbf{Pos}}$ and preserves $\mathcal{M}$ $/\!\!/$ E-maps.

Finally we introduce the arbitrary joins $\bigvee : \mathcal{D}E \to E$ that serve as the "limit" clause of transfinite recursion. These combine with s to define the endofunctor $T :$ **Pos** $/\!\!/$ $E \to$ **Pos** $/\!\!/$ E that we will use together with the foregoing theory to obtain transfinite recursion.

Lemma 4.11 The slice endofunctor $T :$ **Pos** $/\!\!/$ $E \to$ **Pos** $/\!\!/$ E is defined on objects by

$$(f : A \to E) \longmapsto (F : \mathcal{D}A \to E) \qquad \text{where} \qquad FU \equiv \bigvee \{s(fa) \mid a \in U\}$$

and on morphisms by $p \mapsto T_0 p \equiv \mathcal{D}p$. where

$$\mathcal{D}pU \;\equiv\; \{b \mid \exists a \in U.\, b \le pa\}.$$

$$\begin{tikzcd}
A \arrow[rr, "p"] \arrow[dr, "f"'] & \le & B \arrow[dl, "g"] & \longmapsto & \mathcal{D}A \arrow[rr, "\mathcal{D}p"] \arrow[dr, "F"'] & \le ? & \mathcal{D}B \arrow[dl, "G"] \\
& E & & & & E &
\end{tikzcd}$$

It has $T_0 \equiv \mathcal{D} : \mathbf{Pos} \to \mathbf{Pos}$ and T preserves $\mathcal{M} \mathbin{/\!\!/} E$.

Proof Combining the formulae for $\mathcal{D}p$ and G,

$$G(\mathcal{D}pU) \;\equiv\; \bigvee \{s(gb) \mid \exists a \in U.\, b \le pa\} \;=\; \bigvee \{s(g(pa)) \mid a \in U\}.$$

This is because s and g are monotone, so if $b \le pa$ then $s(gb) \le s(g(pa))$, whilst the join includes the case $b = pa$. Since $f \le p\,;g$ we have $s(fa) \le s(g(pa))$ and

$$FU \;\equiv\; \bigvee \{s(fa) \mid a \in U\} \;\le\; \bigvee \{s(g(pa)) \mid a \in U\} \;=\; G(\mathcal{D}pU),$$

so Tp is a $\mathbf{Pos} \mathbin{/\!\!/} E$-morphism.

The case $p \in \mathcal{M} \mathbin{/\!\!/} E$ satisfies $f = p;g$ and so $F = \mathcal{D}p;G$, whilst $\mathcal{D}$ preserves $\mathcal{M}$, so $Tp \in \mathcal{M} \mathbin{/\!\!/} E$. $\square$

Lemma 4.12 A T-***coalgebra***

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "f"'] & \le & \mathcal{D}A \arrow[dl, "F"] \\
& E &
\end{tikzcd}$$

has

$$fa \;\le\; F(\alpha a) \;\equiv\; \bigvee \{s(fb) \mid b \prec a\}.$$

It is extensional iff this is equality and $\alpha \in \mathcal{M}$. It is well founded iff $\alpha : A \to \mathcal{D}A$ is well founded in $\mathbf{Pos}$ and so corresponds by Lemma 3.3 to $(A, \le, \prec)$ with $(\prec)$ well founded in the traditional sense. $\square$

Example 4.13 If $(A, \le, \prec)$ is well founded in $\mathbf{Pos}$ then $\bot : A \to E$ is well founded in the *lax* slice $\mathbf{Pos} \mathbin{/\!\!/} E$.

However, if we had chosen to use the *op-lax* category, *i.e.* with the reverse inequality in Definition 4.1, then we would also have the reverse one in the characterisation of coalgebras,

$$fa \ge \bigvee \{s(fb) \mid b \prec a\}, \quad \text{which is equivalent to} \quad \forall b.\, b \prec a \Longrightarrow s(fb) \le fa,$$

but $\bot : A \to E$ only satisfies this for trivial s. $\square$

From this we deduce transfinite recursion:

Theorem 4.14 The Proposal, that every well founded T-coalgebra $\alpha : (A, f) \to (\mathcal{D}A, F)$ have an $\mathcal{M} \mathbin{/\!\!/} E$-extensional reflection $\overline{\alpha} : \overline{(A, f)} \to \overline{(\mathcal{D}A, F)}$ means that transfinite recursion in complete $\bigvee$-semilattices has a unique solution.

$$\begin{tikzcd}
\mathcal{D}A \arrow[rr] \arrow[dr] & \le & \mathcal{D}\overline{A} \arrow[dl] \\
\ge & E & = \\
A \arrow[uu, "\alpha"] \arrow[ur, "\bot"] \arrow[rr, two heads] & \le & \overline{A} \arrow[uu, hook, "\overline{\alpha}"'] \arrow[ul, "f"']
\end{tikzcd}$$

Remark 4.15 If we take $\mathcal{M} \equiv \mathcal{R}$ then $\mathcal{E} \mathbin{/\!\!/} E$ consists of the pointwise surjections from ${}^{\perp}\mathcal{R}$, together with some inequalities from the target object E. It is therefore still well co-powered and so the extensional reflection may be found within topos logic. □

Example 4.16 If (A, α) is $\omega \equiv (\mathbb{N}, \leq, <)$ or ω^{+} from Example 3.8, or indeed any Cantor Normal Form, then it is already well founded and $\mathcal{R}$-extensional. Therefore $\perp : A \to E$ is a well founded T-coalgebra and its $\mathcal{R} \mathbin{/\!\!/} E$-extensional reflection $f : \overline{A} \to E$ has $\overline{A} = A$ and

$$fa \;=\; \bigvee \{s(fb) \mid b \in \alpha(a)\} \;\equiv\; \bigvee \{s(fb) \mid b \prec a\}.$$

This reduces to the classical situation:

Example 4.17 In the case $(\mathbb{N}, \leq, <)$, we have $\alpha(0) = \emptyset$ and $\alpha(n) \subset \alpha(n+1)$, so the join in the successor case has a greatest member. Therefore we recover Peano recursion:

$$f(0) \;=\; \perp \quad \text{and} \quad f(n+1) \;=\; s(fn) \;\geq\; f(n),$$

Hence, for the single top element of ω^{+} — let's call it ω — the join is directed,

$$f(\omega) \;=\; \bigvee\!\!{}^{\uparrow} \{f(n) \mid n \in \mathbb{N}\},$$

and in general Cantor normal forms evaluate as expected. □

Remark 4.18 If instead we take $\mathcal{M} \equiv \mathcal{L}$, the extensional reflection applies to the base $\mathcal{D}$-coalgebra (A, α) as well as to $A \to E$. This does require the Proposal.

Just as with the one-point successor, the plump successor $\mathcal{D}A$ has a greatest element, to which the join reduces.

This means that the *system* of all plump ordinals has the universal property of the free algebra with all small joins and monotone successor. This was the point of view of Algebraic Set Theory [JM95]. □

The idea now is to extend the transfinite iteration of funct*ions* to funct*ors* by replacing the small poset E with the large preorder **Ens**.

However, using this directly would require universes, which we rejected from the outset. Therefore, in the next section we will replace *indexations* $X_{(-)} : A \to \mathbf{Ens}$ with *opfibrations* $\mathsf{X} \to A$ by means of (a simple form of) the *Grothendieck construction*. However, we treat that as *motivation* rather than a *theorem*, because an indexation is a function from a small object to a large one, which notion we have also rejected. Indeed, that construction is essentially the same as the set-theoretic formulation of the axiom of Replacement that we want to understand.

Another problem is that constructive ordinals define *joins* but not categorical colimits (Warning 3.10), which is the reason for working in the preorder **Ens**. We must now construct a fibred form of it, so that we can work with joins there instead of colimits. We left $\bigvee$ to last here in order to show more clearly how well-foundedness and extensionality work abstractly. This will take advantage of the fact that application of the functor and (certain) colimits remain within this subcategory (Lemma 2.4) and that it is actually a preorder (Proposition 2.10(a)), so that colimits indexed by posets are meaningful.

5 Discrete opfibrations

What is a monotone function $A \to \mathbf{Ens}$? It is a functor $A \to \mathbf{Ens} \to \mathbf{CoAlg} \to \mathbf{Set}$ with additional properties. We consider functors $A \to \mathbf{Set}$ in this section and then $A \to \mathbf{CoAlg}$ and $A \to \mathbf{Ens}$ in the next. So, for the moment, there is *no* endofunctor in *either* the base category or the fibres, let alone iteration of it.

It is enough to consider (endofunctors of) **Set** for our primary goal of constructing the cumulative hierarchy. In fact, **Set** could be replaced in what follows by other categories for the fibres, using standard categorical techniques. We don't elaborate on them because they would distract from the theory that we are developing here and we already need to use fibred category theory *twice* in this section. The introduction by Thomas Streicher [Str05] covers everything that we need.

