Abstract Stone Duality

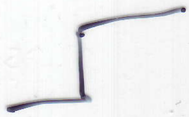
Marshall Stone (1937): "always topologize"

ASD is a <u>direct</u> axiomitisation of topology
<u>not</u> of sets + added topological structure

types are "topological spaces"
terms are "continuous functions"

Continuous for $\mathbb{R}$ — Weierstrass

Continuous ~~for~~ lattices such as $\Sigma^{\mathbb{R}}$ — Scott

"Discontinuous" functions eg 
— use other types.

Recursive analysis / topology

Traditional picture

Recursive Topology

Topology (lattice of "open" sets)

Recursion Theory

Set theory

Turing machines

uncountably many, mainly pathological,
set-theoretic functions
try to control them by cutting down to
continuous functions
computable functions

A horrendous mess!

ASD picture

Natural axiomisation of topology

lattices of open sets etc

programs

recursive "set theory"
(Arithmetic universe)

"Many-sorted algebra" for $\mathbb{R}$.

types: $\mathbb{R}$ itself

$\mathbb{N}$ mainly needed for recursion

 $\equiv \Sigma$ Sierpiński space of truth values $(\top, \bot)$

$\Sigma^{\mathbb{R}}, \Sigma^{\mathbb{N}}$ topologies of $\mathbb{R}$ & $\mathbb{N}$

$\Sigma^{\Sigma^{\mathbb{R}}}$ used for terms denoting compact subspaces.

$\varphi : \Sigma^{\mathbb{R}}$ denotes an <u>open</u> subspace of $\mathbb{R}$ (inverse image of $\top$)

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & 1 \arrow[d, "\top"] \\ \mathbb{R} \arrow[r, "\varphi"] & \Sigma \\ C \arrow[r] \arrow[u, hook] & 1 \arrow[u, "\bot"] \end{tikzcd}$$

also <u>closed</u> subspace (inverse image of $\bot$)

can write

$<$ $>$ $\neq$ on $\mathbb{R}$

$<$ $>$ $\neq$ $\leq$ $\geq$ $=$ on $\mathbb{N}$

as open subspaces

can use

$\top$ $\bot$ $\wedge$ $\vee$ $\exists n : \mathbb{N} \ldots$ $\exists x : \mathbb{R} \ldots$ in the logic

also $\forall x : [0,1] \ldots$

NOT $\neg$ $\Rightarrow$ $\forall n \ldots$ $\forall \varepsilon > 0 \ldots$

"Many sorted algebra" for $\mathbb{R}$

		terms of type:	
	$\mathbb{N}$	$\mathbb{R}$	Σ
variables:	k, n, m	a, b, x, y, z	$\delta, \sigma, \varphi, \psi, \vartheta$
constants:	0	$0, 1, (\frac{1}{2})$	$\top, \bot$
arguments of type: $\mathbb{N}$:	succ	$\wedge$	$=, \leq, \geq, \neq, >, <$
$\mathbb{R}$:		$+, -, \times, (\div)$	$< > \neq$
$\mathbb{N}$ & Σ:			$\exists n \ldots$
$\mathbb{R}$ & Σ:			$\exists x \ldots$ $\forall x{:}[0,1] \ldots$
$\mathbb{N}$ & ?:	primitive recursion at all types		
Σ:	definition by description	Dedekind completeness	$\wedge, \vee$

Heine - Borel theorem (compactness)

(almost) usual definition:

$$K \subset \bigcup^{\uparrow} U_i \quad \Rightarrow \quad \exists i.\ K \subset U_i$$

directed union — one suffices

This says that $\lambda U.\ K \subset U$ is
Scott continuous in U

So it's a value of type $(\mathbb{R} \to \Sigma) \to \Sigma$
(Since U is a value of type $\mathbb{R} \to \Sigma$)

In the definition
"Every open cover has a finite subcover"
the word <u>cover</u> is crucial:
finiteness comes for free (Scott continuity)

We write φ for U
$\forall x : K.\ \varphi x$ for $K \subset U$

Heine-Borel doesn't come for free.

Can interpret "Dedekind cuts" in any category with $\mathbb{N}$, $\Sigma^{(-)}$ and equalisers
$\cong\mathbb{Q}$

$$R \rightarrowtail \Sigma^{\mathbb{Q}} \times \Sigma^{\mathbb{Q}} \rightrightarrows \left(\begin{array}{c}\text{types of equations} \\ \text{for Dedekind cuts}\end{array}\right)$$

Examples:

- Predomains, where order determines topology
- Recursive set theories.

In predomains, R has discrete topology

In type 1 recursive analysis,

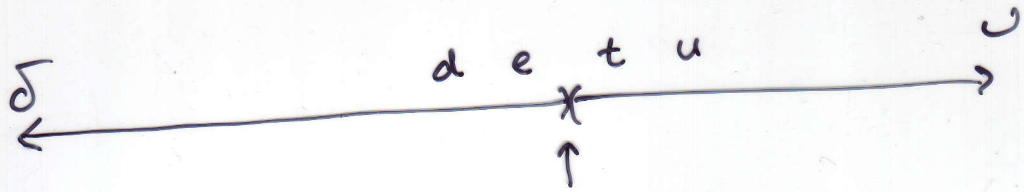
$[0,1] \subset \bigcup U_k$ where $|U_k| < 2^{-k}$

"cover each recursively defined real number"

no finite subset will do.

ASD is not just topology in a cartesian closed category.

Two sided Dedekind cuts.



