COMPLEX VARIABLE

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1. Complex differentiation.

Recall that a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ or $f : \mathbb{C} \to \mathbb{C}$ is <u>differentiable</u> at $\underline{x}$ with <u>derivative</u> $\underline{f}'(\underline{x})$ if in

$$f(\underline{x}+\underline{h}) = f(\underline{x}) + \underline{h} \cdot \underline{f}'(\underline{x}) + \varepsilon(\underline{h})$$

we have $|\varepsilon(\underline{h})|/|\underline{h}| \to 0$ as $\underline{h} \to 0$, where the multiplication is interpreted respectively as an $\mathbb{R}^2$ scalar product or $\mathbb{C}$ product respectively. An $(\mathbb{R}^2 \to \mathbb{R}^2)$-differentiable function is complex-differentiable iff the <u>Cauchy-Riemann</u> equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $f = u + iv$; this is precisely the condition on the matrix $\underline{f}'(\underline{x})$ which makes it represent a complex number, or equivalently which makes it preserve angles (assuming it's nonzero).

If f is $\mathbb{C} \to \mathbb{C}$ differentiable at $z \in \mathbb{C}$, its partial derivatives exist and satisfy the Cauchy-Riemann equations, but not conversely (consider $e^{-z^{-4}}$)

Whereas a function $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{R}^n \to \mathbb{R}^m$) may be r times (continuously) differentiable but not $r+1$ times differentiable for any desired r, or may be differentiable arbitrarily often but not have a "working" Taylor series (eg $e^{-x^{-2}}$), once a function $\mathbb{C} \to \mathbb{C}$ is differentiable once then it is so arbitrarily often and has a Taylor series valid in some neighbourhood.

2. Contour Integration.

In general a <u>curve</u> in a space X is a continuous map $\gamma : [0,1] \longrightarrow X$, where two curves are considered to be the same if one is obtained from the other by reparametrisation, ie precomposition with a strictly monotone bijection $[0,1] \cong [0,1]$. It is quite impossible for us to deal in such generalities:

we shall require the curve (ie the function $[0,1]\to\mathbb{C}$) and the reparametrisation (and its inverse) to be piecewise continuously differentiable. For calculations we shall always use polygons whose edges are either straight lines or circular arcs. Two curves $\gamma,\delta:[0,1]\to\mathbb{C}$ are <u>homotopic</u> if there's a continuous(ly differentiable) $h:[0,1]^2\to\mathbb{C}$ with $h(x,0)=\gamma(x)$, $h(x,1)=\delta(x)$ and both $h(0,y)$, $h(1,y)$ constant functions (ie we keep the endpoints $\gamma(0)=\delta(0)$, $\gamma(1)=\delta(1)$ fixed when deforming γ into δ). By a <u>domain</u> D we shall mean a polygonally-connected open set in $\mathbb{C}$ such that any two curves in D with the same endpoints are homotopic in D (ie $\gamma,\delta:[0,1]\to D$ and $h:[0,1]^2\to D$).

Now we are in a position to define contour integration and state Cauchy's theorem. Let $f:D\to\mathbb{C}$ be differentiable on a domain containing a curve γ. Define

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\,\gamma'(t)dt$$

(where we have implicitly extended the real integral by $\int(u+iv)dx = \int u\,dx + i\int v\,dx$) and check that this is invariant under reparametrisation, is linear in the integrand (ie $\int(\lambda f+\mu g)dz = \lambda\int_\gamma f dz + \mu\int_\gamma g dz$) and satisfies the expected properties vis-à-vis reversal and concatenation of paths. Also check that

$$\int_\gamma (a+bz)dz = \tfrac{1}{2}a(z_2^2-z_1^2)+b(z_2-z_1)$$

(where $z_2=\gamma(1)$ and $z_1=\gamma(0)$ are the endpoints).

Less trivially, we have the important lemma that $|\gamma| = \int_0^1|\gamma'(t)|dt$ – the length of the curve – is finite, and that for continuous $f:D\to\mathbb{C}$, $|\int_\gamma f(z)dz|\le|f||\gamma|$ where $|f| = \sup\{|f(\gamma(t))|: 0\le t\le 1\}$.

Warning: we are interested in $\int_\gamma f(z)dz$, <u>not</u> in $\int_\gamma f(z)|dz| = \int_\gamma f(z)ds$.

<u>Cauchy's theorem</u> now states that in a domain $D\subseteq\mathbb{C}$, (which we have already required to be simply-connected), the integral of a differentiable function f along a closed curve γ in D vanishes: $[\gamma(0)=\gamma(1)]$.

$$\oint_\gamma f(z)dz = 0.$$

3. Cauchy's theorem and Green's theorem.

Before giving a proof of Cauchy's theorem in a special case, we shall discuss the connection with Green's theorem from IA Vector Calculus. Recall that that related the integral of the curl of a function ($\nabla\times\underline{f}$) over a surface ($S$) to the integral of the function ($\underline{f}$) around the boundary (∂S):

$$\oint_{\partial S} \underline{f}\cdot d\underline{s} = \int_S (\nabla\times\underline{f})\cdot d\underline{S}$$

Writing $\underline{f} = (\mathrm{Re} f, \mathrm{Im} f, 0)$, $d\underline{s} = (\mathrm{Re}\, dz, \mathrm{Im}\, dz, 0)$, $d\underline{S} = (0,0,|dS|)$ and $(\nabla\times\underline{f})$ as the determinant

$$\underline{\nabla}\times\underline{f} = \begin{vmatrix} \frac{\partial}{\partial x} & \mathrm{Re} f \\ \frac{\partial}{\partial y} & -\mathrm{Im} f \end{vmatrix} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad \text{by C-R}$$

this clearly corresponds (taking a judicious complex conjugate) to the real part of Cauchy's equation, the imaginary part being obtained similarly. The Cauchy-Riemann equations therefore correspond to the vanishing of a flux, and one may see why the properties of the function in the interior of the (region bounded the closed) curve are relevant to the value of the contour integral.

