Tychonov's theorem in Abstract Stone Duality

Paul Taylor

EPSRC GR/S58522
University of Manchester

www.cs.man.ac.uk/~pt/ASD

temporarily: www.cs.man.ac.uk/~pt/drafts/Tychonov.dvi

Cantor space $2^{\mathbb{N}}$ exists and **is** compact

contradicts Escardó "Synthetic Topology"

ASD: solo full time work since 1997
3 papers published in TAC
4 more complete draft papers on web.

Achievements of ASD

Axiomatise topology directly,
not via sets of points or opens.

Free-standing type theory (λ-calculus)
not based on pre-existing
textbook categories of spaces/locales.

Recursive: $\mathbb{N} \to \mathbb{N}_\perp$ are partial
recursive functions (actually,
equivalence classes of programs)

But recursion theory in ASD comes
naturally, not "bolted on"

Free model is
Equivalent to: locally compact locales
with computable bases.

Extra ("underlying set") axiom $\Rightarrow$
equivalent to locally compact locales
over an elementary topos.

Synthetic proofs of:
direct image of compact is compact
Compact subspace of Hausdorff is closed
closed subspace of compact is compact
compact power of discrete is discrete
etc etc etc.

Extended axiomatisation for bigger categories:
I need some help from experts in PERs,
equilogical spaces etc

The categorical ingredients.

ideas originally due to Marshall Stone

- topology is dual to algebra
- always topologize!

categorically:

$$\begin{tikzcd} \text{algebras} & \mathcal{S}^{op} \arrow[d, shift left=1.5, "\Sigma^{-}"] \\ \text{spaces} & \mathcal{S} \arrow[u, shift left=1.5, "\Sigma^{-}"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

"dual" = opposite categories
"algebras" = monadic adjunction

= sobriety + subspaces

But set theory, order theory, algebraic varieties, quantum computation (?), ... are examples of this too.

For topology, Σ classifies open + closed sets:

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ X \arrow[r, dashed] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r] \arrow[d, hook] & 1 \arrow[d, "\bot"] \\ X \arrow[r, dashed] & \Sigma \end{tikzcd}$$

Comprehension Calculus

$$\frac{X \text{ type} \qquad \text{subtype data on } X}{\{X \mid \text{subtype data}\} \text{ type}}$$

$$\frac{\Gamma \vdash a : X \qquad a \text{ is admissible}}{\text{admit } a : \{X \mid \text{subtype data}\}}$$

$$y : \{X \mid \text{subtype data}\} \vdash iy : X$$

iy is admissible

$$i(\text{admit } a) = a$$

For example, in set theory

subtype data on X = predicate $\vdash \xi : \Sigma^X$

admissibility = $\xi[a] = \top$

Comprehension calculus

$$\frac{X \text{ type} \qquad x:X,\ \varphi:\Sigma^X \vdash E\varphi x:\Sigma \qquad E \text{ is a nucleus}}{\{X \mid E\} \text{ type}}$$

$$\frac{\Gamma \vdash a:X \qquad \Gamma,\ \varphi:\Sigma^X \vdash \varphi a = E\varphi a}{\Gamma \vdash \mathrm{admit}_{X,E}\, a : \{X \mid E\}}$$

$$x:\{X \mid E\} \vdash i_{X,E}\, x : X$$

$$x:\{X \mid E\} \quad \varphi:\Sigma^X \quad \vdash \quad \varphi(i_{X,E}\, x) = E\varphi(i_{X,E}\, x)$$

$$a = i_{X,E}(\mathrm{admit}_{X,E}\, a) : X \qquad \beta$$

$$x = \mathrm{admit}_{X,E}(i_{X,E}\, x) : \{X \mid E\} \qquad \eta$$

$$\vartheta : \Sigma^{\{X \mid E\}} \vdash I_{X,E}\,\vartheta : \Sigma^X$$

$$\varphi:\Sigma^X \vdash I_{X,E}(\lambda x:\{X \mid E\}.\, \varphi(i_{X,E}\, x)) = E\varphi \qquad \beta$$

$$\vartheta:\Sigma^{\{X \mid E\}},\ x:\{X \mid E\} \vdash \vartheta x = (I\vartheta)(i x) \qquad \eta$$

Compact (sub) spaces.

... (long history of compactness) ...

