ELECTRODYNAMICS

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In this course Maxwell's equations, the Lorentz force law, etc., are derived from empirical 'A'-level physics concepts, and many applications at low velocities are considered, ignoring the particulate nature of charge. Heavy use is made of Vector Calculus but the extension to Special Relativity is made only in IB Classical Theory of Fields. Potential theory is also useful.

MAXWELL'S EQUATIONS & EQUIVALENT INTEGRAL FORM.

M1: $\underline{\nabla}\cdot\underline{E}=\rho/\varepsilon_0$	M1′: $\oint_{\partial V}\underline{E}\cdot d\underline{S}=q/\varepsilon_0$	charge is source of $\underline{E}$
M2: $\underline{\nabla}\cdot\underline{B}=0$	M2′: $\oint_{\partial V}\underline{B}\cdot d\underline{S}=0$	no magnetic monopoles
M3: $\underline{\nabla}\wedge\underline{B}=\mu_0\underline{j}+\frac{1}{\mu_0\varepsilon_0}\dot{\underline{E}}$	M3′: $\oint_{\partial S}\underline{B}\cdot d\underline{l}=\mu_0(I+I_{disp})$	current is source of $\underline{B}$
M4: $\underline{\nabla}\wedge\underline{E}+\dot{\underline{B}}=0$	M4′: $emf=-\dot{\Phi}$	Faraday law of induction

ELECTROSTATICS

Electrostatic field due to point charge at the origin (<u>Coulomb's law</u>) $\underline{E}=q\hat{\underline{r}}/4\pi\varepsilon_0 r^2$. Field due to many charges adds vectorially (<u>superposition law</u>), so for density distribution $\rho(\underline{r}')$, field at $\underline{r}$ is $\underline{E}(\underline{r})=\frac{1}{4\pi\varepsilon_0}\int_{V'}\underline{R}\rho(\underline{r}')/R^3$ where $\underline{R}=\underline{r}'-\underline{r}$. Integrating this over V and using the divergence theorem we get <u>Gauss' flux theorem</u> (M1′) and, since this is true for any V, the first Maxwell equation (M1). Note that ∂V is the boundary surface of volume V.

Since $\underline{\nabla}\times\underline{E}=0$ if the field is constant in time, we may define the <u>electrostatic potential</u>, ϕ (in Volts) such that $\underline{E}=-\underline{\nabla}\phi$. For a charge distribution $\rho(\underline{r}')$ choose $\phi=0$ at ∞, so $\phi=\frac{1}{4\pi\varepsilon_0}\int_{V'}\rho(\underline{r}')d^3\underline{r}'/|\underline{r}-\underline{r}'|$. Then $\phi(\underline{b})-\phi(\underline{a})$ is the work done by an outside force on a unit charge against the field to move it from $\underline{a}$ to $\underline{b}$; this is independent of the route [the field is <u>conservative</u>]. M1 gives $\nabla^2\phi=-\rho/\varepsilon_0$, <u>Poisson's law</u> (see Potential Theory). A curve whose tangent is everywhere parallel to $\underline{E}$ is a <u>line of force</u>; it is normal to the <u>equipotentials</u>, which are surfaces of constant ϕ.

<u>Examples of potentials</u>: point charge: $\phi=q/4\pi\varepsilon_0 r$, dipole $\phi=\underline{\mu}\cdot\underline{r}/4\pi\varepsilon_0 r^3$, infinite line of charge (q per unit length) $\phi=\frac{q}{2\pi\varepsilon_0}\log\rho$ [ρ=dist from axis]. Note spherical and cylindrical symmetry.

By considering the work done in bringing each point charge from infinity, the Potential Energy of a system is $W = \frac{1}{2}\sum_{i<j} q_i q_j / 4\pi\varepsilon_0 |\underline{r}_i - \underline{r}_j|$. For a continuous distribution $\rho(\underline{r})$ in potential $\phi(\underline{r})$ this is $\frac{1}{2}\int_V \rho(\underline{r})\phi(\underline{r})\,d^3\underline{r} = \frac{1}{2}\varepsilon_0 \int_V \underline{E}^2 d^3\underline{r}$. For the surface charge σ on a conductor, $W = \frac{1}{2}\varepsilon_0 \oint_{\partial V} \sigma\,\underline{E}\cdot d\underline{S}$. The field between the plates of a large parallel-plate capacitor is perpendicular to the plates of strength $E = V/\tau = \sigma/\varepsilon_0$ where V is the potential difference, τ the separation and σ the surface charge density. By considering a virtual displacement, the force between the plates is $\frac{1}{2}\varepsilon_0 E^2$ per unit area.

