REPRESENTATION THEORY.

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1. Matrix representations of groups.

The process of abstraction, in which the essential features of a problem are identified and studied without reference to the application, is an important one in Pure Mathematics. However another powerful technique is often to look for the ways in which one mathematical object may be realised inside another, better understood, one. This usually involves studying the homomorphisms from the object under investigation to some standard object or set of objects.

The IA group theory course includes one elementary theorem for representing an abstract group as a permutation group. Given any (finite) group G, there is a symmetric group S_n (for some n) and an injective homomorphism $G \to S_n$. In this course our standard objects are not $\{S_n : n = 2,3,\ldots\}$ but $\{GL(n,K) : n = 1,2,\ldots\}$ where K is usually $\mathbb{C}$.

By a <u>representation</u> of a (finite) group G on a (finite dimensional complex) vector space V we mean a map ρ assigning to each element $g \in G$ a linear map $\rho_g : V \to V$ such that $\rho_1 = 1$, the identity on V, $\rho_{gh} = \rho_g \rho_h$ and $\rho_{g^{-1}} = \rho_g^{-1}$. Thus we have a group homomorphism $\rho : G \to GL(V)$. The dimension of V is referred to as the <u>degree</u> of the representation.

The <u>trivial representation</u> has $V = K^1$ and $\rho_g = 1$ for all $g \in G$.

The <u>regular representation</u> is obtained by taking V as the vector space over K with basis the elements of the group, G (we shall restore their multiplication later, rendering V an algebra, the <u>group algebra</u>) and ρ as the linear map determined by the permutation of the basis elements which is given by right multiplicn. by elements of G. Thus

$$\rho_g : \Sigma_{x \in G} \lambda_x \text{"}x\text{"} \longmapsto \Sigma \lambda_x \text{"}xg\text{"}$$

where "x" is the basis element of V corresponding to $x \in G$. Similarly if G acts by permutation on some finite set X, there is a <u>permutation representation</u> defined in the same way.

There is a natural way to define a multiplication operation on the regular representation, namely

$$(\Sigma_{x \in G} \lambda_x \text{"}x\text{"})(\Sigma_{y \in G} \mu_y \text{"}y\text{"}) = \Sigma_{x,y \in G} \lambda_x \mu_y \text{"}xy\text{"}$$

This is associative, but commutative iff G is Abelian.

It's usually written KG and it's clear that $K \le KG$ as a subring; in fact K is in the centre of KG and so we call KG a <u>K-algebra</u>.

We have already related ρ as a system of linear maps to a homomorphism $\rho: G \to GL(V)$. The next step is to recognise that this is equivalent to regarding V as a right module for the noncommutative ring KG. For if we regard ρ as a map

$$\rho: V \times G \to V$$

satisfying the obvious relations, we may replace G by the abstract K-linear combinations of elements of G, viz KG. Thus

$$\rho: V \times KG \to V$$

and this satisfies the module axioms, and conversely (<u>exercise</u>).

At first, however, we shall talk in terms of linear maps and reintroduce the group algebra later.

2. <u>Equivalence of representations; subrepresentations; reducibility</u>

If we are given two representations $\rho: G \to GL(U)$, $\sigma: G \to GL(V)$ such that one can be obtained from the other by means of an isomorphism $\alpha: U \to V$ which simply renames the elements of the vector spaces, we may say that ρ, σ are <u>equivalent</u>. For each $g \in G$ we have a commuting square of isomorphisms:

$$\begin{array}{ccc} U & \xrightarrow{\rho_g} & U \\ \alpha \downarrow & & \downarrow \alpha \\ V & \xrightarrow[\sigma_g]{} & V \end{array}$$

so that $\sigma_g = \alpha^{-1} \rho_g \alpha$. Regarding U, V as KG-modules, α then becomes a module isomorphism. Equivalence of representations is then simply isomorphism in the category of right KG-modules.

Let $\rho: G \to GL(V)$ be a representation and $U \le V$ a vector subspace stabilised by the action of G, so $U\rho_g \subseteq U$ for all $g \in G$. Then by restricting the action ρ_g of elements of G from V to U we have a representation $\rho: G \to GL(U)$. In terms of KG, U is a submodule of V and we call it a <u>subrepresentation</u>.