When we use a category such as **Set** in place of the poset E, the inequalities $a \le f\,;b$ between monotone funct*ions* are replaced with natural transformations between funct*ors*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "X_{(-)}"'] & \arrow[d, phantom, "\overset{\phi}{\Longrightarrow}"] & B \arrow[dl, "Y_{(-)}"] \\ & \mathbf{Set} & \end{tikzcd}$$

that define an (illegitimate) lax slice category. By analogy with $\mathbf{Pos} /\!\!/ E$, we temporarily call it $\mathbf{Pos} /\!\!/ \mathbf{Set}$ since (for what we need) A and B are still posets.

Definition 5.1 Instead of this "indexed" formulation, we collect all of the X_a together (as a disjoint union) into a single object $\Sigma_a X_a$ (sometimes written $\int_a X_a$) that we call X and then indicate where each element came from by a function $\mathsf{X} \equiv \Sigma_a X_a \to A$.

$$\begin{array}{ccccc} X_a \ni x & \le & \exists! y \in X_b & \subset \Sigma_a X_a \equiv \mathsf{X} \\ \downarrow & & \downarrow & \downarrow \\ a & \le & b & \in \quad A \end{array}$$

For each $x \in X_a \subset \mathsf{X}$ and $a \le b$, we decree that $x \le X_{a\le b}(x) \subset \mathsf{X}$, where $X_{a\le b} : X_a \to X_b$ is the function that is the image of $a \le b$ under the functor $X_{(-)} : A \to \mathbf{Set}$.

The resulting monotone function $\mathsf{X} \to A$ is called a ***discrete opfibration***, which means that, whenever x belongs to the op-fibre over a and $a \le b$, there is a unique y over b with $x \le y$, namely $y \equiv X_{a\le b}(x)$. This is known as the ***supine lifting*** of $a \le b$ at x.

Lemma 5.2 A morphism in $\mathbf{Pos} /\!\!/ \mathbf{Set}$ corresponds to a commutative square in **Pos**

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "X_{(-)}"'] & \arrow[d, phantom, "\overset{\phi}{\Longrightarrow}"] & B \arrow[dl, "Y_{(-)}"] \\ & \mathbf{Set} & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{X} \equiv \Sigma_a X_a \arrow[r, "\Sigma_a \phi_a"] \arrow[d] & \Sigma_b Y_b \equiv \mathsf{Y} \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

that is a pullback iff ϕ is invertible, *i.e.* $\forall a \in A.\, \phi_a : X_a \cong Y_{f(a)}$.

Proof On the left, ϕ takes $x \in X_a$ to $\phi_a x \in Y_{fa}$ and on the right $(a, x) \mapsto (fa, \phi_a x)$ over $a \mapsto fa$. Conversely, any function $\mathsf{X} \to \mathsf{Y}$ that makes the square commute restricts to $X_a \to Y_{f(a)}$. If ϕ is invertible then $\forall a.\, \phi_a : X_a \cong Y_{fa}$ making the square a pullback. □

Definition 5.3 The category whose objects are discrete opfibrations and whose morphisms are commutative squares is called **DFPos**. It is a full subcategory of the category $\mathbf{Pos}^{\to}$ of *all* arrows and squares that is closed under pullbacks of the kind in the Lemma.

Lemma 5.4 DFPos has all pullbacks.

Proof In the commutative cube, suppose that the top and bottom faces are pullbacks and that $\mathsf{X} \to A$, $\mathsf{Y} \to B$ and $\mathsf{Z} \to C$ are all discrete opfibrations. We claim that $\mathsf{W} \to P$ is one too and is the pullback in the category whose objects are discrete opfibrations.

$$
\begin{tikzcd}
\mathsf{W} \arrow[rr] \arrow[dr, "\text{pbk}" near start] \arrow[dd] & & \mathsf{Y} \arrow[dr, "G"] \arrow[dd] & \\
& \mathsf{X} \arrow[rr, "F"] \arrow[dd] & & \mathsf{Z} \arrow[dd] \\
P \arrow[rr] \arrow[dr, "\text{pbk}" near start] & & B \arrow[dr, "g"] & \\
& A \arrow[rr, "f"] & & C
\end{tikzcd}
$$

Indeed, taking advantage of the two pullbacks in **Pos**, we are given

$$
\begin{tikzcd}
w = (x, y) \leq \arrow[d, mapsto] & w'? \arrow[d, dotted, mapsto] \\
p = (a, b) & \leq p' = (a', b')
\end{tikzcd}
$$

with $fa = c = gb$ and $Fx = z = Gy$. Then we need to find w'.

Since $\mathsf{X} \to A$ and $\mathsf{Y} \to B$ are discrete opfibrations, there are unique $x' \in \mathsf{X}$ and $y' \in \mathsf{Y}$ with $x \leq x'$ over $a \leq a'$ and $y \leq y'$ over $b \leq b'$. Then

$$
\begin{tikzcd}
\mathsf{Z} \arrow[d] & Fx = z = Gy \arrow[d, mapsto] & \leq & Fx' \arrow[d, mapsto] & z' \arrow[d, mapsto] & Gy' \arrow[d, mapsto] \\
C & fa = c = gb & \leq & fa' = & c' = & gb'
\end{tikzcd}
$$

in which $Fx' = z' = Gy'$ because $\mathsf{Z} \to C$ is a discrete opfibration.

This yields the required $w' \equiv (x', y')$ over $p' = (a'.b')$. $\square$

Corollary 5.5 Pullback within a fibre ($A \equiv C \equiv B$) is pullback in the total category **DFPos**, staying within the fibre $\mathbf{Set}^A$ (so $P = A$ too). $\square$

Four-part factorisation

Now we make the second and more substantial use of fibred category theory, but some explanation of English vocabulary is useful first.

Remark 5.6 It is conventional to draw the inequalities $a \leq b$ in the "indexing" poset A in Definition 5.1 and also the "re"-indexing monotone functions $\Sigma_a \phi_a : X_a \to Y_{f(a)}$ over $f : A \to B$ in the ***base*** category **Pos** ***horizontally***. What happens *over* a single point $a \in A$ or object $A \in \mathbf{Pos}$ is called the ***fibre*** and is drawn ***vertically***.

Then the horizontal inequality $a \leq b$ acts *forwards* in Definition 5.1 and the monotone function $f : A \to B$ will act *backwards* in the following Lemma, in a fashion that is called *orthogonal.* Conveniently there are (Latin and) English words that capture exactly this geometry: ***prone*** and ***supine*** mean horizontal face down and up respectively. These words were introduced into fibred category theory in [Tay99, §9.1] and adopted in [Joh02, §B.1.3] but condemned by Jean Bénabou [Bén05], to replace the historically inappropriate and grossly over-used words "cartesian" and "co-cartesian".

A functor where the backward action has the appropriate property is called a *fibration* and the forward one an *opfibration.* In our case, $\mathsf{X} \to A$ is a discrete *op*fibration over a poset with typical *op*fibre X_a over $a \in A$; it is *discrete* because X_a is just a set. When *categories* are involved, things get more complicated:

Lemma 5.7 The functor $\mathsf{cod} : \mathbf{DFPos} \to \mathbf{Pos}$ is a fibration, whose fibre over $A \in \mathbf{Pos}$ is the functor- or presheaf-category $\mathbf{Set}^A$.

Proof Being a ***fibration*** means that, given any object $(\mathsf{X} \to B)$ in the fibre over B and a morphism $f : A \to B$ in the base category **Pos**, there is a ***prone lifting*** $f^*\mathsf{X} \to \mathsf{X}$ with a certain universal property. In the case of the functor $\mathsf{cod} : \mathbf{DFPos} \to \mathbf{Pos}$, this property is the same as that of the pullback in Lemma 5.2:

$$\begin{tikzcd}
f^*\mathsf{X} \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{X} \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

□

Vertical and prone morphisms in **DFPos** respectively form the orthogonal "epi" and "mono" classes of a factorisation system. This system may be combined with either of those called called $(\mathcal{E}, \mathcal{M})$ in Propositions 3.4 and 3.12 — in *two* ways:

Definition 5.8 Morphisms of **DFPos** have a four-part factorisation:

$$\begin{tikzcd}
\mathsf{X} \arrow[r, two heads] \arrow[d] & \mathsf{U} \arrow[r, tail] \arrow[d] & \mathsf{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{V} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Z} \arrow[d] \\
A \arrow[r, "\mathsf{id}"] & A \arrow[r, "\mathsf{id}"] & A \arrow[r, two heads] & B \arrow[r, tail] & C
\end{tikzcd}$$

This gives rise to three (two-part) factorisation systems by breaking it at any of the intermediate places. Their roles are as follows.

(a) The ***fibre factorisation*** breaks at U and will be used in Section 6 to reflect $\mathbf{CoAlg}^A$ onto $\mathbf{Ens}^A$, fixing the base A.

(b) Breaking at Y expresses the universal property of the fibration $\mathsf{cod} : \mathbf{DFPos} \to \mathbf{Pos}$.