MOVING CHARGES.

Define current density in volume $d^3\underline{r}$ as $\underline{j}(\underline{r})\,d^3\underline{r} = \sum_{d^3 r} q_i \underline{v}_i$ where the i^{th} charge is q_i with velocity $\underline{v}_i$. Conservation of charge then says $\oint_{\partial V} \underline{j}\cdot d\underline{S} = -{}^{d}/_{dt}\int_V \rho\, d^3\underline{r}$ (flow out = − rate of change); thus applying this to infinitesimal volume, $\underline{\nabla}\cdot\underline{j} + \partial\rho/\partial t = 0$, the continuity equation.

In a uniform conductor we have Ohm's law, $\underline{j} = \gamma\underline{E}$ or (in a wire) $I = V/R$. The charge density, ρ, vanishes in a uniform conductor. Between two such conductors, (1) and (2), we have the boundary conditions, (i) $j_{1\perp} = j_{2\perp}$ by conservation, (ii) $\Delta\underline{E}_\perp = \sigma/\varepsilon_0$ as capacitor and (iii) $E_{1/\!/} = E_{2/\!/}$ also by conservation (but less obviously).

The magnetic field, $\underline{B}$, is generated by the moving charges. Empirically, we have the Lorentz force law $\underline{F} = q(\underline{E} + \underline{v}\wedge\underline{B})$ and the force on a conductor in a magnetic field $d\underline{F} = I\,d\underline{\ell}\wedge\underline{B}$. Again empirically, an infinitesimal loop around surface $d\underline{S}$ carrying current I produces a dipole $\underline{B}$-field of moment $I\,d\underline{S}$ (cf electric dipole). There are no magnetic monopoles, hence M2 and M2′. Hence for the current loop $\underline{B} = -\underline{\nabla}\psi$ with $\psi(\underline{r}) = \mu_0 I(\underline{r}-\underline{r}')\cdot d\underline{S}/4\pi|\underline{r}-\underline{r}'|^3$. For a finite current loop S, use superposition, $\psi = (\mu_0 I/4\pi)\int \underline{R}\cdot d\underline{S}'/R^3 = \mu_0 I\Omega/4\pi$ where Ω is the solid angle subtended at $\underline{r}$ by the loop. Now integrate $\underline{B}$ around a test loop $C = \partial S$ not intersecting the wire to get Ampère's law $\oint_{\partial S}\underline{B}\cdot d\underline{\ell} = -\oint_{\partial S}\underline{\nabla}\psi\cdot d\underline{\ell} = -\Delta\psi = \mu_0 I$ whence M3′ and M3 in the static case. Note that ψ exists only where current vanishes; it may be multivalued in a non-simply-connected region.

Since $\nabla \cdot \underline{B} = 0$ we may define the magnetic <u>vector potential</u> $\underline{A}(\underline{r})$ such that $\underline{B} = \nabla \wedge \underline{A}$ (see Vector Calculus or Potential Theory). $\underline{A}$ is not unique: we may add any $\nabla\psi + \underline{A}_0$: verify that we may do so in such a way as to get $\nabla \cdot \underline{A} = 0$ (the <u>Radiation gauge</u>). Then M3 simplifies to $\nabla^2 \underline{A} = -\mu_0 \underline{j}$ (cf $\nabla^2\phi = -\rho/\varepsilon_0$) which is solved in Cartesians by $\underline{A}(\underline{r}) = \mu_0/4\pi \int_{V'} \underline{j}(\underline{r}')d^3\underline{r}'/R$ if the current is steady. By differentiating we get the <u>Biot-Savart Law</u> $\underline{B}(\underline{r}) = \mu_0/4\pi \int_{V'} \underline{j}(\underline{r}') \wedge \underline{R}\, d^3\underline{r}'/R^3$. Also, for a line current, $\underline{B}(\underline{r}) = \mu_0 I/4\pi \int d\underline{\ell} \wedge \underline{R}/R^3$