If we take some subset $S \subseteq V$ and consider the K-vector space spanned by the images $\{s\rho_g : s \in S, g \in G\}$ (equivalently the KG-submodule of V generated by S), we get a subrepresentation $\langle S \rangle \leq V$ (exercise). It's the smallest representation space containing S as a subset. If S consists of a single nonzero vector, $\langle S \rangle$ will have no subrepresentations apart from 0 and itself and is said to be irreducible.

Having described isomorphisms and subobjects in an (Abelian) category, we must next describe the direct sum. Given two representations $\rho: G \to GL(U)$ and $\sigma: G \to GL(V)$, the direct sum is given by $\tau: G \to GL(U \oplus V)$ where $\tau_g: u + v \mapsto u\rho_g + v\sigma_g$. Observe that U, V are subrepresentations of $U \oplus V$ and that they are complementary as vector subspaces. Conversely a representation space possessing complementary subspaces stable under G may be expressed as their direct sum. It's easy to see that this is the same as the direct sum of KG-modules.

It is now natural to ask whether a representation space may be expressed as a direct sum of irreducible subspaces, and if so in what way (if any) it is unique. It cannot be done in general, for consider (exercise) the representation $\rho: \mathbb{Z} \to GL(2, \mathbb{C})$ by

$$n \mapsto \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

which is not irreducible since $\langle (1,0) \rangle$ is a (trivial) subrepresentation but it has no complement. We shall show that it can be done for finite groups.

3. Complete reducibility of finite groups

A representation space which can be expressed as a finite direct sum of irreducible subrepresentations is said to be completely reducible or semisimple [Other synonyms are simple for irreducible and faithful for when $\rho: G \hookrightarrow GL(V)$ is an injective group homomorphism]. We shall show that any finite dimensional complex representation of a finite group is completely reducible; this follows by an easy induction from Maschke's theorem:

Let $\rho: G \to GL(V)$ be a complex representation of a finite group and $U \leq V$ a subresentation. Then there is a complementary subrepresentation $W \leq V$ (so $V = U \oplus W$). Let W' be any vector space complement of U in V and $p': V \twoheadrightarrow W'$ the corresponding projection. Then define $p: V \to V$ by $v \mapsto {}^1/_{|G|} \sum_{g \in G} v\rho_g^{-1} p' \rho_g$ and check that its image, $W \leq V$ is stable under the action of G, is isomorphic as a vector space to W' and intersects U trivially.

We may prove the same result under the assumption that V has an inner product $(-,-)$, ie a sesquilinear Hermitean form $V \otimes V \to \mathbb{C}$ with $(v,v) > 0$ for $v \neq 0$. Replace $(-,-)$ by $\langle -,- \rangle$ where $\langle x,y \rangle = 1/|G| \sum_{g\in G} (x\rho_g, y\rho_g)$ to get a G-invariant inner product and take the orthogonal complement of U in V. [exercise: what is the relationship, logically, between the two proofs?].

Now what about uniqueness? This can't just be "up to permutation", for consider representations of the trivial group (ie just vector spaces with no additional structure). Here the irreducible representations are one-dimensional spaces but whilst $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}$ is the decomposition into irreducibles, the choice of a pair of one-dimensional subspaces is not unique even up to permutation (there are "$PGL(2,\mathbb{C})$" of them not just S_2). However the number of times each of the (isomorphism classes of) irreducibles occur in the decomposition is unique, as we shall see, and if we collect together the occurrences of each irreducible, the resulting less refined decomposition is unique up to order.

4. Schur's lemma and representations of Abelian groups

We shall prove one more basic result about representations before turning to the powerful theory of characters. This result, Schur's lemma, will also tell us that the irreducible representations of Abelian groups are all of degree 1.