Bourbaki Axiom C''': every open cover has a finite subcover

Wilker 1970: compact subspace as filter of open neighbourhoods

... (theory of continuous lattices) ...

locale theory: K compact if
$!_K^*$ ~~Has~~ a Scott continuous right adjoint

ASD: <u>only</u> axiomatise (Scott) continuous functions

K compact if $\Sigma^{!_K}$ has a right adjoint

equivalently:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, "\top"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
\Sigma^K \arrow[r] & \Sigma
\end{tikzcd}$$

what to call this right adjoint?

Compactness as Universal Quantifier

$$\begin{tikzcd} K \arrow[d, "!"'] & \Sigma^K \arrow[d, "\forall_K", shift left=2] \\ 1 & \Sigma \arrow[u, "\Sigma^!", shift left=2] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

$$\frac{(\Gamma,)\ x : K \vdash \sigma \le \varphi x}{(\Gamma) \vdash \sigma \le \forall x.\, \varphi x}$$

$$\begin{tikzcd} \Gamma \times K \arrow[r, "u \times K"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta \times K \arrow[d] \\ \Gamma \arrow[r, "u"] & \Delta \end{tikzcd} \qquad \begin{tikzcd} (\Sigma^K)^\Gamma \arrow[d, "\forall_K^\Gamma"'] & (\Sigma^K)^\Delta \arrow[l, "(\Sigma^u)^K"'] \arrow[d, "\forall_K^\Delta"] \\ \Sigma^\Gamma & \Sigma^\Delta \arrow[l, "\Sigma^u"] \end{tikzcd}$$

Beck-Chevalley condition.

symbolically

$$\frac{\Gamma, x : K \vdash \sigma \le \varphi x}{\Gamma \vdash \sigma \le \forall x.\varphi x}\ \forall\text{-intro} \qquad \frac{\Gamma \vdash \sigma \le \forall x.\varphi x}{\Gamma, x : K \vdash \sigma \le \varphi x}\ \forall\text{-elim}$$

Beck-Chevalley = substitution under $\forall_K$

But also:

$$\forall x.(\varphi x \vee \sigma) = (\forall x.\varphi x) \vee \sigma$$

cf $\quad \exists x(\varphi x \wedge \sigma) = (\exists x.\varphi x) \wedge \sigma$

However

$$\forall \neq \text{for every.}$$

Intuitive universal quantifier on $2^{\mathbb{N}}$

predicate $F : \Sigma^{2^{\mathbb{N}}}$ or $2^{\mathbb{N}} \to \Sigma$

is true "universally" for $s : 2^{\mathbb{N}}$

if <u>either</u> $Fs\downarrow$ without examining s

<u>or</u> $F(0::s)\downarrow$ & $F(1::s)\downarrow$ ditto.

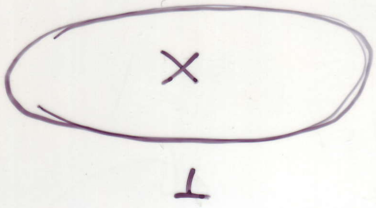
<u>or</u> $F(00::s)\downarrow$ & $F(01::s)\downarrow$ & $F(10::s)\downarrow$ & $F(11::s)\downarrow$

<u>or</u> $F(000::s)\downarrow$ ... $F(111::s)\downarrow$

<u>or</u> 16 cases at depth 4

<u>or</u> etc.

Lifting



$W \subset X_\perp$ is $U \subset X$ with $V \subset \{\perp\}$

such that $\perp \in V \Rightarrow X \subset U$

so $\Sigma^{X_\perp} = \{(\sigma, \varphi) : \Sigma \times \Sigma^X \mid \sigma \leq \varphi\}$

Theorem

$X_\perp$ is the partial map classifier

$$\begin{tikzcd}
U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, hook] \\
Y \arrow[r, dashed] & X_\perp
\end{tikzcd}
\qquad
\frac{Y \longrightarrow X}{Y \longrightarrow X_\perp}$$

this is not obvious
it works for topology, not set theory
it depends on the Phoa principle

$$\sigma : \Sigma \quad F : \Sigma^\Sigma \vdash \quad F\sigma = F\perp \vee \sigma \wedge F\top$$

assuming Scott-continuity axiom
$H : \Sigma^Y \to \Sigma^X$ is Σ^f iff H preserves $\top \perp \wedge \vee$
then the Thm. is easy to prove.

