LINEAR SYSTEMS

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We're concerned with <u>linear</u> operators, $\mathcal{L}$, on a <u>vector space</u> of functions on an <u>interval</u> (a,b) with certain differentiability and boundary conditions. For example $\mathcal{L} = D = d/dx$ is linear since $(\lambda f + \mu g)' = \lambda f' + \mu g'$. In a finite-dimensional space we're familiar with <u>matrices</u> and summation over the coordinates: here certain linear operators will be given by <u>transfer functions</u>, $K(x, x')$, so $(Kf)(x) = \int_a^b K(x,x') f(x') dx'$. Often we have an <u>inner product</u>, $\langle f, g \rangle = \int_a^b f(x) g(x) dx$ (sometimes with a <u>weight function</u>), hence also <u>length</u>, $\|f\| = \langle f, f\rangle^{1/2}$, <u>angle</u> and <u>perpendicularity</u>: $f \perp g$ iff $\langle f, g\rangle = 0$. An <u>orthonormal</u> base (or basis) is one whose vectors have unit length and are mutually perpendicular.

<u>Example</u>: the response of an ammeter to a varying current $I(t)$ is given by $\int_{-\infty}^{t} e^{-(t-t')/t_0} I(t') dt'$, where $1/t_0$ is the decay constant. This is an example of a <u>convolution</u>, where $K(x,x') = g(x-x')$ and $g(x-x') = 0$ for $x \leq x'$. Here $g(x) = e^{-x/t_0}$ for $x > 0$. We write $(g \circ f)(t) = \int_{-\infty}^{t} g(t-t') f(t') dt'$.

Another useful tool is the <u>Wronskian</u> (note spelling), given by

$$W[f_1, f_2, \ldots, f_n](x) = \det \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}$$

If $f_1, f_2, \ldots, f_n$ are linearly dependant, $W(x) = 0$, but not conversely.

A very important linear operator is the <u>Fourier transform</u>, together with its inverse for suitable f (differentiable will do); note the different signs.

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k) dk$$

The functions $e^{ikx}/\sqrt{2\pi}$ for various values of k are orthonormal, and moreover we have <u>Parseval's theorem</u>, $\langle \tilde{f}, \tilde{g} \rangle = \langle f, g \rangle$, that the inner product is preserved. In particular for <u>Fourier series</u>, $f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$, the length is the same:

$$\|f\| = \left[\int_0^{2\pi} f(t)^2 dt\right]^{1/2} = \left[\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]^{1/2} = \|\tilde{f}\|$$

This means that the power content of a signal is given by the sum of the power contents of each component frequency. Finally we have the <u>convolution theorem</u>, $\tilde{f}\tilde{g} = \widetilde{f \circ g}$, ie

$$\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'} g(x')dx'\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} dx \int_{-\infty}^{\infty} f(x-x')g(x')dx'$$

Note that if $f(x)$ is real, $\tilde{f}(-k) = \tilde{f}^*(k)$, that if $f(x)$ is real and even, so also is $\tilde{f}(k)$, and that if $f(x)$ is real and odd, $\tilde{f}(k)$ is imaginary and odd. The behaviour of $\tilde{f}(k)$ for large k (frequency) relates to the smoothness of $f(x)$; if $f(x)$ is only piecewise continuous, the inversion formula gives the average, $\frac{1}{2}[f(x-)+f(x+)]$ at the discontinuity. We have the following results, but note that the last one is proved in the IB Mathematical Methods.

$$g(x) = f(x-a) : \tilde{g}(k) = e^{-ika}\tilde{f}(x) \qquad g(x) = e^{ax}f(x) : \tilde{g}(k) = \tilde{f}(k+ia)$$

$$g(x) = f'(x) : \tilde{g}(k) = ik\tilde{f}(x) \qquad g(x) = xf(x) : \tilde{g}(k) = -i\tilde{f}'(k)$$