Being, then, a result about surfaces, Cauchy's thm, like Green's thm, is proved by disection of areas, not lengths. Moreover the shape of both the curve and the domain will be relevant. Moreover, the connection with Green's theorem leads one to suspect that, in the case where the function fails to be analytic throughout the region bounded by the curve, the integral gives information as to the nature of the singularities enclosed by the curve.

4. A proof of Cauchy's theorem.

In this section Cauchy's theorem for a continuously differentiable closed curve in a star domain (an open set with a point which may be joined to any other point by a straight-line segment lying wholly within the set) will be proved. Small generalisations may easily be made by the reader, eg piecewise continuous differentiability and more complicated domains.

First consider the case of a triangle in a convex open domain. Wlog the edge is of unit length and the integral is $\oint_\Delta f(z)dz = 1$. Bisect the edges and hence quadrisect the area; the integral is easily seen to be the sum of the integrals around the four little triangles, and so one is at least $1/4$. In this fashion choose a sequence of triangles $\Delta = \Delta_0, \Delta_1, \dots$ with $\Delta_n \subset \Delta_{n-1}$ of edge 2^{-n} and integral $\oint_{\Delta_n} f(z)dz \geqslant 2^{-2n}$. From Analysis I these converge to a single point $w \in \Delta_n$.

Now f is differentiable in the domain, and in particular at w, so $f(z) = f(w) + (z-w)f'(w) + \varepsilon(z-w)$ where $\varepsilon(z-w)/(z-w) \to 0$ as $z \to w$. From an observation in §2, $\oint_{\Delta_n}(f(w) + (z-w)f'(w))dz = 0$ so we have only to estimate $\oint_{\Delta_n} \varepsilon(z-w)\,dz$, which by construction is $\geqslant 2^{-2n}$. However there is some $\delta = 2^{-n} > 0$ such that for $|z-w| \leqslant \delta$, $|\varepsilon(z-w)| < \frac{1}{3}.|z-w|$ and so

$$\oint_{\Delta_n} f(z)dz = \oint_{\Delta_n} \varepsilon(z-w)dz \leqslant \sup \varepsilon(z-w) \,.\, |\Delta_n| < \tfrac{1}{3}.2^{-n}.3.2^{-n} < 2^{-2n}$$

contradicting the original assumption that $\oint_\Delta f(z) = 1$.

Now let Γ be any closed continuously differentiable curve in a star-domain Ω in which f is differentiable. We aim to define a function F on Ω which is also differentiable, with derivative f, whence $\oint_\Gamma f(z)dz = \oint_\Gamma F'(z)dz = F(\gamma(1)) - F(\gamma(0)) = 0$.

Let w be the base-point of the star-domain and z any other point, so the straight-line segment ℓ from w to z lies entirely within Ω. Put $F(z) = \int_\ell f(\zeta)d\zeta$. Since Ω is open, some disc about z lies within it, say $\{\zeta : |\zeta - z| < \varepsilon\}$. For $|h| < \varepsilon$, $F(z+h) - F(z)$ is the integral along two sides of a triangle lying wholly within Ω and hence, by Cauchy's theorem for a triangle, is equal to the integral along the third, ie the straight-line segment joining z and $z+h$. Hence

$$F(z+h) - F(z) - hf(z) = \int_z^{z+h} (f(\zeta) - f(z))d\zeta = O(h^2)$$

by the differentiability of f at z. Thus $F'(z) = f(z)$.

5. Laurent and Taylor expansions.

Let f be differentiable in some punctured neighbourhood of a point, say in $\{z : 0<|z|<R\}=\Omega$. Let $z\in\Omega$ and $0<r_1<|z|<r_2<R$. Applying Cauchy's theorem to two curves of the form , the two integrals around the circles of radii r_1 and r_2 of $f(\zeta)/\zeta^{n+1}$ are equal, say to $2\pi i a_n$. On the other hand, the integrals of $f(\zeta)/(\zeta-z)$ differ by

$$\oint_C \frac{f(\zeta)d\zeta}{\zeta-z} = f(z)\oint_C \frac{d\zeta}{\zeta-z} + f'(z)\oint_C d\zeta + o(\text{area } C)$$

which is easily seen to be $2\pi i f(z)$.

$$\begin{aligned}
\text{So}\quad f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(\zeta)d\zeta}{\zeta-z} = \frac{1}{2\pi i}\left(\oint_{r_2}-\oint_{r_1}\right)\frac{f(\zeta)d\zeta}{\zeta-z}\\
&= \frac{1}{2\pi i}\oint_{r_2}\left(1-\frac{z}{\zeta}\right)^{-1}\frac{f(\zeta)d\zeta}{\zeta} + \frac{1}{2\pi i}\oint_{r_1}\left(1-\frac{\zeta}{z}\right)^{-1}\frac{f(\zeta)d\zeta}{z}\\
&= \frac{1}{2\pi i}\oint_{r_2}\sum_{n=0}^{\infty} z^n\frac{f(\zeta)d\zeta}{\zeta^{n+1}} + \frac{1}{2\pi i}\oint_{r_2}\sum_{n=-\infty}^{-1} z^n\frac{f(\zeta)d\zeta}{\zeta^{n+1}}\\
&= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} z^n\oint \zeta^{-n-1}f(\zeta)d\zeta \qquad \text{by uniform convergence of geometric series}\\
&= \sum_{n=-\infty}^{\infty} a_n z^n
\end{aligned}$$

Moreover this is uniform in any closed subset of Ω. This is the <u>Laurent expansion</u> of f about 0.