it's true without this, but is not easy to prove
central idea: modular law

Lifting a compact space

$$
\begin{tikzcd}
K_\perp \\
K \text{ compact} \arrow[u, hook, "i"', "\text{open}"] \arrow[d, "!"'] \\
1
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=large]
\Sigma^{K_\perp} = \Sigma \downarrow \Sigma^K \arrow[d, "\Sigma^i \equiv \pi_1"'] \\
\Sigma^K \arrow[u, bend left=70, "\exists_i \equiv (\bot, \mathrm{id})"] \arrow[u, bend right=70, "R_i \equiv (\forall_K, \mathrm{id})"'] \arrow[d, shift left=2, "\forall_K"] \\
\Sigma \arrow[u, shift left=2, "\Sigma^!"]
\end{tikzcd}
$$

$K \subset K_\perp$ is an open subspace (definition of $K_\perp$)

$K \subset K_\perp$ is a compact subspace

recovered using nucleus

$$J \equiv R_i \cdot \Sigma^i : (\sigma, \varphi) \mapsto (\forall k. \varphi k, \varphi)$$

in terms of $\Sigma^{K_\perp}$

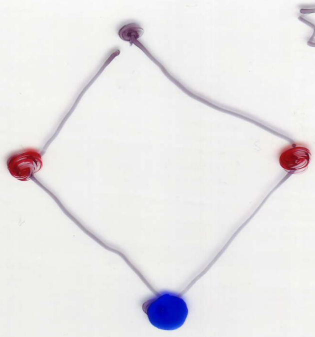
$$J \psi x = \psi x \vee \forall k. \psi(ik)$$

Construction of Cantor Space

$K = 2$ $\quad 2^N \underset{\text{compact}}{\rightarrowtail} (2_\perp)^N \underset{\text{closed}}{\hookrightarrow} (\Sigma \times \Sigma)^N$

N^{th} power of $2 \underset{\substack{\text{open \&}\\ \text{compact}}}{\hookrightarrow} 2_\perp \hookrightarrow \Sigma \times \Sigma$

$\Sigma \times \Sigma$



$$
\begin{tikzcd}
2_\perp \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\perp"] \\
\Sigma \times \Sigma \arrow[r, "(\sigma, \tau) \mapsto \sigma \wedge \tau"] & \Sigma
\end{tikzcd}
\qquad
\begin{tikzcd}
2_\perp^N \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\perp"] \\
\Sigma^N \times \Sigma^N \arrow[r, "(\varphi, \psi) \mapsto \exists n.\, \varphi n \wedge \psi n"] & \Sigma
\end{tikzcd}
$$

$2^N \subset 2_\perp^N$ is N-fold

intersection of compact open subspaces

so it's compact, but not open

given by "join" of nuclei
(Hofmann-Mislove).

N = either $\mathbb{N}$ or any overt discrete Hausdorff

Cantor space as a nucleus.

J_n nucleus on $(K_\perp)^N$.
defining subspace of $f : K_\perp^N$
such that $fn\downarrow$ ie $fn \in K \subset K_\perp$

$n:N \quad f : \Omega \equiv K_\perp^N \quad F : \Sigma^\Omega$

$\vdash \quad J_n F f \equiv F f \vee \forall k:K . F(\lambda m. \begin{array}{l}\text{if } m=n \text{ then } k \\ \text{else } fm\end{array})$

Lemma: J_n is a (family of) nuclei
(in both sense of locales
& sense of ASD) and $id \le J_n$

Lemma: f is admissible wrt J_n
ie $F : \Sigma^\Omega \vdash J_n F f = F f$
equivalently $J_n F f \le F f$
iff $fn\downarrow$

idea: $f \in K^N$ iff $J_n F f \le F f$
for all n.

so nucleus is "join" of J_n.

Cantor space as a nucleus.

Lemma J_n, J_m commute.

Lemma $J_n \cdot J_m \equiv J_m \cdot J_n$ is a nucleus

f admissible ~~wrt~~ $J_n \cdot J_m$
iff it's admissible wrt both J_n, J_m

Lemma $\left\{ \begin{array}{l} J_0 = id \\ J_{n::l} = J_n \cdot J_l \end{array} \right\}$ is a family of nuclei

Lemma $J \equiv \exists l.\, J_l$ (directed join)
is a nucleus.