(c) The ***base factorisation***, breaking at V, will be used to construct transfinite iteration in Section 8. □

Definition 4.4 did not have any analogue of (a), because it used the poset E instead of the category **Set**. It did mention the other two factorisations, but concentrated on the one that corresponds to (c).

So, as well as using fibred category theory twice, we will also use the theory of well founded and extensional coalgebras twice, namely with the base and fibre factorisation systems (a) and (c). This theory is not applicable to (b) because its "monos" do not satisfy the cancellation property that plain monos do.

Lemma 5.9 The fibre factorisation (a), breaking at U, is orthogonal.

$$\begin{tikzcd}
\mathsf{X} \arrow[rrr, two heads, "e"] \arrow[dr, "u"] \arrow[dd, "p"'] & & & \mathsf{Q} \arrow[dll, dotted, "k"'] \arrow[dl, dotted, "h"] \arrow[dr, "z"] \arrow[dd, "s"] & \\
& \mathsf{U} \arrow[r, tail, "m"'] \arrow[dd, "q"'] & \mathsf{Y} \arrow[rr, "w"] \arrow[dd, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{Z} \arrow[dd, "t"] \\
A \arrow[rrr, "\mathsf{id}"] \arrow[dr, "f"'] & & & A \arrow[dll, dotted, "f"'] \arrow[dr, "f\,;\,g"] & \\
& B \arrow[r, "\mathsf{id}"'] & B \arrow[rr, "g"] & & C
\end{tikzcd}$$

Proof Fill in dotted f trivially, then h to the pullback and finally k by the given factorisation in **Pos**. Then the orthogonality mediator is (k, f). This is equivalent to orthogonality of $\mathsf{X} \twoheadrightarrow \mathsf{Q}$ with respect to $f^*\mathsf{U} \rightarrowtail f^*\mathsf{Y}$ in the fibre $\mathbf{Set}^A$ over A. □

Lemma 5.10 The factorisation (b), breaking at Y, is orthogonal.

Proof This is obtained by omitting $m : \mathsf{U} \rightarrowtail \mathsf{Y}$ from the previous result or $e : A \twoheadrightarrow B$ from the next. The result is well known and will not be needed here. $\square$

Lemma 5.11 The base factorisation (c), breaking at V, is orthogonal and has the cancellation property for monos. It agrees *via* Lemma 5.2 with Definition 4.4.

$$\begin{tikzcd}
\mathsf{X} \arrow[rr, two heads] \arrow[dd] \arrow[dr] & & U \arrow[r, tail] \arrow[dd] & \mathsf{Y} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathsf{V} \arrow[dd] \arrow[dr] \arrow[dlll, dotted, tail] & \\
& \mathsf{W} \arrow[rrrr, tail] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & & & \mathsf{Z} \arrow[dd] \\
A \arrow[rr, "\mathsf{id}"] \arrow[dr] & & A \arrow[r, "\mathsf{id}"] & A \arrow[r, two heads, "e"] & B \arrow[dr] \arrow[dlll, dotted, tail] & \\
& C \arrow[rrrr, tail] & & & & D
\end{tikzcd}$$

Proof The unique mediator $B \to C$ is given by orthogonality of $\mathcal{E}, \mathcal{M} \subset$ **Pos**. Then $\mathsf{V} \to \mathsf{W}$ is the mediator to the front pullback, this being part of the condition that it is in $\mathcal{M} /\!\!/ E$. Such monos have the cancellation property because those in $\mathcal{M} \subset$ **Pos** have it. $\square$

Lemma 5.12 DFPos with the fibre and base factorisations derived from either $\mathcal{R}$ or $\mathcal{L} \subset$ **Pos** satisfy Assumptions 2.1. $\square$

Remark 5.13 Base monos, *i.e.* the pullbacks from V to C, obviously have the cancellation property in Assumption 2.3(b). However, fibre monos (the squares from U to C) only obey this over identities or at least monos in the base:

$$\begin{tikzcd}
\mathsf{W} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & \mathsf{U} \arrow[r, tail] \arrow[d] & \mathsf{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Z} \arrow[d] \\
A \arrow[r] & B \arrow[r, "\mathsf{id}"] & B \arrow[r] & C
\end{tikzcd}$$

We will address this issue in Remark 6.12.

6 Fibred coalgebras

Now that we understand functors $A \to$ **Set** as opfibrations $\mathsf{X} \to A$, we consider functors $A \to$ **CoAlg** and then impose well-foundedness and extensionality to yield $A \to$ **Ens**. This uses the fibre factorisation at U in Definition 5.8(a).

The fibred functor

Remark 6.1 The endofunctor S that we want to iterate corresponds to s in Section 4. It acts on the fibres $\mathbf{Set}^A$, taking one discrete opfibration $\mathsf{X} \to A$ to another $\mathsf{SX} \to A$, leaving the indexing object A of the base category **Pos** alone.

$$\begin{tikzcd}
& A \arrow[dl, "X_{(-)}"'] \arrow[dr, "SX_{(-)}"] & \\
\mathbf{Set} \arrow[rr, "S"] & & \mathbf{Set}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{X} \equiv \Sigma_a X_a \arrow[d] \arrow[r, mapsto] & S^A\mathsf{X} \equiv \Sigma_a SX_a \arrow[d] \\
A & A
\end{tikzcd}$$

We write $\mathsf{S} : \mathbf{DFPos} \to \mathbf{DFPos}$ to collect the $S^A : \mathbf{Set}^A \to \mathbf{Set}^A$ over all $A \in \mathbf{Pos}$.

Now we consider the effect of the morphism $f : A \to B$ on the opfibration $\mathsf{X} \to A$.

Lemma 6.2 The functor S commutes with prone lifting.

Proof Re-indexing along a morphism $f : A \to B$ in $\mathbf{Pos}$,

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "X_{f(-)}"'] & {=} & B \arrow[dl, "X_{(-)}"] \\
& \mathbf{Set} \arrow[d, "S"'] & \\
& \mathbf{Set} &
\end{tikzcd}$$

is expressed in terms of fibrations using Lemma 5.2 as the pullbacks

$$\begin{tikzcd}
f^*\mathsf{X} \equiv \Sigma_a X_{fa} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_b X_b \equiv \mathsf{X} \arrow[d] \arrow[r, phantom, "\overset{S}{\longmapsto}"] & \Sigma_a SX_{fa} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_b SX_b \equiv S^B\mathsf{X} \arrow[d] \\
A \arrow[r, "f"] & B & A \arrow[r, "f"] & B
\end{tikzcd}$$

so $\Sigma_a SX_{fa}$ has as fibrational names both $f^*\mathsf{S}\mathsf{X}$ and $\mathsf{S}f^*\mathsf{X}$, which must then be isomorphic. $\square$

Definition 6.3 A ***fibred functor*** between fibrations is one that commutes with them and preserves prone liftings like this [Str05, Def. 2.3].

$$\begin{tikzcd}
\mathbf{Set}^A \arrow[r, "S^A"] \arrow[d, tail] & \mathbf{Set}^A \arrow[d, tail] \\
\mathbf{DFPos} \arrow[r, "\mathsf{S}"] \arrow[d, "\mathsf{cod}"'] & \mathbf{DFPos} \arrow[d, "\mathsf{cod}"] \\
\mathbf{Pos} \arrow[r, "\mathsf{id}"] & \mathbf{Pos}
\end{tikzcd}$$

In Section 8 we will need to consider a more general situation with a non-trivial endofunctor of **Pos**.

Lemma 6.4 If the given functor $S : \mathbf{Set} \to \mathbf{Set}$ preserves monos, as the covariant powerset does, then S^A and S preserve fibre monos.

Proof S also preserves prone maps by Lemma 6.2. $\square$

Fibred coalgebras

Next we consider coalgebras for the functors S, S^A and S.

Remark 6.5 An A-indexed family of S-coalgebras $\xi_a : X_a \to SX_a$ is a system of triangles:

$$\begin{tikzcd}
\Sigma_a X_a \equiv \mathsf{X} \arrow[dd] & X_a \arrow[rr] \arrow[dd, mapsto] \arrow[dr, "\xi_a"] & & X_b \arrow[dd, mapsto] \arrow[dr, "\xi_b"] & \\
& & SX_a \arrow[rr] \arrow[dl, mapsto] & & SX_b \arrow[dl, mapsto] \\
A & a & \leq & b &
\end{tikzcd}$$

But the reason why we want to consider such systems is as *diagrams* $A \to \mathbf{CoAlg}$ of coalgebras *and homomorphisms*, so the parallelogram must also commute. That is, $\xi_{(-)} : X_{(-)} \to SX_{(-)}$ is *natural* as the subscript varies over the poset A.