<u>Examples</u> (1) long straight wire $\underline{B} = \mu_0 \underline{I} \wedge \underline{r}/2\pi r^2$. The force per unit length between parallel wires carrying currents I, I' is $\mu_0 II'/2\pi d$ where d is the separation; this is attractive if the currents are in the same direction. Note that this is used as the definition of the Amp.
(2) Circular loop, radius a in xy-plane; at distance z along z-axis, $\underline{B} = \frac{1}{2}\mu_0 I a^2 (a^2+z^2)^{-3/2} \hat{\underline{z}}$. (3) Inside a long solenoid with n turns per unit length, carrying current I, $\underline{B} = \mu_0 n I \hat{\underline{z}}$ – note that this is uniform.
(4) Inside a long cylindrical conductor $\underline{B} = \mu_0 \underline{I} \wedge \underline{r}/2\pi a^2$; the force is inwards, hence the plasma pinch effect.
(5) Force on current loop, dipole moment $\underline{\mu} = J d\underline{S}$: force like dipole $\underline{F} = \underline{\mu} \cdot \nabla \underline{B}$, couple $\underline{G} = \underline{\mu} \wedge \underline{B}$ (cf $\underline{p} \wedge \underline{E}$ for electric dipole).

Maxwell guessed the existence of the displacement current $\varepsilon_0 \dot{\underline{E}}$, whose theoretical justification really comes from Special Relativity (see IB Fields). This is negligible unless the time variation is very rapid, since $\mu_0\varepsilon_0 = 1/c^2 \sim 10^{-17}\, m^{-2}s^2$. Applying div ($\nabla \cdot$) to M3 then gives conservation of charge again.

INDUCTION.

$\Phi = \int_S \underline{B} \cdot d\underline{S}$ is the <u>flux</u> through surface S: this is the number of lines of force since M2 says they're conserved. If $C = \partial S$ is a wire, this produces an emf $-\dot{\Phi}$ around C: negative since current opposes change in B-field. The emf caused by a varying current is $V = -L\dot{I}$ where L, a constant, is the <u>inductance</u>. Discharging a coil generates a spark: this is the energy $\frac{1}{2}LI^2$ which is stored in the B-field, $\frac{1}{2\mu_0}\int_V B^2 d^3\underline{r}$ (cf $\frac{1}{2}\varepsilon_0 \int_V E^2 d^3\underline{r}$). Also $V = \oint \underline{E} \cdot d\underline{\ell}$, whence M4' and M4.

If we have two magnetically-linked circuits, changing the current in one generates an emf $\pm M\dot{I}$ in the other where M is the <u>mutual inductance</u>, which is symmetric. $L_i = \mu_0 n_i^2 \ell A$ (where n_i = no of turns per unit length on i^{th} coil, $i=1,2$) and $M^2 \leq L_1 L_2$. Changing either $\underline{B}$ or the geometry generates an emf. Eg (1) rotating plane circuit in uniform field, $\Phi = AB\cos\theta$, current $AB\omega\sin\theta / R$ where A = area, ω = angular velocity and R = circuit electrical resistance. (2) Faraday rotating disc: emf $\frac{1}{2}\omega a^2 B$ between centre and rim (a = radius). (3) Moving coil microphone: emf $= -L\dot{I} - \dot{L}I$.

ELECTROMAGNETIC ENERGY

An electrostatic field does work on a current at rate $\int_V \underline{E}\cdot\underline{j}\, d^3\underline{r}$, the <u>Ohmic heating</u>. If $\underline{j} = \gamma\underline{E}$ this gives $\frac{1}{\gamma}\int_V j^2 d^3\underline{r}$. Using M3 and M4 we get

$$-\frac{d}{dt}\int_V \tfrac{1}{2}(\varepsilon_0 E^2 + B^2/\mu_0)\, d^3\underline{r} = \int_V \underline{j}\cdot\underline{E}\, d^3\underline{r} + \oint_{\partial V} \frac{1}{\mu_0}\underline{E}\wedge\underline{B}\cdot d\underline{S},$$