A homomorphism of representation spaces is just a KG-module homomorphism, ie a linear map $\alpha: U \to V$ (where $\rho: G \to GL(U)$ and $\sigma: G \to GL(V)$ are the representations) making the square commute for all $g \in G$:

$$\begin{array}{ccc} U & \xrightarrow{\rho_g} & U \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ V & \xrightarrow{\sigma_g} & V \end{array}$$

(We saw a similar diagram in §2 with α an isomorphism). Schur's lemma says that a homomorphism between irreducible representations is either zero or an isomorphism [Exercise: what is the corresponding result for groups?]. For $\ker \alpha \leq U$ and $\operatorname{im} \alpha \leq V$ are subrepresentations so must be zero or the whole thing.

One can say more than this in the case $\alpha: U \to U$.

Here α must be scalar multiplication by some factor $\lambda \in K$ (possibly zero). For $\alpha: U \to U$ is an endomorphism of complex vector spaces and so must have an eigenvalue, λ [exercise]. Then $\alpha - \lambda: U \to U$ is still an endomorphism of representation spaces so, by the first part, must be zero or an automorphism. However λ is an eigenvalue of α, say with eigenvector $u \in U$, so $u \in \operatorname{Ker}(\alpha - \lambda) \neq 0$. Thus $\alpha - \lambda = 0$.

This easy result is of fundamental significance in the theory since it will enable us to "sort out" the irreducible subrepresentations of a representation space.

Now let $\rho: G \to GL(V)$ be an irreducible representation of an Abelian group. We can use Schur's lemma to show that it has degree 1. The converse, that if all of the irreducible representations have degree 1 then the group is Abelian, will follow easily later.

In the second part of Schur's lemma put $\alpha = \rho_g$ for $g \in G$: the lemma applies since g commutes with every $h \in G$. Then $\rho_g = \lambda_g$ for some $\lambda_g \in \mathbb{C}$ and it is clear that $\lambda: G \to GL(1, \mathbb{C}) = \mathbb{C}^*$ is a 1-dimensional representation which is isomorphic to a subrepresentation of V. Indeed G acts like λ on V and on any subspace of it. Thus if V is irreducible it is of dimension 1.

[Exercise: by considering the decomposition of the regular representation, KG, into irreducible (ie 1-dimensional) subrepresentations, prove the converse result directly.]

[Exercise: by showing that the irreducible representations of a finite Abelian group form an Abelian group of the same order called the dual group, develop the theory of double duals, tensor products and endomorphism spaces for Abelian groups analogous to that for vector spaces. Show that the results coincide with those obtained by viewing an Abelian group as a $\mathbb{Z}$-module.]

5. Characters

We shall now find that all we want to know about a representation can be given by a sequence of complex numbers. The map $\rho_g: U \to U$ is an endomorphism and so has a trace, namely the sum of the eigenvalues. Write $\chi(g)$ for this complex number. Now since $\operatorname{tr}(\alpha\beta) = \operatorname{tr}(\beta\alpha)$ we have $\chi(x^{-1}gx) = \operatorname{tr}(\rho_{x^{-1}}\rho_g\rho_x) = \operatorname{tr}(\rho_g\rho_x\rho_{x^{-1}}) = \operatorname{tr}\rho_g = \chi(g)$

Thus $\chi(g)$ depends only on the conjugacy class of g in G. A (set) function $f: G \to \mathbb{C}$ with this property is called a <u>class function</u>; the class functions on a group form a $\mathbb{C}$-algebra (under pointwise addition and multiplication) of complex dimension h, where h is the number of conjugacy classes in G.

However a character is a rather special kind of class function. If G is a finite group, the eigenvalues of any representing matrices must be (complex) roots of unity, so $\chi(g)$ is a sum of m complex n^{th} roots of unity where m is the degree of the representation and n the order of g. In particular $\chi(1)$ is a sum of m first roots of unity, ie $\chi(1) = m$. Finally, the eigenvalues of $\rho_{g^{-1}}$ are the <u>complex</u> conjugates of those of ρ_g, so $\chi(g^{-1}) = \overline{\chi(g)}$.