$$f(x) = \begin{cases} e^{-ax} & (x>0) \\ 0 & (\text{otherwise}) \end{cases} : \tilde{f}(k) = \frac{1}{a+ik} \qquad f(x) = \begin{cases} 1 & (-a<x<a) \\ 0 & (\text{otherwise}) \end{cases} : \tilde{f}(k) = \frac{2\sin ak}{k}$$

$$f(x) = e^{-a|x|} : \tilde{f}(k) = \frac{a\sqrt{2/\pi}}{a^2+k^2} \qquad f(x) = \frac{1}{a^2+x^2} : \tilde{f}(k) = \frac{\sqrt{2\pi}}{a} e^{-a|k|}$$

$$f(x) = \frac{\lambda}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2x^2} : \tilde{f}(k) = \frac{\sqrt{2\pi}}{\lambda} e^{-\frac{1}{2}k^2/\lambda^2}$$

<u>Ordinary differential equations</u> write $Ly = f(D)y$ for $y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y$ where $D = d/dx$ and $a_{n-1}, \dots, a_0$ are constants. We aim to solve the <u>homogeneous</u> equation $Ly = 0$ and the corresponding <u>inhomogeneous</u> equation $Ly = g(x)$. The solutions of the former form an n-dimensional vector space, V, the <u>kernel</u> of L; moreover if y_0 is any solution to the latter, the general solution is of the form $\{y_0 + y : y \in V\} = y_0 + y$. y_0 is a <u>particular integral</u> and y a <u>complementary function</u>.

If λ is a root of the polynomial equation $f(t)=0$, $y = e^{\lambda x}$ is a solution of $Ly = 0$. In the degenerate case where λ is a repeated root, so $(t-\lambda)^m$, say, divides $f(t)$, then $xe^{\lambda x}, x^2e^{\lambda x}, \dots, x^{m-1}e^{\lambda x}$ are also solutions, giving the full set of n linearly independent functions (since the Wronskian doesn't vanish). For it can be shown that if the a_i are constant (or even analytic) and $y = y' = \dots = y^{(n-1)} = 0$ at $x = x_0$ then

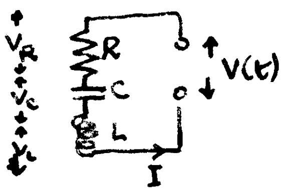
$Ly = 0$ has only the solution $y = 0$. As such, this is an example of a <u>well-posed problem</u>.

In the <u>second order</u> case let $Ly = y'' + 2py' + qy = g(x)$. For $g = 0$ the general solution is

$$y(x) = \begin{cases} ae^{-px}\sin(\omega_0 x + \theta) & \text{if } \omega_0^2 = q - p^2 > 0 \quad [\underline{\text{underdamped}}] \\ (a + bx)e^{-px} & \text{if } q - p^2 = 0 \quad [\underline{\text{critically damped}}] \\ ae^{-\lambda x} + be^{-\mu x} & \text{if } q - p^2 < 0 \quad [\underline{\text{overdamped}}] \end{cases}$$

where a, b and θ are arbitrary constants and $\lambda, \mu = -p \pm \sqrt{q - p^2}$ in the last case. If $g = c$, a constant, a particular integral is $y(x) = c/q$. Finally if $g(x) = r\sin\omega x$ it is $R\sin(\omega x - \varepsilon)$ if $\omega \neq \omega_0$ or $p \neq 0$ and $-\frac{1}{2}rx\cos\omega x$ if $\omega = \omega_0$ and $p = 0$. The amplitude, $R = r/\sqrt{[(p^2 + (\omega^2 - \omega_0^2))^2 + 4p^2\omega_0^2]}$, is greatest at <u>resonance</u>, $\omega = \omega_0$. The phase shift is $\varepsilon = \tan^{-1}[2p\omega/\sqrt{(q - \omega^2)}]$. If, as usual, $p > 0$ we have <u>damping</u>, in which case the (decaying) complementary function is said to be <u>transient</u>.