ie the rate of decrease of electromagnetic energy is equal to the rate of working on the charges plus the rate of flow of energy out across ∂V. The flux is $\underline{P} = \frac{1}{\mu_0}\underline{E}\wedge\underline{B}$, called the <u>Poynting vector</u>: its meaning is nonlocal: it is <u>only</u> meaningful when integrated over a closed surface since the curl of any vector field may be added to it with it affecting the result. Example: steady current in straight wire; then $\frac{1}{\mu_0}\underline{E}\wedge\underline{B} \perp$ wire and nonzero only outside it. In the integration the total contribution is from outside the wire; this is equal to the ohmic heating, but the energy flow is not necessarily entirely outside the conductor.

EXAMPLES.

<u>Field discontinuities</u> between vacuum & conductor: $\Delta E_\perp = \sigma/\varepsilon_0$ (σ = surface charge density) $\Delta B_\perp = 0$ $\Delta \underline{E}_\parallel = 0$ $\Delta \underline{B}_\parallel = \mu_0 \underline{J}\wedge\underline{n}$ ($\underline{J}$ = surface current density, $\underline{n}$ = unit normal). eg cyclindrical cell conductor: current $\perp$ axis (like a coil), $\Delta\underline{B}_\parallel \perp \underline{J}$. Permanent magnet & superconductor: $\underline{B} = 0$ inside; use method of images. Surface current, gives repulsive force equivalent to that due to an equal dipole in the image position.

<u>Relaxation time</u>. Nonzero charge density in a uniform medium decays as $e^{-t\gamma/\varepsilon_0}$ until all charge on surface or at infinity. $\varepsilon_0/\gamma \simeq 10^{-14}$ sec. for a metal, so take $\rho = 0$ inside conductor. Then we have the <u>Equation of telegraphy</u>, $\nabla^2\underline{E} = \gamma\mu_0\dot{\underline{E}} + \frac{1}{c^2}\ddot{\underline{E}}$. Also $\nabla^2\underline{B} = \gamma\mu_0\dot{\underline{B}} + \frac{1}{c^2}\ddot{\underline{B}}$.

Now look for harmonic solutions, $\underline{E}(\underline{r},t) = e^{i\omega t}\underline{E}_0(\underline{r})$ [ie the real part of this, but when dealing with periodic functions linearly we usually omit to say so; however be careful when multiplying or squaring: often a factor of $\frac{1}{2}$ is needed]. At very high frequency, ie $\omega \gg \gamma/\varepsilon_0$, displacement current dominates and the conductivity is irrelevant - we get the wave equation $\nabla^2\underline{E} = \frac{1}{c^2}\ddot{\underline{E}}$. At low frequency conduction dominates and we have the diffusion equation, $\nabla^2\underline{E} = \gamma\mu_0\dot{\underline{E}}$.

Plane waves in vacuo. Two pairs, (E_y, B_x) and (E_x, B_y) are coupled. Polarised solution: $E_y = E_z = B_x = B_z = 0$ and $E_x = \frac{1}{k}B_y = E_0 \exp(i\omega t - ikz)$ where $kc = \omega$. k = wave no., ω = angular frequency, $c = (\mu_0\varepsilon_0)^{-1/2}$ = speed of light. Thus $\underline{E}(\underline{r},t) = \underline{E}_0 \exp i(\omega t - \underline{k}\cdot\underline{r})$, $\underline{B} = \underline{k}\wedge\underline{E}/\omega$, $\underline{E}_0 \perp \underline{k}$, $\underline{E}_0$ & $\underline{k}$ constant. Then $(\underline{E}, \underline{B}, \underline{k})$ form a right-handed orthogonal triad. Oblique reflection by a surface, or transmission through some crystals may affect the two polarisations differently. White light is a mixture of frequencies & polarisations.