If $\rho: G \to GL(U)$ and $\sigma: G \to GL(V)$ are representations with characters χ and ψ respectively, then since the matrices of $\rho \oplus \sigma$ are of the form

$$\begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}$$

and their eigenvalues are just the unions (with multiplicity) of those of ρ and σ, the character of $\rho \oplus \sigma$ on $U \oplus V$ is just $\chi + \psi$.

6. <u>Orthogonality relations</u>

The characters of the irreducible complex representations of a finite group G form an orthonormal basis of the space of class functions, under the inner product given by $(\chi, \psi) = \sum_{g \in G} \chi(g) \overline{\psi(g)} / |G|$. For characters χ, ψ we may replace $\overline{\psi(g)}$ by $\psi(g^{-1})$.

Thus we are required to prove that if χ, ψ are characters of nonisomorphic irreducible complex representations then $(\chi, \chi) = 1$ and $(\chi, \psi) = 0$. This follows from Schur's lemma by elementary but tedious matrix calculation [Ledermann pp. 37-40; Serre pp. 13-15], see also §11 below.

Now since the character of $U \oplus V$ is the sum of those of U and V, (χ, ψ) gives the number of times the character χ of an irreducible representation occurs in the (general) representation (with character) ψ. This means that two representations are equivalent iff their characters are equal [<u>Exercise</u>]

The inner product (χ,ψ) has a direct interpretation in terms of the representation spaces (or KG-modules) U, V to which χ, ψ correspond, namely that it is the dimension (over $K=\mathbb{C}$) of the space of KG-module homomorphisms $U \to V$. It's easy to check directly that the maps $\alpha: U \to V$ making the relevant squares commute (§4) form a vector space; if KG were a commutative ring, this would be a KG-module, but as it is it's just a vector space [Exercise: why?].

Schur's lemma says simply that $\mathrm{Hom}_{KG}(U,V)$ has dimension zero or one for the irreducible rep^{n}s U, V according as they differ or coincide (exercise). One may also verify that $\mathrm{Hom}_{KG}(U \oplus U', V) \cong \mathrm{Hom}_{KG}(U,V) \oplus \mathrm{Hom}_{KG}(U',V)$ and $\mathrm{Hom}_{KG}(U, V \oplus V') \cong \mathrm{Hom}_{KG}(U,V) \oplus \mathrm{Hom}_{KG}(U,V')$. Thus using complete reducibility and the orthogonality relations we have $(\chi,\psi) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(U,V)$ as claimed.

It follows as a corollary that $\mathrm{Hom}_{\mathbb{C}G}(U,V) \cong \mathrm{Hom}_{\mathbb{C}G}(V,U)$ but the isomorphism is not natural: in fact one is the dual of the other [Exercise].

Returning to the characters, since they are orthogonal they must be linearly independent. Consequently there can only be at most h distinct irreducible representations, where h is the number of conjugacy classes of G. In fact there are exactly h.

From the orthogonality relations it follows that $(\chi,\chi) = \Sigma\, m_i^2$ where χ is the character of the representation whose decomposition involves m_i copies of the i^{th} irreducible. In particular χ is irreducible iff $(\chi,\chi) = 1$.

7. The character table and group algebra

It remains to show that the number of irreducible complex representations of a finite group is equal to the number of conjugacy classes. In view of the foregoing section, it will suffice to prove that if f is a class function on G with $(f,\chi) = 0$ for every irreducible character χ then $f = 0$. We shall work inside the group algebra, KG, which is the space of abstract K-linear combinations of elements of the group, with the multiplication inherited from G using the distributive law. (§1).

Put $x = \Sigma_{g \in G}\, f(g^{-1})g \in KG$. Since f is a class function, $hx = \Sigma\, f(g^{-1})hg = \Sigma\, f(hg^{-1}h^{-1})(hgh^{-1})h =$ (by change of variables) $\Sigma\, f(g^{-1})gh = xh$. Thus x is in the centre of KG. Hence for any KG-module (representation space) V, $v \mapsto vx$

is a KG-module homomorphism. In particular if V is irreducible, by Schur's lemma it is multiplication by a scalar, λ. But (exercise) $(f,\chi)=0$ means that $\lambda=0$, so x acts like 0 on V and all the other irreducible representations. Now KG can be decomposed into irreducibles, so x acts like 0 on KG, whence $x=0$. Hence $f=0$.