If f is analytic and bounded in a punctured neighbourhood of a point, the negative terms in its Laurent expansion vanish. Then wlog it takes the value a_0 at the point (which is called a <u>removeable singularity</u>). The expansion is then called a <u>Taylor series</u> about the point, and we have shown that this is valid on the largest open disc about the point on which f is analytic (ie the radius of convergence is the distance of the nearest singularity). It follows that f is infinitely differentiable. Moreover we have <u>Cauchy's formula</u>:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint\frac{f(\zeta)d\zeta}{(\zeta-z)^{n+1}}$$

for some suitable contour containing z.

6. Morera's theorem.

We shall now prove the converse to Cauchy's theorem. Let f be a continuous function on a domain Ω with $\oint f(z)dz=0$ for every closed curve in Ω; then f is differentiable on Ω. Wlog Ω is connected.

Once again we shall define F on Ω with $F'=f$. By §5, since F is once differentiable it is infinitely so, whence f is differentiable as required.

Choose $w\in\Omega$ and for $z\in\Omega$ let Γ be any curve in Ω with endpoints w and z; then put $F(z)=\int_\Gamma f(z)dz$. Since $\oint f(z)dz=0$ this is independent of the curve chosen. Now $F(z+h)-F(z)$ is the integral along a curve from z to $z+h$; since Ω is open, for sufficiently small h this may as well be a straight line. Hence

$$F(z+h)-F(z)-hf(z)=\int_z^{z+h}(f(\zeta)-f(z))\,dz=o(h)$$

by the continuity of f, so $F'=f$ as required.

We have now demonstrated the equivalence of the following conditions on a continuous function defined on a domain Ω; a function satisfying them is called <u>analytic</u>:

(i) $\mathbb{C}$-differentiable at every point
(ii) $\mathbb{R}^2$-differentiable and satisfying the Cauchy-Riemann equations at every point
(iii) $\oint f(z)dz=0$ for every closed curve entirely contained in Ω (ie including its interior)
(iv) arbitrarily $\mathbb{C}$-differentiable at every point
(v) has a valid Taylor expansion on some neighbourhood of each point
(vi) has continuous partial derivatives satisfying the Cauchy-Riemann equations at each point (from Analysis II)

The real (or imaginary) part of an analytic function is <u>harmonic</u> (satisfies <u>Laplace's equation</u> $\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$): this is easily deduced from repeated differentiability and the Cauchy-Riemann equations. Conversely, given a harmonic $u:\Omega\to\mathbb{R}$ let $g(x+iy)=u_x(x,y)+iu_y(x,y)$ and f be its integral.

(Check that g satisfies (ii) and hence by (iii) there's an analytic f with $f'=g$). Then $f = c+u+iv$ for some constant c and some (real) harmonic v on Ω. Moreover f is unique up to some additive constant (c). This method is used in Part IB Fluids.

7. Classification of isolated singularities.

Again suppose that f is analytic on some punctured neighbourhood, say $\{z : 0 < |z| < R\}$. Consider its Laurent expansion $f(z) = \sum_{\mathbb{Z}} a_n z^n$ and let n_-, n_+ be the least and greatest values of n such that $a_n \neq 0$. Then if

(i) $n_- \geq 0$: f has a removeable singularity at the point
(ii) $-\infty < n_- < 0$: f has a pole of order $|n_-|$ and residue a_{n_-}
(iii) $n_- = -\infty$: f has an essential singularity
(iv) $n_- > 0$: f has a zero of order n_-
(v) $0 \leq n_- \leq n_+ < \infty$: f is a polynomial of degree n_+

An example of a removeable singularity is (at 0) $f(z) = \sin z / z$: we may simply redefine $f(0) = \lim_{z \to 0} \sin z / z = 1$. If f has a pole of order $\leq n$ at 0, $f(z)z^n$ has a removeable singularity and wlog $f(z)z^n = a_{-n}$. At a pole, $|f(z)| \to \infty$ as $z \to 0$. A function $f : \mathbb{C} \to \mathbb{C}$ which is analytic everywhere except for isolated poles is called meromorphic; an everywhere analytic function is holomorphic.

$e^{1/z}$ has an essential singularity at 0. In any neighbourhood of an essential singularity a function takes values arbitrarily close to any given value. For suppose that $f(z)$ is bounded away from some $c \in \mathbb{C}$; then put $g(z) = (f(z)-c)^{-1}$: this is bounded in the given neighbourhood and hence has a removeable singularity where f has an essential one. But g cannot be identically zero, so has a zero of order n $(0 \leq n < \infty)$ and $f(z) = g(z)^{-1} + c$ has a pole of order n, contrary to hypothesis. In fact in any neighbourhood of an essential singularity every value with at most two exceptions is actually attained infinitely often ($e^{1/z}$ misses zero) but this is far outside the scope of the IB course.