This is all happening in a single fibre $\mathbf{Set}^A \subset \mathbf{DFPos}$, so $\xi : \mathsf{X} \to \mathsf{SX}$ is a vertical map in $\mathbf{DFPos}$ over $A \in \mathbf{Pos}$. $\square$

Notation 6.6 We write $\mathbf{CoAlg}^A$ for the category of natural A-indexed families of S-coalgebras, as in the previous diagram, and natural families of homomorphisms. That is, of functors $A \to \mathbf{CoAlg}$ and natural transformations between them, expressed as discrete opfibrations as we have done.

What is the effect of a base morphism $f : A \to B$ on such a coalgebra?

Lemma 6.7 Prone liftings of coalgebra structure maps are pullbacks.

Proof Let $\xi : \mathsf{X} \to \mathsf{SX}$ be a coalgebra in $\mathbf{Set}^B$ and $f : A \to B$ in $\mathbf{Pos}$. Its prone lifting is $f^*\mathsf{X} \to f^*\mathsf{SX} \cong \mathsf{S}f^*\mathsf{X}$ in $\mathbf{Set}^A$.

$$\begin{tikzcd}
\Gamma \arrow[drr, "\gamma"] \arrow[ddrrrr] \arrow[ddr, dotted] \arrow[ddddrr] & & & & & \\
& & \mathsf{S}\Gamma \arrow[ddrrr] \arrow[ddr, dotted] \arrow[ddd] & & & \\
& f^*\mathsf{X} \arrow[rrr] \arrow[drr, "f^*\xi"'] \arrow[ddr] & & & \mathsf{X} \arrow[dr, "\xi"] \arrow[dd] & \\
& & & f^*\mathsf{SX} = \mathsf{S}f^*\mathsf{X} \arrow[rr] \arrow[dl] & & \mathsf{SX} \arrow[dl] \\
& & A \arrow[rr, "f"] & & B &
\end{tikzcd}$$

Since prone liftings are pullbacks by Lemma 6.2, $f^*\mathsf{X} \to \mathsf{X}$ and $f^*\mathsf{SX} = \mathsf{S}f^*\mathsf{X} \to \mathsf{SX}$ are pullbacks over $f : A \to B$. Hence so is the parallelogram that they form themselves.

That is in $\mathbf{DFPos}$, but we may also compare it with another *coalgebra* $\gamma : \Gamma \to \mathsf{S}\Gamma$ that makes the diagram of solid lines commute. Since $f^*\mathsf{X}$ and $f^*\mathsf{SX}$ are pullbacks, there are unique mediators as shown dotted that make the entire diagram commute. This means that the coalgebra $f^*\xi$ is also the pullback and prone lifting in CoAlg. $\square$

Notation 6.8 We write $\mathsf{CoAlg} \to \mathbf{Pos}$, $\mathsf{WfCoAlg} \to \mathbf{Pos}$ and $\mathsf{Ens} \to \mathbf{Pos}$ for the fibrations with fibres $\mathbf{CoAlg}^A$, $\mathbf{WfCoAlg}^A$ and $\mathbf{Ens}^A$ over $A \in \mathbf{Pos}$. In the same notation, $\mathbf{DFPos} \to \mathbf{Pos}$ would be called $\mathsf{Set} \to \mathbf{Pos}$. The analogous fibred forms of a category $\mathcal{C}$ over $\mathbf{Set}$ are called $\mathsf{Fam}(\mathcal{C}) \to \mathbf{Set}$ in [Str05] and elsewhere.

Corollary 6.9 $\mathsf{CoAlg} \to \mathbf{Pos}$ is a fibration with fibres $\mathbf{CoAlg}^A$ over $A \in \mathbf{Pos}$ and the forgetful functor $\mathsf{CoAlg} \to \mathbf{DFPos}$ is fibred. $\square$

Well-foundedness

Now we consider well-foundedness and extensionality of these coalgebras and show that they may be seen as examples of these concepts *straight from* [Tay21], applied to the endofunctor S of the category $\mathbf{DFPos}$ with the fibre factorisation (at U) in Definition 5.8(a).

Remark 6.10 We may in principle define well-foundedness (Definition 1.1) either

(a) "pointwise" in each op-fibre (a copy of $\mathbf{Set}$) over $a \in A$ separately, as defined by the broken pullbacks that are drawn with solid lines below, from which we would deduce that *each* $U_a \rightarrowtail X_a$ is invertible;

(b) "uniformly" for all $a \in A$ in a particular poset A, *i.e.* for the category $\mathbf{Set}^A$, where we use families of subobjects $(U_a \rightarrowtail X_a)$ that are *natural* in the sense of requiring the diagram including the dotted maps to commute;

$$
\begin{tikzcd}
SU_a \arrow[rrrr] \arrow[dr, dotted] & & & & SX_a \arrow[dr] & \\
 & SU_b \arrow[rrrr] & & & & SX_b \\
K_a \arrow[uu] \arrow[rr] \arrow[dr, dotted] \arrow[uur, phantom, "\lrcorner" very near start] & & U_a \arrow[rr, tail] \arrow[dr, dotted] & & X_a \arrow[uu, "\xi_a"'] \arrow[dr] & \\
 & K_b \arrow[uu] \arrow[rr] \arrow[uur, phantom, "\lrcorner" very near start] & & U_b \arrow[rr] & & X_b \arrow[uu, "\xi_b"']
\end{tikzcd}
$$

(c) uniformly in this way for *all* $A \in \mathbf{Pos}$; or

(d) "globally" in the category **DFPos**, using the fibre "monos" of the factorisation in Definition 5.8(a).

Since the uniform version (b) provides induction for *fewer* systems $U_{(-)}$, namely just those that are natural, it is *â priori* weaker than the pointwise form (a).

There is no reason why uniform well-foundedness (b) in $\mathbf{CoAlg}^A$ should imply the same in $\mathbf{CoAlg}^B$ when $f : A \to B$, because the coalgebra over B may be bigger. The reverse direction amounts to whether coalgebra homomorphisms reflect well-foundedness, which they do in **Set** so long as the functor S preserves inverse image diagrams [Tay21, §9].

Lemma 6.11 The uniform definition for all A in version (c) is equivalent to the global one (d).

Proof Version (b) amounts to using the usual broken pullback in each category $\mathbf{Set}^A$, which is a single fibre of **DFPos**, with the same pullbacks, where $\mathsf{U} \rightarrowtail \mathsf{X}$ is a *vertical* mono. Version (c) says this for all $A \in \mathbf{Pos}$.

However, we may equivalently see the latter situation using the *general* fibre monos of the factorisation in Definition 5.8(a). These consist of a vertical mono followed by the prone lifting of an arbitrary map $f : A \to B$ in **Pos**.

$$
\begin{tikzcd}
\mathsf{SU} \arrow[rr] & & \mathsf{S}f^*\mathsf{X} \cong f^*\mathsf{SX} \arrow[r] & \mathsf{SX} \\
\mathsf{K} \arrow[u] \arrow[r, tail] \arrow[ur, phantom, "\lrcorner" very near start] & \mathsf{U} \arrow[r, tail] & f^*\mathsf{X} \arrow[u, "f^*\xi"] \arrow[r] \arrow[ur, phantom, "\lrcorner" very near start] & \mathsf{X} \arrow[u, "\xi"']
\end{tikzcd}
$$

$$
A \xrightarrow{\quad f \quad} B
$$

The coalgebra $\xi : \mathsf{X} \to \mathsf{SX}$ is in the fibre over B, but we're testing its well-foundedness with a predicate in the fibre over A, related by $f : A \to B$. The objects SU and K also lie over A. This is the familiar situation in symbolic logic, where the predicate to which induction is applied may have parameters, which are inherited by the induction hypothesis.

So we bring X and SX into the same fibre as the rest of the diagram by means of a prone lifting. In predicate calculus this is called *weakening*: pretending that the structure X depends on the additional parameters.

This operation factors $\mathsf{U} \rightarrowtail \mathsf{X}$ into its vertical and prone parts, whilst the prone lifting of the coalgebra ξ is a pullback by Lemma 6.7. This pullback may now be detached from the "broken" one in the definition of well-foundedness by the familiar lemma for cancelling them.

This leaves a diagram that is entirely over A and is version (c) of well-foundedness. □

Remark 6.12 Whilst $\mathsf{U} \rightarrowtail \mathsf{X}$ in this argument is a *general* fibre mono over $A \to B$ that need not be plain mono in **DFPos**, the argument brings it into the fibre $\mathbf{Set}^A$ over A as $\mathsf{U} \rightarrowtail f^*\mathsf{X}$

over id_A. Then a vertical splitting is enough to make it an isomorphism in the fibre, so that the original $\mathsf{U} \rightarrowtail \mathsf{X}$ is prone in **DFPos**.

This explanation needs to be improved but the situation is quite familiar: we may use induction to prove a property with more parameters than the structure that it describes Therefore the object that is defined by the predicate is not isomorphic to the ambient structure but related to it by a projection that encodes the additional parameters.

Extensionality

Lemma 6.13 For extensionality, the analogous four versions are the same.