Normal reflection off perfect conductor. Incident wave has $\underline{E} = \hat{\underline{x}}E_0 \exp i(\omega t - kz)$, $\underline{B} = \hat{\underline{y}}\,{}^{E_0}/_c \exp i(\omega t - kz)$; reflected wave has $\underline{E} = -\hat{\underline{x}}\exp i(\omega t + kz)$, $\underline{B} = \hat{\underline{y}}\,{}^{E_0}/_c \exp i(\omega t + kz)$, so total field has $E_x = 2E_0 \sin\omega t \sin kz$ $B_y = {}^{2E_0}/_c \cos\omega t \cos kz$. The reflected wave is caused by an oscillating surface current $J_x = -{}^{2E_0}/_{\mu_0 c} \cos\omega t$. This causes a radiation pressure of $\varepsilon_0 E_0^2$ in time-average. This is interpreted as the momentum carried by the incident wave which is reversed by the reflected wave. This agrees with the treatment using the Poynting vector; the energy density in front of the plate is $W = \frac{1}{2}\varepsilon_0 E_0^2$, so energy travels with speed c.

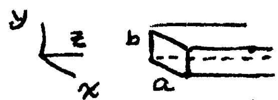
Imperfect conductor. Now there is a penetrating wave which decays as $e^{-z/\tau}$ where $\tau = {}^{2\varepsilon_0 c}/_\gamma + {}^{\gamma c}/_{4\varepsilon_0\omega^2} - O(\omega^{-4})$ with speed $<c$. The surface current is now spread throughout the penetrating depth τ. For a conducting sheet of finite thickness, calculate the incident, reflected, transmitted and two internal waves. The radiation pressure on the front is now less than that for a perfect conductor.

Parallel-plate capacitor at high frequency. Solve iteratively using the displacement current. Then $E = E_0 e^{i\omega t} J_0(\omega r/2c)$ at distance r from axis, where J_0 is the zero-order Bessel function (see Potential Theory).

The correction terms are negligible even at radio frequencies (RF). Radiation cavity: capacitor with isolated plates. Resonant frequencies $n\pi c/a$ $(n=1,2,\dots)$. Standing waves $\underline{E} = \Sigma_n \underline{E}_n^0 \cos\frac{n\pi ct}{a} \sin\frac{n\pi z}{a}$ and similarly $\underline{B}$. Such a cavity forms part of a laser (Part II Quantum Electronics)

Waveguide: pipe with perfectly-conducting inner wall; wave eqn. inside. There are three modes of solution: (i) transverse electric (TE) $E_{\parallel}=0$ $\underline{B}_{\parallel} \neq 0$; (ii) transverse magnetic (TM) $\underline{B}_{\parallel}=0$ $\underline{E}_{\parallel} \neq 0$; (iii) transverse electromagnetic: $\underline{E}_{\parallel} = \underline{B}_{\parallel} = 0$ — similar to free space, but needs two concentric pipes in order to satisfy boundary conditions nontrivially. Optical waveguide (glass fibre) works similarly, using total internal reflection.

TE mode in rectangular pipe: $E_z = 0$.



$E_x = Am a \cos\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \exp i(\omega t - kz)$,
$E_y = -Am b \sin\frac{n\pi x}{a} \cos\frac{m\pi y}{b} \exp i(\omega t - kz)$ $\quad m,n = 0,1,2,\dots$ $\quad a \geq b$

k, ω now satisfy $n^2\pi^2/a^2 + m^2\pi^2/b^2 + k^2 = \omega^2/c^2$, with the general solution obtained by superposition. Any frequency $\omega > c\pi/a$ can propagate, with discrete values of k for each. Phase velocity (propagation of wave "pattern") is $c_p = \omega/k > c$, but group velocity (with which energy propagates) $c_g = c^2 k/\omega < c$. [See Part II Waves]. Get $\underline{B}$ from $\nabla \wedge \underline{E} = -\dot{\underline{B}} = -i\omega \underline{B}$: all components nonzero. There are current sheets in the walls. The TM mode is treated similarly.

Hence also the standing waves in a cubical cavity of dimensions (a_1, a_2, a_3): there are now discrete values of ω allowed, namely $\omega = \frac{1}{c}(\Sigma_1^3 \, l_i^2 \pi^2 / a_i^2)^{1/2}$ for $l_i = 0,1,2,\dots$ $(i=1,2,3)$.

Note: This course summary was based upon Peter Landshoff's excellent lectures of Lent 1980. After my showing him the original form of this summary in February 1982, Keith Moffatt explained briefly his alternative treatment, as given in his own lectures. I had intended to revise this summary to include an account of that treatment, but was unable to do so in time for this edition.