By Cauchy's theorem, a contour integral for a function with only isolated singularities is given by a sum of residues for the singularities enclosed by the curve (the contours traditionally used for calculations consist of circular arcs and straight line segments, but for general curves some version of the Jordan curve theorem will be needed). Recall, therefore Cauchy's formula:

$$\oint_{(w)} f(z)\,dz = \mathop{\mathrm{Lt}}_{z\to w} f(z).(z-w)$$

(L'Hôpital's rule may be useful here).

8. "Multivalued functions"

We would like to define $\log z = \int_1^z \frac{dw}{w}$ but since $\oint \frac{dw}{w} = 2\pi i$ for a curve enclosing 0 this is not well-defined. However if Γ is any curve from 0 to ∞ (often the negative real axis is chosen) then log may be defined on $\mathbb{C}\setminus\Gamma$. Such a curve is called a branch cut and 0 is a branch point.

Another example is $\sqrt{z}$, or in general z^c for $c\in\mathbb{C}$. Again there is no continuous definition of these functions for the whole of $\mathbb{C}$, but they may be defined on $\mathbb{C}\setminus\Gamma$ for suitable Γ. In this case there is a continuous definition of 0^c (for $c\geq 0$).

Branch cuts need not extend to ∞: consider $f(z)=\sqrt{(z^2-z)}$. In this case a cut from 0 to 1 will suffice, although in case of uncertainty two cuts (from 0 to ∞ and from 1 to ∞) may be made.

This is clearly an ad hoc solution to the problem: the more sophisticated approach is to define a Riemann Surface. Take many copies of $\mathbb{C}$ with the branch-cuts drawn in, labelled with the various possible "routes" (ie sequences of crossings of branch-cuts) from a base point, and glue them together at the branch cuts. The Riemann Surface for log is a spiral: there are sheets indexed by $\mathbb{Z}$, whose values differ by $2\pi i$. That for $z^{p/q}$ (with $p,q\in\mathbb{Z}$, $q\neq 0$ and $(p,q)=1$) is a q-fold cover of $\mathbb{C}$, wherein a q-fold circuit of the origin is homotopic to a point. The surfaces for $\sqrt{(z^2-z)}$ and $\sqrt{(z^3-z)}$ are homeomorphic to a cylinder and a punctured torus respectively.

9. Two contour lemmas

Definite integrals give rise to curves which are frequently either (half or all of) the real axis, or circles crossing branch-cuts. In order to evaluate these we need to build them into (closed) contours which enclose known singularities but not branch points. This involves adding extra arcs whose radii tend to either 0 or ∞.

The result for large arcs is called Jordan's lemma: we shall give two different versions of it. Let $\Gamma(r)$ be the semicircle $\{re^{i\theta} : 0 \leq \theta \leq \pi\}$ and $f : \mathbb{C} \to \mathbb{C}$ have only finitely many singularities in the upper half plane (ie none outside some radius R_0). Suppose (i) $|zf(z)| \to 0$ or (ii) $|g(z)| \to 0$ as $z \to \infty$, ie given $\varepsilon > 0$ there's some R such that $|zf(z)| < \varepsilon$ (respectively $|g(z)| < \varepsilon$) for $|z| > R$. Then (i) $\int_{\Gamma(r)} f(z)\,dz \to 0$ and (ii) $\int_{\Gamma(r)} e^{-imz} g(z)\,dz \to 0$ as $r \to \infty$ for $m > 0$. Let $r > R > R_0$ as in the condition; then (i) $\left|\int_{\Gamma(r)} f(z)\,dz\right| = \left|\int_0^\pi re^{i\theta} f(re^{i\theta})\,d\theta\right| < 2\pi\varepsilon$ and (ii) (replacing $\Gamma(r)$ by the circumscribed rectangle) $\left|\int_{\Gamma(r)} e^{imz} f(z)\,dz\right| < 2\varepsilon\left[(1-e^{-mr})/m + re^{-mr}\right] < 2\varepsilon/m$

For small arcs we want to calculate $\operatorname*{Lt}_{r\to 0} \int_{\Gamma(r)} f(z)\,dz$ where $\Gamma(r)$ is the arc $\{re^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}$. For $f(z) = 1/z$ this is easily seen to be $i(\theta_2 - \theta_1)$. More generally, suppose $f(z) = a/z + g(z)$ for some analytic g; then clearly $\int_{\Gamma(r)} g(z)\,dz \to 0$ as $r \to 0$, so the limit for f is $ia(\theta_2 - \theta_1)$

10. Evaluation of definite integrals (a recipe book).

This is largely a question (as far as simple – ie Tripos – problems are concerned) of chasing the right contour. Try change of variable, eg

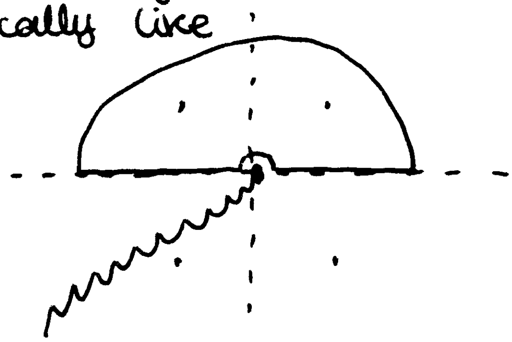
$$\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \oint \frac{-iz^{-1}dz}{3+z+z^{-1}} \qquad \int_0^{\infty} \frac{x^{m-1}dx}{1+x^n} = \frac{1}{n}\int_{-\infty}^{\infty} \frac{e^{(m/n)z}}{1+e^z}dz$$

where $z=e^{i\theta}$ and $z=n\log x$ respectively. Also try duplicating the range of integration, eg

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} \qquad \int_0^{\infty} \frac{x^{1/3}\sin x}{1+x^4}dx = \frac{1}{1-e^{i\pi/3}}\int_{-\infty}^{\infty} \frac{x^{1/3}\sin x}{1+x^4}dx$$

where the latter will also involve a branch cut.