Proof A natural transformation $\xi : \mathsf{id} \to S : \mathbf{Set}^A \to \mathbf{Set}^A$ is "uniformly" mono (for $\mathbf{Set}^A$) iff it is "pointwise" mono (for each $a \in A$). Since the functor acts within fibres, the prone part of the "global" mono must be the identity, so it is determined by its effect on fibres. □

In any case, the reason why we are using well-foundedness and extensionality is that the given endofunctor S and colimits preserve these properties in the global sense and hence so does transfinite iteration of S (Proposition 2.6).

Proposition 6.14 The forgetful functors from the categories $\mathsf{WfCoAlg}$ and Ens, as defined in the "global" senses,

$$\mathsf{Ens} \to \mathsf{WfCoAlg} \to \mathsf{CoAlg} \to \mathbf{DFPos} \to \mathbf{Pos}$$

are fibrations over $\mathbf{Pos}$, whilst the intermediate functors are fibred.

Proof The functors all the way to $\mathbf{Pos}$ are fibrations: from $\mathbf{DFPos}$ by Lemma 5.7, from CoAlg by Corollary 6.9, from $\mathsf{WfCoAlg}$ by Lemma 6.11 and from Ens by Lemma 6.13. The intermediate functors are fibred because prone lifting preserves the various properties. □

7 Fibred coproducts

So far we have only used *finitary* categorical structure, but the "limit" case of transfinite iteration requires infinitary colimits of some kind.

Fibred category theory provides co*products* of sets and coalgebras very simply, whilst the situation for other categories is a little more complicated [Str05, §6]. Also, finite colimits such as pushouts and coequalisers must be considered separately in order to provide general infinitary colimits.

Our application requires joins in Ens in order to define the endofunctor in Definition 8.7 that takes

$$\xi_a : X_a \to SX_a \text{ for } a \in A \qquad \text{to} \qquad \bigvee \{SX_b \mid b \in U\} \text{ for } U \in \mathcal{D}A.$$

We will obtain these joins from coproducts using the reflection $\mathsf{WfCoAlg} \to \mathsf{Ens}$ and relying on the fact that the fibres $\mathbf{Ens}^A$ are preorders (Proposition 2.10).

Lemma 7.1 Discrete opfibrations compose and provide fibred coproducts in $\mathbf{DFPos}$.

Proof Let $\mathsf{X} \xrightarrow{p} \mathsf{A} \xrightarrow{f} B$ be discrete opfibrations with $x \in \mathsf{X}$ and $f(px) \leq b'$ in B. Then there are unique supine liftings $px \leq a'$ in A with $fa' = b'$ and $x \leq x'$ in X with $px' = a'$. Hence $f(px') = b'$ as required.

If also $x \leq x''$ with $f(px'') = b'$ then $px \leq px''$, so $px' = a' = px''$ and $x' = x''$ by uniqueness of supine liftings along $\mathsf{A} \to B$ and $\mathsf{X} \to \mathsf{A}$.

Notice that, over fixed $b \in B$, any inequality $a \leq a'$ or $x \leq x'$ is equality.

Now suppose that the square

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{\phi} & \mathsf{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathsf{A} & \xrightarrow{f} & B \end{array}$$

commutes, where q is also a discrete opfibration. By Lemma 5.2, ϕ corresponds to a family $\phi_a : X_a \to Y_b$ with $b \equiv fa$. This factors uniquely as

$$X_a \xrightarrow{\nu_a} \Sigma_{a \in A_b} X_a \xrightarrow{\psi_b} Y_b,$$

expressing the universal property of a B-indexed family of coproducts over A_b. □

Notation 7.2 Considering X as a discrete opfibration over B, we write

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\nu"] \arrow[d] & \Sigma_f \mathsf{X} \arrow[d] \\
\mathsf{A} \arrow[r, "f"] & B
\end{tikzcd}$$

where ν is in fact id_{X} in **Pos**, although it is not preserved by fibred functors.

The following is known as the ***Beck–Chevalley condition***:

Lemma 7.3 Fibred coproducts commute with prone lifting.

$$\begin{tikzcd}
h^*\mathsf{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{X} \; = \; \Sigma_f \mathsf{X} \arrow[d] \\
h^*\mathsf{A} \arrow[r, "k"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
C \arrow[r, "h"] & B
\end{tikzcd}$$

Proof Let $\mathsf{X} \longrightarrow \mathsf{A} \xrightarrow{f} B$ be discrete opfibrations as before, so that their composite provides the fibred coproduct $\Sigma_f \mathsf{X}$.

Now let $h : C \to B$ be any map in **Pos** and form the pullbacks as shown. By Lemma 5.4, the maps on the left are also discrete opfibrations.

This means that $h^*\mathsf{X}$ also plays the roles of $k^*\mathsf{X}$, $h^*\Sigma_f\mathsf{X}$ and $\Sigma_g k^*\mathsf{X}$. □

Lemma 7.4 $\mathsf{CoAlg} \to$ **DFPos** creates fibred coproducts.

$$\begin{tikzcd}
\mathsf{X} \arrow[rrrr, "h"] \arrow[drrr, "\nu_X X"] \arrow[ddd, "\xi"'] & & & & \mathsf{Y} \arrow[ddd, "\upsilon"] \\
& & & \Sigma_f \mathsf{X} \arrow[ur, dotted, "k"] \arrow[d, dotted, "\zeta"'] & \\
& & \Sigma_f \mathsf{SX} \arrow[r, dotted, "\ell"] & \mathsf{S}\Sigma_f \mathsf{X} \arrow[dr, dotted, "\mathsf{S}k"] & \\
\mathsf{SX} \arrow[urr, "\nu_{\mathsf{SX}}"] \arrow[urrr, "S\nu_{\mathsf{X}}"'] \arrow[rrrr, "\mathsf{S}h"] & & & & \mathsf{SY}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "f"] & B
\end{tikzcd}$$

Proof Let $h : \mathsf{X} \to \mathsf{Y}$ be a coalgebra homomorphism over $f : \mathsf{A} \to B$, so the outer rectangle commutes.

Let $\Sigma_f\mathsf{X}$ and $\Sigma_X\mathsf{SX}$ be fibre coproducts, with mediators k, ℓ and ζ that (together with functoriality of S) make all of the triangles and the left hand trapezium commute.

By uniqueness of coproduct mediators, the right hand trapezium also commutes. This says that $k : \Sigma_f\mathsf{X} \to \mathsf{Y}$ is a coalgebra homomorphism that is the coproduct mediator in CoAlg. □

Lemma 7.5 $\mathsf{WfCoAlg} \subset \mathsf{CoAlg}$ is closed under fibred coproducts and so creates them.

$$\begin{tikzcd}
& S^B\mathsf{U} \arrow[rrrr] & & & & S^B\Sigma_f\mathsf{X} \\
f^*S^B\mathsf{U} = S^{\mathsf{A}}f^*\mathsf{U} \arrow[rrrr] \arrow[ur, dotted] & & & & S^{\mathsf{A}}\mathsf{X} \arrow[ur] & \\
& \mathsf{K} \arrow[uu] \arrow[rr, tail] & & \mathsf{U} \arrow[rr, tail] & & \Sigma_f\mathsf{X} \arrow[uu] \arrow[d] \\
\mathsf{H} \arrow[uu] \arrow[ur, dotted] \arrow[rr, dotted] & & f^*\mathsf{U} \arrow[ur] \arrow[rr, tail] & & \mathsf{X} \arrow[uu] \arrow[ur, "\nu"] \arrow[d] & B \\
& & & & \mathsf{A} \arrow[ur, "f"] &
\end{tikzcd}$$

Proof Let $\mathsf{X} \to S^{\mathsf{A}}\mathsf{X}$ be a well founded coalgebra in the fibre $\mathbf{Set}^{\mathsf{A}}$ (at the front) and the upper right parallelogram be its fibred coproduct in $\mathbf{Set}^B$ (at the back), along $f : \mathsf{A} \to B$.

Suppose that $\mathsf{U} \rightarrowtail \Sigma_f\mathsf{X}$ gives a broken pullback (Definition 1.1) in $\mathbf{Set}^B$,

Let $f^*\mathsf{U} \to \mathsf{U}$ be the prone lifting and form $S^{\mathsf{A}}f^*\mathsf{U}$ and H in $\mathbf{Set}^{\mathsf{A}}$. This means that the front rectangle (rather than the bottom left parallelogram) is a pullback.

By Lemma 7.3, $S^{\mathsf{A}}f^*\mathsf{U} = f^*S^B\mathsf{U}$, from which the dotted map is the prone lifting to $S^B\mathsf{U}$ and makes the diagram commute.

Then there are mediators $\mathsf{H} \to \mathsf{K}$ to the pullback and $\mathsf{H} \to f^*\mathsf{U}$ to the prone lifting.