Examples of this case include <u>springs</u> with frictive or viscous terms: $my'' + 2\eta y' + ky = g(t)$, where m is the mass, η the friction, k the spring constant and $g(t)$ the applied force; also <u>LCR-circuits</u>



$V_L = L\dot{I} \quad I = \dot{Q} \quad V_C = Q/C \quad V_R = IR$

so $\ddot{Q} + R/L\,\dot{Q} + 1/LC\,Q = V(t)$

and the critical (<u>resonant</u>) frequency is $\omega = (L/C - R^2)^{1/2}/L$.

A general system of linear ODEs may always be reduced to a system of n first order linear ODEs where n is the total order. For instance put $y_k = y^{(k)}$, so $Dy_k - y_{k+1} = 0 \quad (k = 0, 1, \ldots, n-2)$ and

$Dy_{n-1} + a_{n-1}y_{n-1} + a_{n-2}y_{n-2} + \cdots + a_1 y_1 + a_0 y_0 = g(x)$.

This may then be expressed as a matrix: $Dy_i = \sum a_{ij} y_j + b_i g$. The roots λ_i are the <u>eigenvalues</u> of this matrix, $A = (a_{ij})$. If, by changing the basis to consist of <u>eigenvectors</u>, we can <u>diagonalise</u> the matrix, so $a_{ij} = \lambda_i \delta_{ij}$, we can solve it easily by $y_i = e^{\lambda_i x}$. In the degenerate case diagonalisation is not possible and we need the <u>Jordan canonical form</u> from Algebra III. In the second order case above, $\omega^2 = \det A$, the determinant of the matrix of coefficients.

This is the method for solving coupled systems of equations: reduce them to the matrix form and diagonalise by a change of basis. The solutions obtained thus are the normal modes of the system.

ODEs: some more tricks.

1: Reduction of order may be achieved if we know one solution, say u, to the equation $Ly=0$. For example if $Ly = y'' + 2py' + qy = 0$, put $y = uv$. Then $(u'' + 2pu' + qu)v + 2u'v' + uv'' + 2puv' = 0$, so another (linearly independent) solution is $u\int u^{-2}e^{-2\int p\,dx}\,dx$.

2: Abel's equation. Let $W(x)$ be the n^{th} order Wronskian of a system of solutions to $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$; then we find that $W'(x) + a_{n-1}'(x)W(x) = 0$, so in particular if $a_{n-1}(x)$ is constant so is $W(x)$. Thus if we have $n-1$ solutions we may find the last by solving this equation for it.

3: Euler type equations. To solve $\sum a_r x^r y^{(r)} = 0$ where $a_r = \text{const.}$, try $y = x^k$, $y^{(i)} = (k!/(k-i)!)x^{k-i}$, to get a polynomial equation for k. Eg $x^2y'' + 2pxy' + qy = 0$, then $k(k-1) + 2pk + q = 0$, so $k = -p + \frac{1}{2} \pm \sqrt{[(p-\frac{1}{2})^2 - q]}$. If equal roots use either (1) or (2) above: in this case general solution is $(A + B\log x)x^{-p+\frac{1}{2}}$.

4: First order. $y' + py = q$. Multiply by $e^{\int p\,dx}$ to give "perfect differential", so $[e^{\int p\,dx}y]' = e^{\int p\,dx}q$, and $y = e^{-\int p\,dx}\int e^{\int p\,dx}q\,dx$. There is unfortunately no analogue for simultaneous systems.

5: Fourier transform with constant coefficients. Take Fourier transform of both sides, remembering $\widetilde{y^{(r)}} = (ik)^r\tilde{y}$. This reduces to a linear algebraic equation for $\tilde{y}$. eg $y'' + 2py' + qy = g$ becomes $(-k^2 + 2ipk + q)\tilde{y} = \tilde{g}$. Get y by inverse transformation on $\tilde{g}/f(ik)$: note zeroes correspond to normal mode solutions, $x^r e^{-\lambda x}$, found before. Usually need contour integration (IB Mathematical Methods) for all but trivial $g(x)$.

6: Fourier transform with polynomial coefficients of degree $\leq (n-1)$ in x. Here the Fourier transform leads to another linear ODE, for $\tilde{y}$, of lower degree since $\widetilde{xy} = -i\tilde{y}'$, which may be easier to solve.