Now clear trigonometrical functions. This is not just a manipulative trick as it is in A-level problems: it is necessary to make Jordan's lemma work – e^{-ikz} $(k>0)$ means that the large semicircle in the upper half plane won't vanish. Of course Jordan's semicircle mustn't cross any branch cuts. After locating critical points (and steering clear of suspicious ones), we have a diagram typically like



This gives us an equation such as

$$\begin{matrix}\text{big}\\ \text{semicircles}\end{matrix} + \begin{matrix}\text{little}\\ \text{semicircles}\end{matrix} + \begin{matrix}\text{required}\\ \text{integral}\end{matrix} = \begin{matrix}\text{residues}\\ \text{enclosed}\end{matrix}$$

where the big and little semicircles are evaluated as in §9 and the residues using Cauchy's formula. Notice that choosing to go to one side of a pole rather than the other has the effect of moving its residue across the equation.

Finally, dont forget the factors which arose from changing variable or duplicating the range. It is very rare for Tripos problems to involve real-variable methods, and most supervisors have forgotten them and will not thank you for reminding them of them.

11. Isolated zeroes and poles.

Let f be an analytic function on a domain Ω which is not identically zero. Then the zeroes of f are isolated. Clearly an accumulation point of zeroes is itself a zero (by continuity), so suppose $f(z_0)=0$ and $\forall \varepsilon>0\ \exists z\in\Omega\ f(z)=0\ \&\ |z-z_0|<\varepsilon$. Since f is not identically zero, z_0 is a zero of order n, say, so $f(z)=a(z-z_0)^n+g(z)$ where $g(z)/(z-z_0)^n\to 0$ as $z\to z_0$. Choose ε such that $|g(z)/(z-z_0)|^n<\frac{1}{2}|a|$ on $\{z: 0<|z-z_0|<\varepsilon\}$; then if z is another zero within ε, $g(z)/(z-z_0)^n=-a$ which is a contradiction since $a\neq 0$.

Likewise suppose f has no essential singularity in Ω; then its poles are isolated. For $g(z)=1/f(z)$ is analytic on Ω (since wlog f has no zero in Ω) and so has isolated zeroes, being the poles of f. In other words, an accumulation of singularities is necessarily essential; we shall not deal with such cases in this course.

Suppose $f:\mathbb{C}\to\mathbb{C}$ has no singularities whatever. Then by §5 it has a Taylor series valid throughout $\mathbb{C}$. Hence it is a polynomial or has terms of arbitrarily high degree. In the latter case $g(z)=f(1/z)$ has an essential singularity at 0, so we may say that f has one at ∞ (something we shall make formal in §); for example $f(z)=e^z$.

Suppose f has no essential singularities on $\mathbb{C}$ or at ∞ (in the above sense). Then since the poles and zeroes of f and g ($g(z)=f(1/z)$) are isolated, there are only finitely many in the closed unit disc (by compactness, ie Bolzano-Weierstraß) and so (adding these together) on $\mathbb{C}$. Let the zeroes be $a_i(z-z_i)^{n_i}$ and the poles $b_j(z-p_j)^{-n_j}$ and put $h(z)=\sum a_i(z-z_i)^{n_i}+\sum b_j(z-p_j)^{-n_j}$. Then $f-h$ has no zeroes, poles or essential singularities and is hence a polynomial. However by the <u>fundamental theorem of algebra</u> (below) a polynomial has a zero unless it is constant. It follows that f is a <u>rational function</u>, ie a quotient of two polynomials.

We may prove the fundamental theorem of algebra by this method too. If f is a polynomial with no zeroes, both $f(z)$ and $g(z) = 1/f(z)$ are analytic functions without poles, zeroes or essential singularities (the last even at infinity), so g is also a polynomial. But $\deg f.g = \deg f + \deg g \neq 0$ unless one of f, g is constant. Similarly Louiville's thm: if $f: \mathbb{C} \to \mathbb{C}$ is analytic & bounded it's constant; for it's a polynomial of degree zero since otherwise $|z^n| \to \infty$ as $z \to \infty$.

12. The argument principle and Rouché's theorem

We now know quite a lot about zeroes and poles, but not how to count them. It will become apparent that a zero of order n should count as n (simple) zeroes, and a pole of order n should count as $-n$. A function which gives this is $\frac{1}{2\pi i}\oint f'(z)/f(z).\,dz$ around the critical point (check this for $f(z) = az^n$ and hence $f(z) = az^n + g(z)$ where $g(z)/z^n \to 0$ as $z \to 0$). It follows that this contour integral counts the zeroes and poles enclosed according to the multiplicity scheme above.

The justification of this manner of counting is Rouché's theorem: adding a small perturbation to f may "split" the zeroes, but does not change the number of them. Specifically, if $|g(z)| < |f(z)|$ in Ω, $\oint (f'+g')/(f+g)\,dz = \oint f'/f.\,dz$ for any contour in Ω. For put $\zeta(\lambda) = \frac{1}{2\pi i}\oint (f'+\lambda g')/(f+\lambda g).\,dz$: we want $\zeta(0) = \zeta(1)$. However the conditions imply that the integrand is analytic on and near the contour (in particular $f + \lambda g \neq 0$) so $\zeta: [0,1] \to \mathbb{Z}$ is continuous and hence constant.