$a = (\text{the } du.\ \delta d \wedge \upsilon u)$

(rounded)

$\exists e.\ \delta e \wedge (d<e) \Leftrightarrow \delta d$ $\qquad \upsilon u \Leftrightarrow \exists t.\ (t<u) \wedge \upsilon u$

$\exists d.\ \delta d$ (bounded) $\qquad \exists u.\ \upsilon u$

disjoint: $\delta d \wedge \upsilon u \Rightarrow d<u$

order-located: $\delta d \vee \upsilon u \Leftarrow d<u$

arithmetically located:

$\varepsilon > 0 \Rightarrow \exists du.\ \delta d \wedge \upsilon u \wedge |u-d| < \varepsilon$

in practice:

rounded, bounded & disjoint: easy

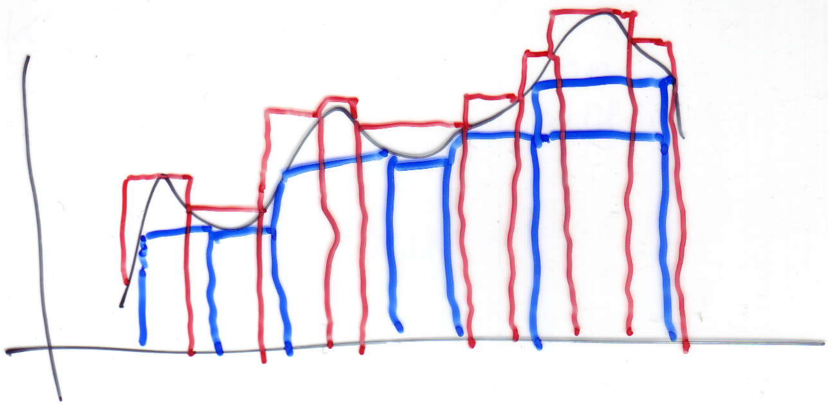
located: difficult

∴ useful to consider "pseudo-cuts" that are rounded, bounded & disjoint

classically pseudo-cuts = intervals.

Dedekind cuts in analysis

Integration:



$$d < \int_a^b fx\,dx < u \quad \text{iff}$$

$\exists$ elementarily integrable functions $f^\pm$
eg rectangles.

$$d < \sum_a^b f^- \;\wedge\; \forall x:[a,b].\, f^-x < fx < f^+x \;\wedge\; \sum_a^b f^+ < u$$

Limits: Cauchy sequence (a_n) with
$|a_m - a_n| < 2^{-\min(n,m)}$

$$d < \lim a_n < u \quad \text{iff} \quad \exists n.\, d < a_n - 2^{-n} < a_n + 2^{-n} < u$$

Differentiation:

$e_0 < fx < t_0$ & $e_1 < f'x < t_1$ iff

$$\exists \delta > 0.\, \forall h:[0,\delta].\; e_0 + e_1 h < f(x+h) < t_0 + t_1 h$$
$$\wedge\; e_0 - e_1 h < f(x-h) < t_0 - e_1 h$$

introduction rules for

λ-abstraction, description & Dedekind

$$\frac{\begin{array}{c}[P]\\ \vdots\\ Q\end{array}}{P\Rightarrow Q}\qquad\qquad \frac{\begin{array}{c}[x:P]\\ \vdots\\ t:Q\end{array}}{\lambda x.t : P\to Q}$$

$$\frac{\begin{array}{c}[n:\mathbb{N}]\\ \vdots\\ \varphi n:\Sigma\end{array}\qquad \exists n.\varphi n\Leftrightarrow T\qquad \begin{array}{c}\left[\begin{array}{c}n; m:\mathbb{N}\\ \varphi n\Leftrightarrow\varphi n\Leftrightarrow T\end{array}\right]\\ \vdots\\ n=m\end{array}}{(\text{the } n.\ \varphi n):\mathbb{N}}$$

$$\frac{\begin{array}{c}[d:\mathbb{R}]\\ \vdots\\ \delta d:\Sigma\end{array}\qquad \begin{array}{c}[u:\mathbb{R}]\\ \vdots\\ \upsilon u:\Sigma\end{array}\qquad \text{axioms for Dedekind cut}}{(\text{cut } d\,u.\ \delta d\wedge\upsilon u):\mathbb{R}}$$

elimination, β- and η- rules

$$E: \quad \frac{a : P \qquad f : P \to Q}{fa : Q}$$

$$\beta: \quad (\lambda x.\ f(x))\, a \overset{\text{substitution}}{=} f(a)$$

$$\eta: \quad (\lambda x.\ f x) = f$$

$$E: \quad \frac{(\text{the } n.\varphi n) : \mathbb{N}}{\exists n.\, \varphi n \Leftrightarrow T} \qquad \frac{(\text{the } n.\varphi n) : \mathbb{N} \qquad \varphi m_1 \Leftrightarrow \varphi m_2 \Leftrightarrow T}{m_1 = m_2}$$

$$\beta: \quad \frac{(\text{the } n.\varphi n) : \mathbb{N}}{\varphi(\text{the } n.\varphi n) \Leftrightarrow T} \qquad \frac{(\text{the } n.\varphi n) : \mathbb{N} \qquad \varphi m}{(\text{the } n.\varphi n) = m}$$

$$\eta: \quad (\text{the } n.\ n = m) = m$$

E: if $(\text{cut } du.\ \delta d \wedge \upsilon u) : \mathbb{R}$ is well formed
δ and υ are a Dedekind cut

β: $e < (\text{cut } du.\ \delta d \wedge \upsilon u) < t \Leftrightarrow \delta e \wedge \upsilon t$

η: $\delta d \equiv (d < a) \qquad \upsilon u \equiv (a < u)$
define a Dedekind cut.