We may now display the irreducible represent^{n}s of a finite group as an $h \times h$ array of complex numbers, with a column for each conjugacy class and a row for each irreducible character. It is usual to put the class of the identity as the first column (which then contains the degrees of the representations) and the trivial character (which is identically 1) as the first row, with the others in ascending order of degree.

What is the character of the regular representation? Well, if χ is the character of any permutation rep^n, $\chi(g)$ is equal to the number of points (basis elements) fixed by g. In particular for the regular representation, g has no fixed points unless $g=1$, so $\chi(1)=|G|$, and $\chi(g)=0$ for $g\neq 1$. This enables us to describe the decomposition of KG, the regular representation, into irreducibles, V_i (with character χ_i). For $m_i=(\chi,\chi_i)=$ $1/|G| \sum_{g\in G} \chi(g)\chi_i(g^{-1}) = 1/|G| \cdot |G|\chi_i(1) = \chi_i(1) = f_i = \dim V_i$. Thus each irreducible representation occurs with multiplicity equal to its degree.

Thus in a sense "most" of KG consists of copies of the "most complicated" representations. It also follows that $|G| = \sum f_i^2$ Recall from IA group theory that the size of any conjugacy class divides the order of the group. It is also true that the degree of an irreducible complex representation divides the order of the group.

8 Restricted and induced representations: Frobenius reciprocity

So far in this treatment the group G has been fixed. We have investigated the complex linear representations, in other words essentially the category $\mathrm{Mod}_{\mathbb{C}G}$ of right $\mathbb{C}G$-modules. We shall now turn to the relationship between the representations of G and those of a subgroup H and find we have a pair of adjoint functors $\mathrm{Mod}_{\mathbb{C}G} \rightleftarrows \mathrm{Mod}_{\mathbb{C}H}$ called restriction and induction. This is a special case of a group homomorphism $\theta: H\to G$ where θ is injective: the same theory may also be applied to general θ.

Given a representation $\rho: G \to GL(V)$ (ie a KG-module V) and a subgroup $H \leq G$, we have a representation $\rho|_H: H \to GL(V)$, so that V is a KH-module. The character of $\rho|_H$ is just that of ρ restricted to H since each conjugacy class of H is contained in a unique conjugacy class of G. More generally the group homomorphism $\theta: H \to G$ will give the representation $\theta\rho: H \to GL(V)$ for H and render V a KH-module.

Now suppose we have a representation $\sigma: H \to GL(U)$ for $H \leq G$. G acts on the right cosets $\{Ht : t \in G\}$ of H by permutation; choose a transversal $T = \{t_1 = 1, t_2, \ldots, t_k\}$ for $G:H$, ie a subset of G such that Ht_i and Ht_j are disjoint for $i \neq j$ and $k = (G:H) = |G|/|H|$. Now take abstract $\mathbb{C}$-vector spaces $V_i =$ "Ut_i" and let G act on $V = \bigoplus V_i$ by $g: \sum v_i \mapsto \sum v_{i'} h_i$ where $t_i g = h_i t_{i'}$ in G. Then V becomes a KG-module with $\dim_{\mathbb{C}} V = (\dim_{\mathbb{C}} U)(G:H)$.

We can express this more clearly in terms of KG-modules. First observe that KG itself, as well as being a (right) KG-module, is also a (left) KH-module. Thus we may take the tensor product $U \otimes_{KH} KG$, which is then a right KG-module and so a representation space. This is constructed in the same way as the tensor product $U \otimes_K KG$, but with the additional relations that $u \otimes hx = uh \otimes x$ for $u \in U$, $x \in KG$ and $h \in H$. The dimension over $K = \mathbb{C}$ is then $\dim U . \dim KG / \dim KH = (\dim U)(G:H)$. It is now an easy <u>exercise</u> to show that $U \otimes_{KH} KG \cong V$.