This gives another proof of the fundamental theorem of algebra: let f be a polynomial and $f+g = z^n$, then for $\Omega = \{z : |z| > R\}$ for suitable R, clearly $\frac{1}{2\pi i}\oint (f'+g')/(f+g).\,dz = n$, so f has a zero in $\{z : |z| \leq R\}$.

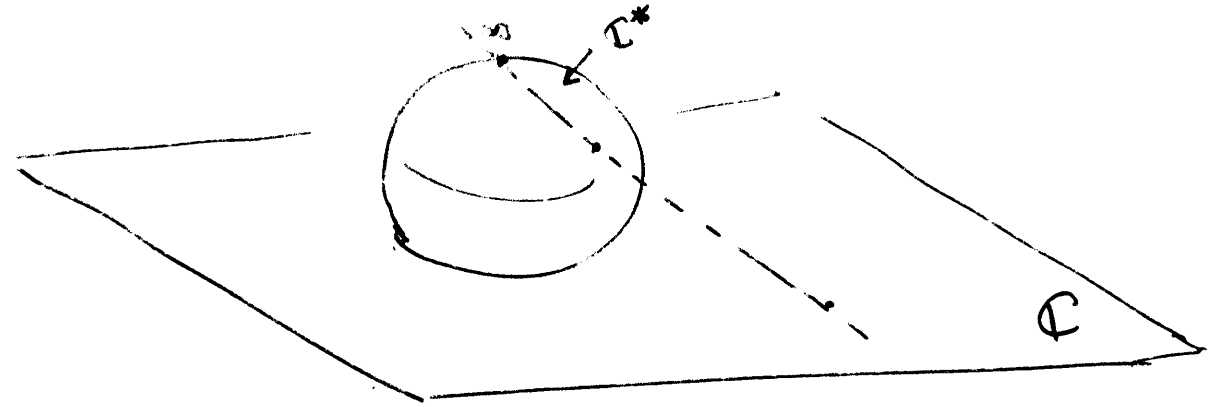
Now suppose $f: \Omega \to \mathbb{C}$ is analytic and $f'(z_0) \neq 0$ for some $z_0 \in \Omega$. Then there's some open set $z_0 \in U \subseteq \Omega$ on which f is injective and, moreover, the inverse $f(U) = V \to U$ is analytic. Then there's an analytic h on some neighbourhood of z_0 with $f(z) = (z - z_0)h(z)$, and $h(z_0) = f'(z_0) \neq 0$ so wlog we may cut down Ω st. $f: \Omega \to \mathbb{C}\setminus\{0\}$ is analytic. Choose $\varepsilon > 0$ and let $C = \{z : |z - z_0| = \varepsilon\} \subset \Omega$, where $|h(z_0)| > \frac{1}{2}|f'(z_0)|$ on C; put $\delta = \frac{1}{2}\varepsilon|f'(z_0)|$. Then if $|w - f(z_0)| < \delta$, $g(z) = f(z_0) - w$

(ie a constant function) satisfies $|g(z)| < |f(z)-f(z_0)|$ and so $(f(z)-f(z_0))+g(z) = f(z)-w$ has the same number of zeroes within C as does $f(z)-f(z_0)$, ie one. Hence if we put $V = \{w : |f(z_0)-w| < \delta\}$ and $U = f^{-1}(V)$ we have two open sets with $f: U \to V$ analytic and bijective. Let $g: V \to U$ be the inverse function: we want to show that g is differentiable, say at $w = f(z) \in V$. Let $w+k = f(z+h) \in V$ with $z, z+h \in U$ (this is possible for sufficiently small k since V is open); then $g(w+k)-g(w) = z+h-z = h$ and $f(z+h)-f(z) = w+k-w = k$, but the latter is $hf'(z)+o(h)$ by the differentiability of f, and moreover we have ensured $f'(z) \neq 0$. We have $k/h = f'(z)+o(h)$, so $h/k = 1/f'(z) + o(h) = 1/f'(z) + o(k/f'(z)+o(h))$ which is clearly $o(k)$; so g is differentiable and $g'(w) = 1/f'(g(w))$ as expected.

The foregoing argument is really a special case of the Inverse Function Theorem in Analysis III: a function $\mathbb{R}^n \to \mathbb{R}^m$ which is continuously differentiable and has a nonsingular derivative at some point has a continuously differentiable inverse in some neighbourhood of the point.

13. The Riemann Sphere and other Surfaces.

We have already (in §11) talked of "adding the point at infinity". Geometrically this may be described by the stereographic projection:



where the pole represents ∞. Notice that, unlike the extended real line, we only add one point. The sphere is compact, so any discrete set (such as zeroes or poles of an analytic function $\mathbb{C}^* \to \mathbb{C}^*$) is finite.

As will be familiar from terrestrial cartography there is no single chart which covers the sphere. We may conveniently use two charts, discussing "$f(z)$" and "$f(1/z)$" separately, as we did in §11. For the torus, which arose in §8, we need three charts. The formalisation of this is done in Part II <u>Riemann Surfaces</u>.

In this picture, a pole is merely a point which is mapped to infinity. An analytic function $\mathbb{C}^* \to \mathbb{C}^*$ we shall call a <u>meromorphic function</u>. In §11 we showed that such a function is actually rational. Note that the rational function $f(z) = z$ has a pole at ∞. The numbers of poles and zeroes are equal; calculate $\oint f'/f . dz$ for a "small" circle enclosing ∞ (ie a large circle enclosing all zeroes and poles in $\mathbb{C}$). More generally, any two values in $\mathbb{C}^*$ are achieved equally often (with suitably-defined multiplicity), unless (of course) the function is constant. Specifically $f(z) = z^n$ gives an $|n|$-fold cover.