<u>δ-function</u> When we express linear operators (which are sometimes called "functionals") as transfer functions, thus

$$f(x) \longmapsto (Kf)(x) = \int_{-\infty}^{\infty} K(x,x')f(x')\,dx',$$

to what does the identity operator correspond? It's called the δ-function, $\delta(x-x') = K(x,x')$, but since $\delta(x-x')=0$ for all $x \neq x'$ and yet $\int_{-\infty}^{\infty}\delta(x-x')\,dx' = 1$ it is not a function, but a <u>generalised function</u> or <u>distribution</u>. It only has a meaning when it's integrated with a function which is continuous at its "spike", thus

$$\int_{x-\varepsilon}^{x+\varepsilon} \delta(x-x')f(x')\,dx' = f(x)$$

The formal definition of such "functions" is not undertaken in this course, but they may de defined in terms of the <u>dual space</u> (see Algebra II) of the
δ - given space of functions, ie in fact as linear operators. When it appears on the right of an inhomogeneous equation, $Ly = \delta(x)$, it represents physically an instantaneous impulsive "kick" to the system of unit strength. Thus an impulsive force is written $I\delta(t)$, a point charge $q\delta^3(\underset{\sim}{x}) = q\delta(x)\delta(y)\delta(z)$, where $\delta^3(\underset{\sim}{x})$ is the vector δ-function.

The function $H(x) = \int_{-\infty}^{x}\delta(x')\,dx'$ is called the <u>step function</u>. $H(x)=0$ for $x<0$ and $H(x)=1$ for $x>0$. If the probability that some random variable is less than x is $F(x)$ (so $F(-\infty)=0$, $F(\infty)=1$ and F is increasing), then $f(x)=F'(x)$ is the <u>probability distribution</u>. So if $F(x)=H(x)$, $f(x)=\delta(x)$ and there is a finite probability (certainty) that the random variable takes the value exactly zero.

If $\delta(x)$ represents a charge, $\delta'(x)$ represents a <u>dipole</u>; we have $\delta'(x)=0$ for $x\neq 0$ and

$$\int_{-\infty}^{\infty} f(x')\delta'(x-x')\,dx' = f'(x)$$

if f is differentiable at x. The fourier transforms of $\delta(x)$, $H(x)$ and $\delta'(x)$ are $1/\sqrt{2\pi}$, $-ik/\sqrt{2\pi}$ and $i/k\sqrt{2\pi}$ respectively.

$\delta(x)$ may be thought of as a "limit" of a sequence of functions, eg

(i) $$f_n(x) = \begin{cases} \tfrac{1}{2}n & (-1/n < x < 1/n) \\ 0 & (\text{otherwise}) \end{cases}$$

(ii) $$f_n(x) = \frac{n}{\pi(1+n^2x^2)}$$

(iii) $f_n(x) = \frac{1}{2} n e^{-n|x|}$

(iv) $f_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2}$ (cf diffusion equation)

(v) $f_n(x) = \dfrac{\sin nx}{\pi x}$

Mathematically we may view the δ-function as giving a "coordinate basis" for our space of functions, because analogously to $\underset{\sim}{v} = v_1 \hat{\underset{\sim}{x}} + v_2 \hat{\underset{\sim}{y}} + v_3 \hat{\underset{\sim}{z}}$ we can write $f(x) = \int_{-\infty}^{\infty} f(x')\,\delta(x-x')\,dx'$, and just as we might solve $A\underset{\sim}{x} = \underset{\sim}{b}$ first for $\underset{\sim}{b} = (1,0,0)$, $(0,1,0)$ and $(0,0,1)$, in this case we solve $Ly = g(x)$ for $Ly = \delta(x-x')$ and get the general solution by integration.