Since X is well founded, $f^*\mathsf{U} \cong \mathsf{X}$ in $\mathbf{Set}^{\mathsf{A}}$, but in fact $\nu : \mathsf{X} = \Sigma_f\mathsf{X}$, so $\mathsf{U} \rightarrowtail \Sigma_f\mathsf{X}$ is split, as required. □

Before discussing fibre coproducts (joins) in Ens, we need a better picture of it as a fibred category over $\mathbf{Pos}$, since it combines the properties of *both* its fibres $\mathbf{Ens}^A$ *and the base category.*

Remark 7.6 An object of $\mathbf{Ens}^A \subset \mathsf{Ens}$, which corresponds to a diagram $A \to \mathbf{Ens}$.

$$\begin{tikzcd}
\mathsf{X} \arrow[r, hook] \arrow[dr] & \mathsf{SX} \arrow[d] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
SX_a \arrow[r, hook] & SX_b \\
X_a \arrow[r, hook] \arrow[u, hook] & X_b \arrow[u, hook] \\
a \arrow[r, phantom, "\leq"] & b
\end{tikzcd}$$

The fibration $\mathbf{DFPos} \to \mathbf{Pos}$, unlike the more familiar $\mathbf{Set}^{\to} \to \mathbf{Set}$, has inequalities $a \leq b \in A$ in the base objects, whose supine liftings are functions $X_a \to X_b$.

In CoAlg and $\mathsf{WfCoAlg}$ these are coalgebra homomorphisms. In Ens they and the structure maps $X_a \hookrightarrow SX_a$ are monos, as illustrated in the square on the right. Putting all of these together for $a \in A$, we get the mono in the triangle, which is *vertical* in the sense of Remark 5.6 and so *fibre* mono in Definition 5.8.

Remark 7.7 Now consider a general morphism of Ens, over a general morphism of $\mathbf{Pos}$. That is, over a monotone function $f : A \to C$ that factorises as $f = e\,; m$ according to whichever system we adopt in $\mathbf{Pos}$.

Looking at the individual fibres over $a \in A$, the square above becomes a commutative cube of monos:

$$\begin{tikzcd}
& SX_b \arrow[rr, hook] & & SY_{fb} \\
SX_a \arrow[ur, hook] \arrow[rr, hook] & & SY_{fa} \arrow[ur, hook] & \\
& X_b \arrow[uu, hook] \arrow[rr, hook] & & Y_{fb} \arrow[uu, hook] \\
X_a \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, hook] & & Y_{fa} \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

As a fibre morphism, this cube corresponds to the leftmost square in

$$\begin{tikzcd}
\mathsf{X} \arrow[r, hook] \arrow[d] & f^*\mathsf{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & m^*\mathsf{Y} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Y} \arrow[d] \\
A \arrow[r, "\mathsf{id}"] & A \arrow[r, two heads, "e"] & B \arrow[r, hook, "m"] & C.
\end{tikzcd}$$

This is the same as the four-part factorisation in Definition 5.8, except that the vertical epi now vanishes.

Therefore Proposition is adapted to Ens by saying that

(a) each *fibre* $\mathbf{Ens}^A$ is a preorder, but Ens is not;

(b) the structure maps and homomorphisms in $\mathbf{Ens}^A$ are monos;

(c) there is at most one morphism $\mathsf{X} \to \mathsf{Y}$ *over any given base map* $A \to C$; and

(d) it is then a *fibre* mono, *i.e.* the composite of a vertical mono with a prone map. □

Since we still have base epis ($e : A \twoheadrightarrow B$):

Example 7.8 Forming the fibred coproduct of coalgebras does not preserve extensionality.

Proof Let $\mathsf{X} \equiv \mathsf{A} \equiv \mathbf{2} \equiv \{0,1\}$ and $B \equiv \mathbf{1}$, so $X_0 \equiv X_1 \equiv \mathbf{1}$.

The singleton $\mathbf{1}$ has a unique well founded relation, namely the empty one, whose coalgebra structure is $\bot : \mathbf{1} \to \mathcal{D}\mathbf{1} \equiv \Omega$. ($\Omega$ is the object of truth-values, which you may take to be $\{\bot, \top\}$ if you wish.)

The two copies of this form an extensional coalgebra in the fibre $\mathbf{Set}^{\mathbf{2}}$. However, their fibre coproduct in $\mathbf{Set}^{\mathbf{1}}$ is $\mathbf{2}$, still with the empty relation, which is no longer extensional: as a coalgebra it is $\alpha \equiv \bot : \mathbf{2} \to \Omega^{\mathbf{2}}$, so both elements $0, 1 \in \mathbf{2}$ have the same set of predecessors $\alpha(0) = \alpha(1) = \emptyset$. □

Notation 7.9 Now we invoke the Proposal to repair this, still using the *fibre* factorisation, so that the base object remains the same. For any fibred well founded coalgebra, write

$$\begin{tikzcd}
\mathsf{X} \arrow[rr, two heads] \arrow[dr, "\xi"] \arrow[dd] & & \overline{\mathsf{X}} \arrow[dr, hook, "\overline{\xi}"] \arrow[dd] & \\
& \mathsf{SX} \arrow[rr] \arrow[dl] & & \overline{\mathsf{SX}} \arrow[dl] \\
A \arrow[rr, equal] & & A &
\end{tikzcd}$$

for its extensional reflection, although we often omit the coalgebra structure to simplify the diagrams.

Proposition 7.10 Ens has fibred joins.

$$\begin{tikzcd}
\mathsf{X} \arrow[rrr] \arrow[dr, "\nu"'] \arrow[dd] & & & \mathsf{Y} \arrow[dd] \\
& \Sigma_f\mathsf{X} \arrow[r, two heads] \arrow[dd] & \overline{\Sigma_f\mathsf{X}} \arrow[ur, dotted] \arrow[dd] & \\
\mathsf{A} \arrow[dr, "f"] & & & \mathsf{B} \\
& B \arrow[r, "\mathsf{id}"] & B \arrow[ur, "\mathsf{id}"] &
\end{tikzcd}$$

Proof Suppose that the outside of the figure commutes, where $\mathsf{X} \to \mathsf{A}$ and $\mathsf{Y} \to B$ are in Ens (with coalgebra structures that can easily be filled in) and $f : \mathsf{A} \to B$ is also a discrete opfibration. Because of the Example, the fibred coproduct $\Sigma_f \mathsf{X} \to B$ is in $\mathsf{WfCoAlg}$ but in general not Ens, so let $\overline{\Sigma_f \mathsf{X}} \to B$ be its reflection into Ens by the Proposal. Its universal property as a reflection provides the coproduct mediator to Y. However, this coproduct is in a preorder by Lemma 7.7, so it is in fact a join. $\square$

Now we introduce the particular join that we require.

Lemma 7.11 For any poset $(A, \leq)$, define

$$(\in_A) \quad \equiv \quad \{(b, U) \mid b \in U\} \subset A \times \mathcal{D}A,$$

equipped with the order relation

$$(a, U) \leq (b, V) \quad \equiv \quad a = b \ \wedge \ U \subset V.$$

Then $\pi_1 : (\in_A) \to \mathcal{D}A$ by $(a, U) \mapsto U$ is a discrete opfibration. $\square$

This order relation is rather unnatural, given that $a, b \in A$ in a *po*set. Changing it to $(a \leq b) \wedge (U \subset V)$ would still give an opfibration, but it's no longer discrete.

Nevertheless, the Lemma provides what we need:

Proposition 7.12 The fibred coproduct

$$\begin{tikzcd}
 & & \mathsf{C} \equiv & \mathsf{J} \equiv \\
\mathsf{SX} \arrow[d] & \pi_0^*\mathsf{SX} \arrow[l] \arrow[r, "\nu"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & \Sigma_{\pi_1}\pi_0^*\mathsf{SX} \arrow[r, two heads] \arrow[d] & \overline{\Sigma_{\pi_1}\pi_0^*\mathsf{SX}} \arrow[d] \\
A & (\in_A) \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & \mathcal{D}A \arrow[r, "\mathsf{id}"] & \mathcal{D}A
\end{tikzcd}$$

(again with coalgebra structure to be filled in)) is the join $\bigvee \{SX_b \mid b \in U\}$ for $U \in \mathcal{D}A$. $\square$

In the next section we will compare our construction with the classical situation. For this, we will we need to evaluate J at particular $U \subset_{\downarrow} A$.

Lemma 7.13 $J_U = \bigvee \{SX_b \mid b \in U\}$.