What are the bianalytic maps $\mathbb{C}^* \to \mathbb{C}^*$? Clearly they are the rational maps with at most one zero and at most one pole in $\mathbb{C}$, ie those of the form $z \mapsto (az+b)/(cz+d)$ with $ad-bc \neq 0$ (wlog $ad-bc = 1$), which are called the <u>Möbius transformations</u>. They must, of course, form a group, which is easily seen to be generated by translations ($z \mapsto z+b$), rotations ($z \mapsto ze^{i\theta}$), enlargements ($z \mapsto |a|z$) and inversion ($z \mapsto 1/z$).

Using another coordinate system we may easily identify this group as $PGL(2,\mathbb{C})$ (or $PSL(2,\mathbb{C})$). Consider $\mathbb{C}^2 \setminus \{\underline{0}\}$ and idenaify $(x:y)$ with $(\lambda x : \lambda y)$ for any $\lambda \in \mathbb{C} \setminus \{0\}$; $\mathbb{C}$ is embedded in this by $z \mapsto (z:1)$, whilst $\infty = (1:0)$. It's easy to check that the topologies coincide. The Möbius transformation $z \mapsto (az+b)/(cz+d)$ now corresponds to the (nonsingular) matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over $\mathbb{C}$. This gives the group $GL(2,\mathbb{C})$, but the <u>homotheties</u> $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $(\lambda \neq 0)$ act on $\mathbb{C}^*$ as the identity (which no other matrix does). These actually form the centre of $GL(2,\mathbb{C})$ (Algebra II) and the quotient group is $PGL(2,\mathbb{C})$, the <u>projective general</u>

<u>linear group</u>, because the space we constructed is the <u>complex projective line</u>. Similar things may be done in any dimension over any field, but it is a coincidence (arising from the existence of $\sqrt{-1}$) that $PGL(2,\mathbb{C}) \cong PSL(2,\mathbb{C})$. If we say z and $-1/z^*$ are "antipodal", the subgroup preserving this structure is SO_3, the rotations of a sphere.

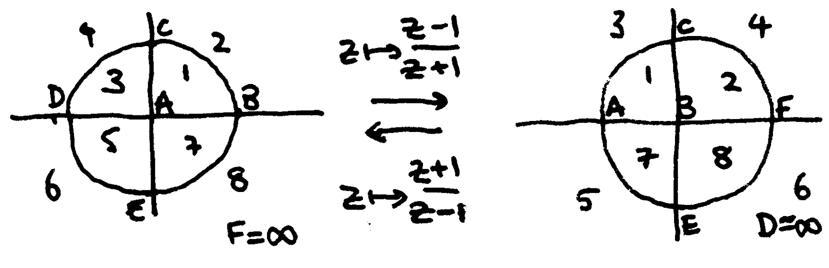
14. Conformal mappings

A bijective analytic map $f: U \to V$ has an analytic inverse, so that U and V are isomorphic from the point of view of complex analysis. Moreover f locally behaves as a composite of a translation (by $f(z_0)-z_0$), an enlargement (by $|f'(z_0)|$) and a rotation (by $\arg f'(z_0)$) so that geometric properties such as angles between curves are preserved. Such a function is said to be <u>conformal</u>. Notice that the Cauchy-Riemann equations are precisely what we need to preserve angles.

In §12 we showed that a sufficient condition for this is $f'(z_0) \neq 0$. This is also necessary since otherwise (assuming $z_0 = f(z_0) = 0$ and f is analytic but not constant near z_0) f behaves locally like z^n, which may easily be seen to multiply angles by n.

Conformally equivalent domains are clearly topologically equivalent, but the converse is false. For $\mathbb{C}$ is inequivalent to Δ, the open unit disc: otherwise $f: \mathbb{C} \to \Delta$ would be a bounded non-constant analytic function. However a great variety of subsets of the plane are conformally equivalent, as demonstrated in many Tripos questions.

Of particular importance are Möbius transformations: it's easy to verify that they map circles/straight lines to circles/straight lines. The following is worth remembering (as of use in Tripos questions):



Three tricks are (i) send inconvenient points to infinity by inversion at them (ii) multiply angles in sectors (with straight arms) by a power ($z \mapsto z^n$) and (iii) turn sectors (eg upper half plane) into strips by log (example gives $\{z : 0 < \operatorname{im} z < \pi\}$). Note that $z \mapsto z^4$ sends the open first quadrant to the plane *less the positive real axis.*

15. <u>Maximum modulus principle</u>.