To solve the <u>boundary value problem</u> $Ly = \delta(x-x_0)$ in an interval $[a,b]$ where y and its derivatives are prescribed at the endpoints and $a < x_0 < b$, we solve it in stead separately in $[a, x_0]$ and $[x_0, b]$. Now if y is a solution which, together with its first $n-1$ derivatives, is bounded near x_0, we have $\int_{x_0-\varepsilon}^{x_0+\varepsilon} (Ly)(x)\,dx = 1$, where $(Ly)(x) = y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \ldots + a_0 y(x)$ and $\int_{x_0-\varepsilon}^{x_0+\varepsilon} y^{(n)}(x) = \Delta y^{(n-1)}(x_0) = \operatorname{Lt}_{\varepsilon\to 0^+} y^{(n-1)}(x_0+\varepsilon) - \operatorname{Lt}_{\varepsilon\to 0^+} y^{(n-1)}(x_0-\varepsilon)$.
But $\left| \int_{x_0-\varepsilon}^{x_0+\varepsilon} a_{n-i}\, y^{(n-i)}(x)\,dx \right| \le |a_{n-i}| . 2\varepsilon . \sup |y^{(n-i)}(x)| \to 0$ $(i \ge 1)$ as $\varepsilon \to 0$, so we have $\Delta y^{(n-1)}(x_0) = 1$ and $\Delta y^{(i)}(x_0) = 0$ $(i = 0, 1, \ldots, n-2)$. This gives the required n extra boundary conditions to solve the equation.

The solution, $y(x)$, of course depends also on x_0, and so we write $y(x) = G(x, x_0)$, which is called the <u>Green's function</u> for the problem. Since the original inhomogeneous equation was linear, $y(x) = \int_{-\infty}^{\infty} G(x,x')g(x')\,dx'$ is a solution of

$$Ly = g(x) = \int_{-\infty}^{\infty} \delta(x-x')g(x')\,dx' = \int_{-\infty}^{\infty} (LG)(x,x')g(x')\,dx .$$

Thus remembering that $\delta(x-x')$ was our "identity" transform, I, we have $LG = I$, so we may regard G as an <u>inverse operator</u>, L^{-1}, to L. Thus $Ly = g$ is solved by $y = L^{-1}g = Gg$. Note that Green's or transfer functions may sometimes also be called <u>kernel functions</u>, but don't confuse this with the algebraic meaning. If the independant variable is time, t, and the coefficients are constant, G is <u>causal</u>: $G(x,x') = 0$ for $x < x'$, ie the output of the system depends only on the input at earlier times; in fact $G(x,x') = h(x-x')$, a convolution.

If the system is an <u>electrical circuit</u> with constant capacitors, resistors and inductors, it has a governing equation $RI = V$, where R (or L or X) is a differential operator called resistance or <u>reactance</u>, $V(t)$ is the applied potential and $I(t)$ is the output current. If I is sinusoidal, $I = \tilde{I} \sin \omega t$, where ω is a fixed frequency, X becomes the familiar complex reactance $X = R + i\omega L - i/\omega C$. This is the same as taking the Fourier transform, giving $X(\omega)\,\tilde{I}(\omega) = \tilde{V}(\omega)$. The reciprocal, $Y(\omega) = 1/X(\omega)$, is called the <u>admittance</u> and is the Fourier transform of the Green's function of the equation. The circuit is a <u>high-</u> or <u>low-pass filter</u> if $Y(\omega)$ has its peak at high or low values of ω, respectively.

The <u>Diffusion equation</u>, $\partial T/\partial t = c\,\partial^2 T/\partial x^2$, governs the temperature, $T(x,t)$, in a metal bar, where $c>0$ is a constant. To solve this we try <u>separation of variables</u>, ie put $T(x,t) = u(x)v(t)$. Then $cu''/u = v'/v$, where the LHS is independant of t and the RHS of x, so both sides are constant, $-\lambda$, say. Then if $\alpha^2 = \lambda/c$, $T(x,t) = Ae^{-\lambda t}\sin\alpha(x-x_0)$. Suppose now $T(x,0) = g(x)$ is given, and also $T(\pm a, t) = 0$. Then we have odd and even solutions

$$T_{2n+1}(x,t) = e^{-\lambda_{2n+1}}\sin\alpha_{2n+1}x \qquad T_{2n}(x,t) = e^{-\lambda_{2n}}\cos\alpha_{2n}x$$

where $\lambda_m = (m+1)^2\pi^2 c/4a^2$ and $\alpha_m = \pi(m+1)/2a$. If we express $g(x)$ as a Fourier series, also in terms of $\sin\alpha_{2n+1}x$ and $\cos\alpha_{2n}x$, the same coefficients give the coefficients of $T_m(x,t)$ in the general solution.