Proof By J_U we mean the fibre of $\mathsf{J} \to \mathcal{D}A$ obtained by pulling back along the "name" $\ulcorner U \urcorner : \mathbf{1} \to \mathcal{D}A$ of $U \subset_{\downarrow} A$, giving the right end parallelogram.

$$\begin{tikzcd}
\mathsf{SX} \arrow[dd] & & \pi_0^*\mathsf{SX} \arrow[ll] \arrow[rr, "\nu_{\pi_1}"] \arrow[dd] & & \mathsf{C} \arrow[rr, two heads] \arrow[dd] & & \mathsf{J} \arrow[dd] \\
 & i^*\mathsf{SX} \arrow[ul] \arrow[ur] \arrow[rr, "\nu_!"] \arrow[dd] & & C_U \arrow[ur] \arrow[rr, two heads] \arrow[dd] & & J_U \arrow[ur] \arrow[dd] & \\
A & & (\in_A) \arrow[ll, "\pi_0"'] \arrow[rr, "\pi_1"] & & \mathcal{D}A \arrow[rr, "\mathsf{id}"] & & \mathcal{D}A \\
 & {|U|} \arrow[ul, "i"] \arrow[ur] \arrow[rr, "!"] & & \mathbf{1} \arrow[ur, "\ulcorner U \urcorner"'] \arrow[rr] & & \mathbf{1} \arrow[ur, "\ulcorner U \urcorner"'] &
\end{tikzcd}$$

By analogy with the universal property of the powerset, the pullback of $(\in_X)$ should give $U \subset_{\downarrow} A$. However, because of the way in which we defined $(\in_A)$ in Lemma 7.11, the subobject of it that $\ulcorner U \urcorner$ classifies has the *discrete* order instead, for which we write $|U|$. So, even though $U \subset_{\downarrow} A$ was lower (in $\mathcal{L}$), the composite $i : |U| \to (\in_A) \to A$ is not even in $\mathcal{R}$.

Nevertheless, the objects $i^*\mathsf{SX}$, C_U and J_U are prone liftings, giving more pullbacks. Then the cancellation property makes the top parallelograms into pullbacks too. The back left square ($\pi_0^*\mathsf{SX}$) is a pullback by construction and so is the left face by composition. However, the other two back squares and the two at the front need not be pullbacks.

By the Beck–Chevalley condition,

$$C_U \;=\; \Sigma_{a\in U} SX_a \qquad \text{and so} \qquad J_U \;=\; \bigvee_{a\in U} SX_a. \qquad \square$$

Lemma 7.14 $J_\emptyset = \emptyset$.

Proof Using Assumption 2.1(c): Since $i^*\mathsf{SX} \to U \equiv \emptyset$, we must have $i^*\mathsf{SX} = \emptyset$ too. Then its fibre coproduct $C_\emptyset$ is also $\emptyset$. Finally, $\emptyset \twoheadrightarrow J_\emptyset$, which is an isomorphism too. $\square$

Lemma 7.15 If $U \subset_\downarrow V \subset_\downarrow A$ then $J_U \hookrightarrow J_V$ by Remark 7.6. $\square$

Lemma 7.16 If U has a greatest element a then $J_U = SX_a$.

Proof Any $b \in U$ has $b \le a$ and so $SX_b \longrightarrow SX_a$ by Remark 7.6. Hence there is a coproduct mediator $C_U \longrightarrow SX_a$ but SX_a is extensional, so this factors

$$SX_a \hookrightarrow C_U \twoheadrightarrow J_U \hookrightarrow SX_a,$$

making $J_U \cong SX_a$. $\square$

Remark 7.17 Instead of using (infinitary) coproducts, [Tay21] resolved general colimits into pushouts and filtered colimits. The forgetful functor $\mathbf{Ens}^A \to \mathbf{Set}^A$ or $\mathsf{Ens} \to \mathbf{DFPos}$ *i.e.* to the category of discrete fibrations, creates filtered colimits of monos of the chosen class (Section 2).

When the underlying category is **Set** or **DFPos**, pushouts behave nicely, *cf.* [Tay21, §10]. If additionally the functor preserves inverse image diagrams (as the covariant powerset functor does),

(a) binary meets in **Ens** are constructed as pullbacks of discrete fibrations and
(b) binary joins in **Ens** are constructed as pushouts over pullbacks, yielding the "overlapping unions" from set theory.

8 Transfinite iteration of functors

In this section we will achieve transfinite iteration of the endofunctor $S : \mathbf{Set} \to \mathbf{Set}$ over a given $\mathcal{R}$- or $\mathcal{L}$-ordinal.

Remark 8.1 Beware that this will involve *two* functors and two uses of our theory:

(a) the functor S that we want to iterate, with fibred versions S^A and S, whose fibred category Ens of extensional well founded coalgebras we defined in Section 6 using the *fibre* factorisation in Definition 5.8(a);
(b) the new endofunctor T of Ens that we defined in Proposition 7.12, whose extensional well founded coalgebras according to the *base* factorisation system in Definition 5.8(c) will encode transfinite iteration of S.

In order to avoid lots of subscripts and prefixes, it is understood that CoAlg, $\mathsf{WfCoAlg}$ and Ens continue to refer to the categories of coalgebras for S. They will play a *passive* role in this section. When we want to talk about T-coalgebras, we will say so.

Fibred endofunctors

As in Section 4, it is clearer in the first instance to understand the meaning of T-well-foundedness and T-extensionality with respect to the base factorisation by considering a *generic* endofunctor T. Without the difficulties introduced by colimits or joins, we can do this first in **DFPos**, before lifting the ideas to the other categories..

Definition 8.2 A ***fibred endofunctor*** $\mathsf{T} : \mathbf{DFPos} \to \mathbf{DFPos}$ over a non-trivial functor $T_0 : \mathbf{Pos} \to \mathbf{Pos}$ is one that makes the square commute and preserves prone morphisms.

$$\begin{tikzcd}
\mathbf{DFPos} \arrow[r, "\mathsf{T}"] \arrow[d, "\mathsf{cod}"'] & \mathbf{DFPos} \arrow[d, "\mathsf{cod}"] \\
\mathbf{Pos} \arrow[r, "T_0"] & \mathbf{Pos}
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*\mathsf{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{X} \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}
\longmapsto
\begin{tikzcd}
(T_0 f)^*(\mathsf{TX}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{TX} \arrow[d] \\
T_0 A \arrow[r, "T_0 f"] & T_0 B
\end{tikzcd}$$

This means that for each $(\mathsf{X} \to B)$ in **DFPos** and $f : A \to B$ in **Pos**, we have

$$T_0(f)^*\mathsf{X} \cong \mathsf{T}(f^*\mathsf{X}),$$

where $(-)^*$ denotes prone lifting.

Lemma 8.3 T preserves base monos iff T_0 preserves $\mathcal{M} \subset \mathbf{Pos}$, because fibred endofunctors preserve prone maps. □

This simple observation has the result that fibred endofunctors *amplify* our notions of well-foundedness and extensionality.

Lemma 8.4 A T-coalgebra in **DFPos** is T-well founded iff its underlying T_0-coalgebra is T_0-well founded in **Pos**, *cf.* Lemma 4.9.

$$\begin{tikzcd}
\mathsf{TW} \arrow[rrrr, hook] \arrow[dr] & & & & \mathsf{TX} \arrow[dl] \\
& T_0 U \arrow[rr, hook, "T_0 i"] & & T_0 A & \\
& K \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\urcorner", very near start] & U \arrow[r, hook, "i"] & A \arrow[u, "\alpha"'] & \\
\mathsf{Y} \arrow[uuu] \arrow[ur] \arrow[rr] \arrow[ur, phantom, "\urcorner", very near start] & & \mathsf{W} \arrow[u] \arrow[rr, hook] \arrow[ur, phantom, "\urcorner", very near start] & & \mathsf{X} \arrow[ul] \arrow[uuu]
\end{tikzcd}$$

Proof The "predicate" testing T-well-foundedness in **DFPos** is the subobject defined by the lower right quadrilateral. The central sub-diagram from $T_0 U$ to A is a broken pullback in **Pos**, so that T_0-well-foundedness there makes $i : U \cong A$. Then its pullback $\mathsf{W} \to \mathsf{X}$ is also an isomorphism. □

We can now lift this to T-coalgebras whose underlying objects are S-coalgebras.

Corollary 8.5 A T-coalgebra in CoAlg, $\mathsf{WfCoAlg}$ or Ens is T-well founded iff its underlying T_0-coalgebra in **Pos** is T_0-well founded.