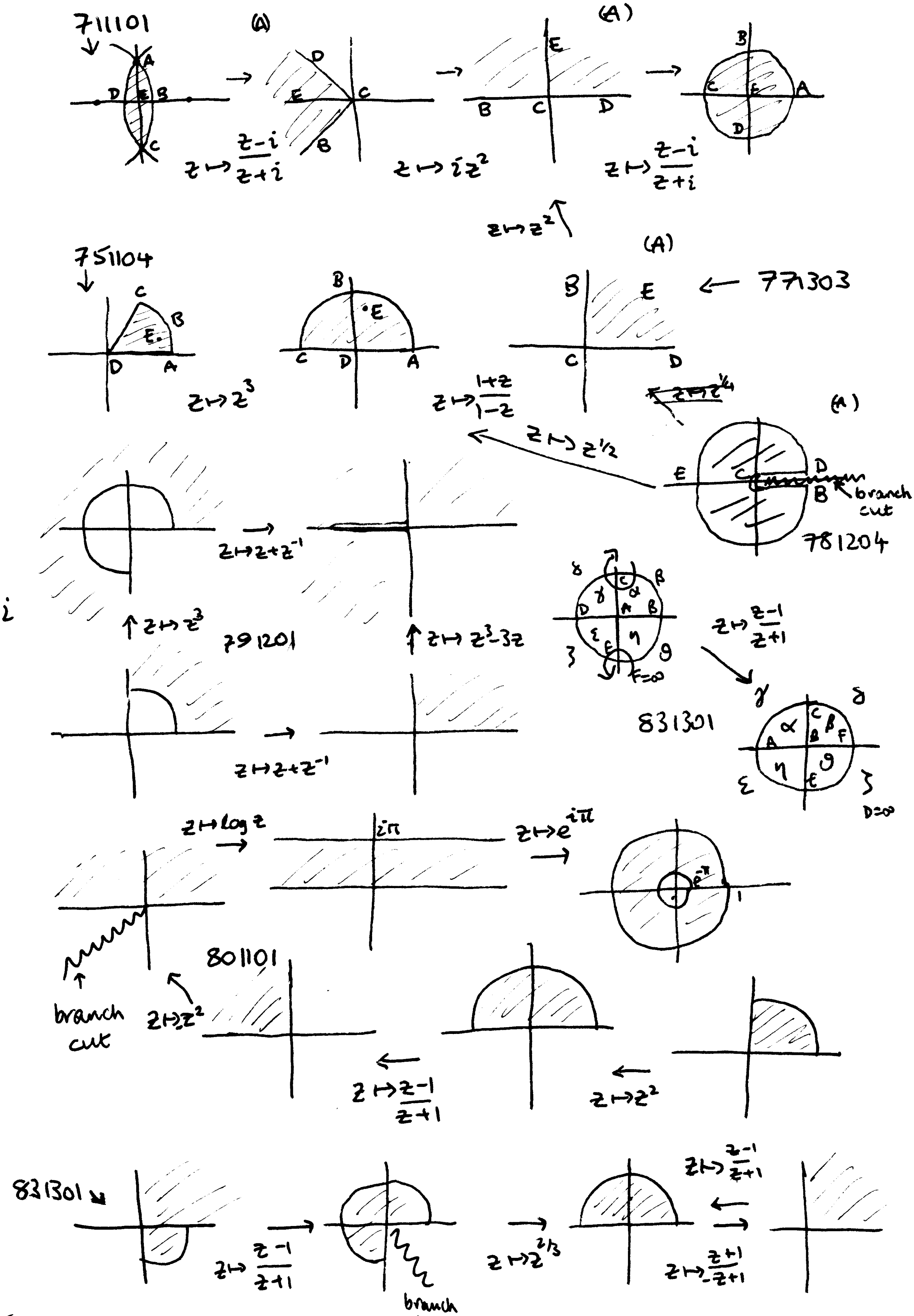
Let f be analytic on a domain $\Omega \subseteq \mathbb{C}$ and let $A \subseteq \Omega$ be a compact (ie closed & bounded) set. From Analysis I we know that $|f(z)|$ is bounded on A and attains its bounds, but in fact we can show (in two different ways) that (assuming it's not constant) it is on the boundary of A (ie $A \setminus \operatorname{int} A$) that it achieves its bounds.

From Cauchy's theorem, $f(z) = \oint \frac{f(w)}{w-z} dw / 2\pi i$ for a curve enclosing z. Let this be a circle centre z and estimate the integral, so $|f(z)| \leq \sup\{|f(w)| : |w-z| = r\}$. Let $z_0 \in \operatorname{int} A$ be a point at which $|f(z)|$ is greatest and let r be the radius of the largest circle centre z_0 contained in A. Then $|f(z_0)| \leq \sup |f(w)| \leq |f(z_0)|$, so $|f(w)| = |f(z_0)|$ on this circle. However the circle is not contained in int A by the choice of r, so it meets the boundary, and $|f(w)| = |f(z_0)| = \max |f(\zeta) : \zeta \in A|$ for some $w \in A \setminus \operatorname{int} A$.

Alternatively we may use Rouché's theorem, or rather the corollary in §12. Suppose again that $|f(z)|$ achieves its bounds at z_0 and that $f(w) = f(z) + (w - z_0)^n g(w)$ where $g(z_0) \neq 0$ and g is analytic. Choose $z_0 \in U_1 \subset A \subset \Omega$ st. $g(w) \neq 0$ on U_1. Then there's an analytic function h on U_1 with $h(w)^n = (w - z_0)^n g(w)$ and, moreover, $h'(z_0) \neq 0$. It follows that there's some $z_0 \in U \subseteq U_1 \subset A \subset \Omega$ and $0 \in V \subseteq \Omega$ so that $h : U \to V$ is bianalytic. Let $f(z_0) = Re^{i\theta}$ and choose $\varepsilon e^{i\theta/n} \in V$; then if $h(w) = \varepsilon e^{i\theta/n}$, $f(w) = (R + \varepsilon^n) e^{i\theta}$ so $|f(w)| = R + \varepsilon^n > R = |f(z_0)|$. This then proves the stronger result that the bound is not attained in int A, unless f is constant.

Conformal mappings.

$\frac{i-1}{i+1} = -i$



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