Who should get credit
for the "Bourbaki–Witt" Theorem?

Paul Taylor

8 February 2026

The "Bourbaki–Witt" theorem says:

- Let X be a poset

- with joins of chains,

- in particular a least element $\perp$, and

- $s : X \to X$ an endofunction of X

- such that $\forall x \in X.\, x \leq sx$.

- Then s has a least fixed point.

The key argument in this, which I call the "catenary lemma". says that, for the subset $K \subset X$ generated by $\perp$, s and whatever joins exist,

$$\forall xy \in K. \quad x \leq sy \quad \vee \quad x = y \quad \vee \quad y \leq sx.$$

I am trying to work out who gave the first correct proof of this.

In fact it was first stated and proved in the case where X is either a full powerset $\mathcal{P}(M)$ or at least a family of subsets of M and the order $\leq$ was either $\subseteq$ or $\supseteq$. However, that was because abstract order theory had not yet been developed: the notion of *partially ordered set* was first defined in *Mengenlehre.* I do not consider that *new* credit is due for merely re-writing a set-theoretic result in oder-theoretic notation.

The candidates for original credit are not the Bourbaki group [Bou39b, Bou49] or Ernst Witt [Wit50, Wit51] but

- Ernst Zermelo *Neuer Beweis* [Zer08],

- Gerhard Hessenberg *Kettentheorie* [Hes09]

- Felix Hausdorff *Mengenlehre* [Hau14] and

- Kazimierz Kuratowski *Élimination* [Kur22].

Specifically the issue is whether §§144–8 of *Kettentheorie* contains a valid proof of the catenary lemma.

Zermelo, Hessenberg and the philosopher Leonard Nelson were collaborators.

Zermelo wrote *Neuer Beweis* in Chesières in the Swiss Alps, where he was recovering from tuberculosis, and dated it 14 July 1907. Hessenberg wrote to Nelson on 7 September, saying that he had seen the printer's proofs of *Neuer Beweis* on the previous day [EP07, §2.8.1] and it was published in mid 1908.

Neuer Beweis has a valid proof that the axiom of choice defines a well-ordering on any set. It uses a version of the catenary lemma, but this is only valid for this specific application, since it relies on the fact that the successor of a set differs from the set by exactly one element.

The key part of the proof is in the paragraph beginning "Now let A..." in the English translation in [Ebb10], specifically

> every V_A', since it cannot be a U_A and yet must be an element of M, is either A itself or a V_A

So far as I am aware, Zermelo originated this argument and so he deserves credit, but only partially, because this version is not adequate to prove the abstract general form of the catenary lemma and so the "Bourbaki–Witt" theorem.

Hesseneberg signed off *Kettentheorie* in Bonn in August 1908 and it was published in December 1909, giving full citations for all of Zermelo's paper. In other words, it was written in immediate response to *Neuer Beweis*. Unfortunately, its notation is utterly idiosyncratic. §§144-9 might be a treatment of the catenary lemma, but only only someone who has studied the notation in detail will be able to confirm this.

Even so, did Hessenberg recognise the limitation of Zermelo's proof and give the more general one? Surely if he had done this then he would have said what the problem and the correction was.

The next version is Hausdorff's in *Mengenlehre*, four or five years later. This book appeared in several German editions and the English translation is based on the third, but much of the original material was omitted or re-written.

The catenary lemma appears in Chapter V, §1, rather more clearly than in *Neuer Beweis*. However, Zermelo's limitation is still there, in paragraph V.

After that we have Kuratowski's *Élimination*, which includes the catenary lemma as Theorem III. This version is valid in general, so unless the proof in *Kettentheorie* can be shown to be correct, Kuratowski deserves credit for this as well as for "Zorn's Lemma" [Cam78] and "Tarksi's Theorem".

After 1922, there seems to be no mention whatever of the catenary lemma until the early draft [Bou39b] of Bourbaki's original 1939 *Théorie des Ensembles* [Bou39a]. The scanned typescript of this draft is available in the online Bourbaki archives but the published *Fascicule des Résultats* is unobtainable.

This draft contains (apparently) the same proof that appeared ten years later in a paper [Bou49]. This proof seems to me to be the clearest of all the published versions.

The draft credits Hessenberg but the paper Zermelo.

Another approach to the catenary lemma also began in 1939 with a paper by Arthur Milgram that I cannot access [Mil39]. This was given at a colloquium hosted by Karl Menger, who had left Vienna in 1935, where he had run a similar colloquium that was loosely linked to the Vienna Circle of mathematicians, philosophers and economists,

Milgram's methods were taken up by others in the 1950s and these papers were reviewed by Walter Felscher [Fel62], who subsequently commented on the *Historia Mathematica* email forum on 17 April 2000, shortly before he died:

> *Kettentheorie und Wohlordnung* can hardly be underestimated in its importance. Not that it was understood by his contemporaries. But Hessenberg, analyzing Zermelo's second proof of the well-ordering theorem, studied the general ways to construct well ordered subsets of ordered sets — with the one restriction that order always was inclusion and ordered sets were subfamilies of power sets. In the course of this, Hessenberg stated and proved the fixpoint theorem which thirty years later was rediscovered — for ordered sets now — by Nicolas Bourbaki. The amazing thing is that Hessenberg's proof is precisely the same as that given by Bourbaki! (only that at one small point a simpler argument can be used due to the circumstance that Hessenberg's order is inclusion).

This is apparently a claim that he had studied the proof in *Kettentheorie* and verified that the Bourbaki proof was the same.

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