Proof In order to adapt the previous argument, we must replace the radial discrete opfibrations in the diagram with S-coalgebras. These are triangles whose radial sides are still discrete fibrations (Remark 6.5). In particular, the lower right pullback becomes (upside down)

$$\begin{tikzcd}
\mathsf{W} \arrow[rr] \arrow[dr, "i^*\xi"] \arrow[ddr] & & \mathsf{X} \arrow[dr, "\xi"] \arrow[ddr] & \\
& \mathsf{SW} \arrow[rr] \arrow[d] & & \mathsf{SX} \arrow[d] \\
& U \arrow[rr, "i"] & & A
\end{tikzcd}$$

The parallelograms in this are pullbacks, as in Lemma 6.7. Therefore, the same argument makes $\mathsf{W} \cong \mathsf{X}$. It also applies to the categories $\mathsf{WfCoAlg}$ and Ens. □

Lemma 8.6 A T-coalgebra in **DFPos**, CoAlg, $\mathsf{WfCoAlg}$ or Ens is T-extensional iff the underlying T_0-coalgebra $\alpha : A \to T_0 A$ in **Pos** is T_0-extensional ($\alpha \in \mathcal{M}$) and the T-coalgebra structure maps are prone.

$$
\begin{tikzcd}
\mathsf{X} \arrow[rr, hook] \arrow[dr, "\xi"] \arrow[ddr] & & \mathsf{TX} \arrow[dr, "\mathsf{T}_1\xi"] \arrow[dd] & \\
& \mathsf{SX} \arrow[rr, hook] \arrow[d] & & \mathsf{ST}_1\mathsf{X} \arrow[dl] \\
& A \arrow[r, hook, "\alpha"] & \mathsf{T}_0 A &
\end{tikzcd}
$$

Proof This is Definition 5.8(c) for base monos in **DFPos** and Lemma 6.7 showed that the same pullbacks are used in CoAlg, $\mathsf{WfCoAlg}$ and Ens. □

Transfinite iteration

Now we are finally ready to apply the extensional reflection arising from the base factorisation in Definition 5.8(c) to the particular endofunctor T of Ens that we defined in Proposition 7.12. This will provide transfinite iteration of the given funct*or* $S : \mathbf{Set} \to \mathbf{Set}$, just as Theorem 4.14 gave transfinite iteration of the funct*ion* $s : E \to E$.

Definition 8.7 T has $T_0 \cong \mathcal{D}$ and $(\mathsf{TX})_U \equiv J_U \equiv \bigvee_{\mathbf{Ens}} \{SX_b \mid b \in U\}$ in

$$
\begin{tikzcd}
X_a \arrow[rr] \arrow[dr, "\xi_a"] \arrow[ddr, mapsto] & & (TX)_{\alpha a} = \bigvee \{SX_b \mid b \prec a\} \arrow[dr, "(\mathsf{T}\xi)_{\alpha a}"] \arrow[dd, mapsto] & \\
& SX_a \arrow[rr] \arrow[d, mapsto] & & S(TX)_{\alpha a} \arrow[dl, mapsto] \\
& a \arrow[r, mapsto, "\alpha"] & \alpha a \equiv \{b \mid b \prec a\} &
\end{tikzcd}
$$

We apply Corollary 8.5 and Lemma 8.6 to this:

Lemma 8.8 A T-coalgebra in Ens is T-well founded iff its underlying $\mathcal{D}$-coalgebra in **Pos** corresponds to a binary relation that is well founded in the traditional sense.

Proof Lemma 3.3 and Corollary 8.5. □

Lemma 8.9 A T-coalgebra in Ens is T-extensional iff

(a) its underlying $\mathcal{D}$-coalgebra $\alpha : A \to \mathcal{D}A$ is an $\mathcal{R}$- or $\mathcal{L}$-ordinal; and

(b) the top map in the diagram in Definition 8.7 is a bijection for the op-fibre over each $a \in A$:

$$X_a \;\cong\; (\mathsf{TX})_{\alpha a} \;=\; \bigvee \{SX_b \mid b \prec a\}.$$

Proof Lemmas 3.5, 3.14 and 8.6 and Proposition 7.12. □

Theorem 8.10 The Proposal yields the cumulative hierarchy, *i.e.* the transfinite iteration of the covariant powerset functor. □

As in Example 4.17 we now apply this to the familiar situation of transfinite iteration.

Example 8.11 The following is a T-coalgebra in Ens that is T-well founded so long as (A, α) corresponds to a well founded binary relation:

$$
\begin{tikzcd}
\emptyset \arrow[rr, hook] \arrow[dr] \arrow[ddr] & & \mathsf{Z} \arrow[dr] \arrow[dd] & \\
& \mathsf{S}\emptyset \arrow[rr, hook] \arrow[d] & & \mathsf{SZ} \arrow[dl] \\
& A \arrow[r, hook, "\alpha"] & \mathcal{D}A &
\end{tikzcd}
$$

where $\mathsf{Z} = \mathsf{T}\emptyset$, so
(a) $Z_\emptyset = \emptyset$ by Lemma 7.14; and
(b) $Z_U = S\emptyset$ if U is inhabited, essentially by Lemma 7.16. □

Notation 8.12 Using the Proposal, we write its T-extensional reflection like this:

$$\begin{tikzcd}
\mathsf{X} \arrow[rr, hook] \arrow[dr] \arrow[dd] & & \mathsf{J} \arrow[dr] \arrow[dd] & \\
& \mathsf{SX} \arrow[rr, hook, crossing over] \arrow[dl] & & \mathsf{SJ} \arrow[dl] \\
\overline{A} \arrow[rr, hook, "\overline{\alpha}"] & & \mathcal{D}\overline{A} &
\end{tikzcd}$$

except that if we are using $\mathcal{M} \equiv \mathcal{R}$ and (A, α) is already $\mathcal{R}$-extensional (for example if it is a Cantor Normal Form) then $\overline{A} = A$, *etc.*

Lemma 8.13 $X_0 \ = \ J_\emptyset \ \equiv \ \emptyset$ by Lemma 7.14, since $\alpha(0) = \emptyset$. □

Lemma 8.14 If $a \leq b \in A$ then $X_a \hookrightarrow X_b$ by Lemma 7.15. □

Lemma 8.15 $X_{a^+} \ = \ SX_a$ by Lemma 7.16, if $\alpha(a^+)$ has a greatest element a, in particular if a^+ is either the one-point or plump successor. □

Proposition 8.16 Using $\mathcal{M} \equiv \mathcal{R}$, if (A, α) is $\omega \equiv (\mathbb{N}, \leq, <)$ or ω^+ from Example 3.8 then

$$X_0 \ = \ \emptyset, \qquad X_{n+1} \ = \ SX_n \hookleftarrow X_n \qquad \text{and} \qquad X_\omega \ = \ \mathsf{colim}^\uparrow X_n.$$

Proof The first two parts are the previous lemmas. Consequently, the join in the third is directed and $SX_n = X_{n+1}$, so the join with either n or $n+1$ is the same. □

Remark 8.17 Classically, $\mathcal{P}(X) \cong \mathbf{2}^X$ and there is a bijection $X_\omega \cong \mathbb{N}$, so this can be constructed in an elementary topos with $\mathbb{N}$.

However, we need Replacement or the Proposal to obtain the transfinite iteration or extensional reflection in the case where A is $\omega \cdot 2 + 1$, with top value

$$X_{\omega \cdot 2} \ = \ \mathsf{colim}^\uparrow S^n(X_\omega).$$

Remark 8.18 We said that we were not going "to do something infinitely often, and then some more" but now that we have actually proved the Theorem from the Proposal we can "let our hair down" and give some intuition.

The recursive argument that we have given constructs the *entire sequence* and not just a particular value. As in Example 8.11, it begins with the constantly $\emptyset$ sequence in the first row.

Applying T to the whole of a row yields the next row down. Its value at a is the colimit of

$\{SX_b \mid b \prec a\}$, where X_b is taken from the previous row:

$$\begin{tikzcd}
\underline{\emptyset} \arrow[r, "\mathsf{id}"] & \emptyset \arrow[r, "\mathsf{id}"] & \emptyset \arrow[r, "\mathsf{id}"] & \emptyset \arrow[r, "\mathsf{id}"] & \cdots \arrow[r, "\mathsf{id}"] & \emptyset \\
\emptyset \arrow[r, "\eta_\emptyset"] & \underline{S\emptyset} \arrow[r, "\mathsf{id}"] & S\emptyset \arrow[r, "\mathsf{id}"] & S\emptyset \arrow[r, "\mathsf{id}"] & \cdots \arrow[r, "\mathsf{id}"] & S\emptyset \\
\emptyset \arrow[r, "\eta_\emptyset"] & S\emptyset \arrow[r, "S\eta_\emptyset"] & \underline{S^2\emptyset} \arrow[r, "\mathsf{id}"] & S^2\emptyset \arrow[r, "\mathsf{id}"] & \cdots \arrow[r, "\mathsf{id}"] & S^2\emptyset \\
\emptyset \arrow[r, "\eta_\emptyset"] & S\emptyset \arrow[r, "S\eta_\emptyset"] & S^2\emptyset \arrow[r, "S^2\eta_\emptyset"] & \underline{S^3\emptyset} \arrow[r, "\mathsf{id}"] & \cdots \arrow[r, "\mathsf{id}"] & S^3\emptyset \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\emptyset \arrow[r, "\eta_\emptyset"] \arrow[d, mapsto] & S\emptyset \arrow[r, "S\eta_\emptyset"] \arrow[d, mapsto] & S^2\emptyset \arrow[r, "S^2\eta_\emptyset"] \arrow[d, mapsto] & S^3\emptyset \arrow[r, "S^3\eta_\emptyset"] \arrow[d, mapsto] & \cdots \arrow[r] \arrow[d, mapsto] & ? \arrow[d, mapsto] \\
0 & 1 & 2 & 3 & 4 & \omega
\end{tikzcd}$$

This provides the final value of the iteration at one more place in each row, as indicated by underlining. In the last row we form the directed colimit, providing the final value, (?). $\square$

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