Proposition. for $\Gamma \vdash f : K_\perp^N$ ~~[illegible]~~ TFAE

- $\Gamma \vdash (\lambda n.\ f n \perp = \top) : \Sigma^N$
- $\Gamma, n{:}N \vdash f n \perp = \top : \Sigma$
- $\Gamma, n{:}N, F : \Sigma^\Omega \vdash J_n F f = F f$
- $\Gamma, l : KN, F : \Sigma^\Omega \vdash J_l F f = F f$
- $\Gamma, F : \Sigma^\Omega \vdash J F f = F f$
- $\Gamma \vdash f : \{\Omega \mid J\}$

Cantor space is compact

$$
\begin{tikzcd}
K^{\mathbb{N}} = \{ K_\perp^{\mathbb{N}} \mid J \} \arrow[r, tail, "i"] & K_\perp^{\mathbb{N}} \arrow[r, shift left, "!"] & 1 \arrow[l, shift left, "\perp"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Sigma^{(K^{\mathbb{N}})} \arrow[r, shift right, "\perp"'] \arrow[r, phantom, "\perp"] & \Sigma^{(K_\perp^{\mathbb{N}})} \arrow[l, two heads, shift right, "\Sigma^i"'] \arrow[r, shift right, "ev_\perp"'] \arrow[r, phantom, "\perp"] & \Sigma \arrow[l, shift right, "\Sigma^!"']
\end{tikzcd}
$$

calculus for $\{X \mid J\}$
["Subspaces in ASD", TAC 2002]

provides $i : \{X \mid J\} \rightarrowtail X$
admit
$I : \Sigma^{\{X \mid J\}} \rightarrowtail \Sigma^X$
s.t. $\Sigma^i \cdot I = \mathrm{id}$ $\quad J = I \cdot \Sigma^i$
but $\mathrm{id} \leq J$ so $\Sigma^i \dashv I$.

trivially, $\Sigma^! \dashv ev_\perp$ makes $K_\perp^{\mathbb{N}}$ compact

so $\forall_{K^{\mathbb{N}}} = ev_\perp \cdot I$.

or $\Box_{K^{\mathbb{N}}} = \forall_{K^{\mathbb{N}}} \cdot \Sigma^i = ev_\perp \cdot I \cdot \Sigma^i = ev_\perp \cdot J$

so $\Box F = J F \perp$

the Universal Quantifier as a Program.

For general overt discrete Hausdorff N

$$\begin{aligned} JFf &= \exists l.\, J_l F f \\ &= J_0 F f \vee \exists n l.\, J_{n::l} F f \\ &= \ldots \\ &= JFf \vee \exists n.\, J(\forall k.\, F(\lambda m.\, \begin{smallmatrix}\text{if } m=n \text{ then } k \\ \text{else } fm\end{smallmatrix})) \end{aligned}$$

For $N = \mathbb{N}$

$$JFf = J(\lambda g. F(f0::g) \vee \forall k. F(ik::g))(\text{tail} f)$$

so $\square F = JF\bot = \square(\lambda g.\, \forall k.\, F(ik::g))$

$$= \forall k.\, \square(\lambda g.\, F(ik::g))$$

or for $k = 2$

$$\begin{aligned} \square F &= \square(\lambda g.\, F(0::g) \wedge F(1::g)) \\ &= \square(\lambda g.\, F(0::g)) \wedge \square(\lambda g. F(1::g)) \end{aligned}$$

termination of this as a program?

this is the same as the intuitive universal quantifier.

Kleene trees.

program $f : nat \to bool$

is called a **(total/partial) stream**

denotational semantics

$$[\![f]\!] : 2_\bot^{\mathbb{N}}$$

program $P : (nat \to bool) \to unit$

$$[\![P]\!] : \Sigma^{(2_\bot^{\mathbb{N}})}$$

is called a **drain**

superficial $P(\bot)\downarrow$
shallow $\exists n.\ \forall l.\ |l| = n.\ P(l :: \bot)\downarrow$
deep $\forall s.\ Ps\downarrow$
blocked $\exists s.\ Ps\uparrow$

Can we distinguish the four types of drain?

superficial — succeeds on partial stream $\perp$

$\vee \neq$

fails on some partial stream

shallow

intuitive quantifier succeeds

$\vee \neq$

deep — intuitive quantifier fails

succeeds on every total stream

$\vee \neq$

blocked — fails on some total stream

Do deep drains exist?

Naively no: moment of success on a stream
→ pruning of binary tree

→ Finitely branching tree
with no infinite branch

→ Finite König's lemma.

Wrong! (Kleene)

there's an infinite binary tree
with no computable infinite path

Drain D with $\delta = [\![D]\!] : \Sigma^{2^{\mathbb{N}}_{\bot}}$

such that

$$\frac{\vdash \; s : 2^{\mathbb{N}} \text{ well formed}}{\vdash \; \delta s \downarrow}$$

but $s : 2^{\mathbb{N}} \nvdash \delta s \downarrow$

ie $\nvdash \Sigma^{i}\delta \neq \top : \Sigma^{2^{\mathbb{N}}}$