If we now put $k = 2n\pi/a$ and let $a\to\infty$, choosing $g(x) = \delta(x)$, the coefficients become unity and we have a solution

$$E(x,t) = \int_{-\infty}^{\infty} e^{-k^2ct+ikx}\,dk = \sqrt{2\pi}\,e^{-x^2/4ct} = 2\pi\,\mathrm{erf}\frac{x}{\sqrt{2ct}}$$

We can also get this by taking the Fourier transform of the equation, $\partial\tilde{T}/\partial t = -ck^2\tilde{T}$, so $\tilde{T}(k,t) = \tilde{T}(k,0)e^{-ck^2t}$. Note the behaviour of $E(x,t)$ for various values of $t \geq 0$: it has a δ-function "spike" at $t=0$, which diffuses (as in example (iv) of the functions "converging" to $\delta(x)$) as t increases. As before the general solution is obtained by integration:

$$T(x,t) = \int_{-\infty}^{\infty} E(x',t)\, g(x')\,dx'.$$

Here $E(x',t)$ is the transfer function taking the temperature distribution at time 0 to that at time t.

With the <u>wave equation</u>, $\partial^2\phi/\partial x^2 - 1/c^2\, \partial^2\phi/\partial t^2 = 0$, we may use

1: separation of variables: $\phi(x,t) = u(x)v(t)$, whence $u(x) = \sin kx$, $v(x) = \sin \omega t$, etc., with $k = \omega/c$.

2: Fourier transforms, $-k^2\tilde{\phi} - 1/c^2\, \partial^2\tilde{\phi}/\partial t^2 = 0$, so $\tilde{\phi}(k,t) = \sin kct$.

3: change of coordinates: $\phi(x,t) = \psi(x-ct, x+ct) = \psi(u,v)$, so $\partial^2\phi/\partial u\partial v = 0$, whence $\psi(u,v) = f(u) + g(u)$. This trick is not possible in more dimensions.

The first two methods reduce to looking for <u>normal modes</u>: $\phi(x,t) = \psi(x)e^{i\omega t}$ (ω = const). Then get <u>eigenvalue equation</u>, $\psi'' + \omega^2/c^2\,\psi = 0$. The boundary conditions usually restrict $\lambda = -\omega^2/c^2$ to a countable set of eigenvalues, $\lambda_1, \lambda_2, \ldots$ If these are distinct, the solution space (for ψ) has an orthonormal basis of <u>eigenvectors</u> (or <u>eigenfunctions</u>), ψ_n, when a suitable weight function (usually corresponding physically to a mass or density of some kind) is taken. Then if initial conditions are $\phi(x,0) = g(x)$, general solution is $\phi(x,t) = \sum_{n=1}^{\infty} \langle \psi_n, g\rangle \psi_n(x) e^{i\omega_n t}$, where $\langle \psi_n, g\rangle = \int_{-\infty}^{\infty} \psi_n(x) g(x)\,dx$.

Take a <u>string with fixed ends</u>, so $\rho\,\partial^2\phi/\partial t^2 + T\partial^2\phi/\partial x^2 = F(x,t)$, where ρ and T are the density and tension, ϕ is the displacement and F the applied force (put $F=0$). $\phi(\pm a, t) = 0$ is the boundary condition, so $\psi_n(x)$ are trigonometric functions as before with $\omega_n = n\omega$.
If the string is <u>plucked</u>, it is given an initial displacement $\phi(x,0) = \delta(x)$, so $\phi(x,t) =$
If it is <u>struck</u>, it is given an initial velocity, $\phi_t(x,0) = \delta(x)$, so $\phi(x,t) =$
Note that here the second time derivative is involved, so we must specify both the initial displacement and the initial velocity.