More generally for $\theta: H \to G$ we define the induced KG-module as $U \otimes_{K(\mathrm{Im}\,\theta)} KG$ so that $uh \otimes x = u \otimes (h\theta)x$.

[<u>Exercise</u>: show that restriction and induction are "transitive", ie that you get the same result if you go via an intermediate group $H \leq K \leq G$ or $H \xrightarrow{\alpha} K \xrightarrow{\beta} G$].

Now consider the two K-vector spaces

$$\mathrm{Hom}_{KG}(\mathrm{Ind}_\theta U, V) \text{ and } \mathrm{Hom}_{KH}(U, \mathrm{Res}_\theta V)$$

where U is a KH-module, V a KG-module and $\theta: H \to G$ a group homomorphism (which, for the purposes of the Part II course, is injective). Given a map $\alpha: \mathrm{Ind}_\theta U \to V$ invariant under G, we may restrict attention to the K-subspace $U \otimes_{K(\mathrm{Im}\theta)} K(\mathrm{Im}\theta) \leq U \otimes_{K(\mathrm{Im}\theta)} KG$ (recalling the construction of $\mathrm{Ind}_\theta U$). In the case where θ is just the inclusion of H in G this is simply U and is a KH-module so that α becomes $\alpha|_H : U \to V$ where V is now considered simply as a KH-module (ie as $\mathrm{Res}_\theta V$). Conversely $\beta: U \to \mathrm{Res}_\theta V$ may be uniquely extended to $\beta: \mathrm{Ind}_\theta U \to V$; moreover these operations are linear and mutually inverse. [Exercise fill in the details and extend to general $\theta: H \to G$].

Consequently these two vector spaces are (naturally) isomorphic. This is what it means to say that the functors Ind_θ and Res_θ are adjoint (Exercise: what do they do to homomorphisms of KH- and KG-modules). [Since $\mathrm{Hom}_{KG}(\mathrm{Ind}_\theta U, V)$ is (dually) isomorphic to $\mathrm{Hom}_{KG}(V, \mathrm{Ind}_\theta U)$ and likewise the other two, Ind_θ and Res_θ are also adjoint the other way round.]

Taking dimensions and identifying the inner products of characters, we have for ψ, χ characters for H, G respectively, the Frobenius reciprocity formula:

$$(\mathrm{Ind}_\theta \psi, \chi)_G = (\psi, \mathrm{Res}_\theta \chi)_H$$

We have already seen that $(\mathrm{Res}_\theta \chi)(h) = \chi(h\theta)$. The induced character is given by (exercise):

$$\psi^G(g) = \frac{1}{|H|} \sum_{\substack{h \in H \\ x \in G}} \{\psi(h) : h\theta = x^{-1} g x\}$$

Alternatively, for $H \leq G$ and T a transversal,

$$\psi^G(g) = \sum_i \psi(t_i^{-1} g t_i)$$

where it is understood that the terms are zero if their arguments fall outside H.

9. Tensor products; symmetric and alternating squares

Given two representation spaces U, V for G we may take their tensor product as vector spaces and impose a KG-module structure by

$$(u \otimes v)g = (ug) \otimes vg$$

Note that this is not the tensor product as KG-mods. In fact it is the restriction to the diagonal subgroup $G \leq G \times G$ as a special case of the tensor product construction for representations U, V of groups G, H by $(u \otimes v)(g,h) = (ug) \otimes (vh)$ giving the $KG \otimes KH$-module $U \otimes V$.

Anyway, the K-dimension of $U \otimes V$ as a KG-module is $(\dim U . \dim V)$ and the set of eigenvalues of $U \otimes V$ for each $g \in G$ is the product of the sets of eigenvalues of U and of V. Thus the character is the product of the two characters.

Putting $U = V$, $U \otimes U$ is an irreducible representation space iff U has dimension 1 : in general $U\otimes U$ is highly reducible and breaks up (at least) into the symmetric and alternating squares, $S^2(U)$ and $\Lambda^2(U)$. However the product is relevant in the case of Abelian groups, since it provides the composition for the dual group (§4).

$S^2(U)$ and $\Lambda^2(U)$ are the $+1$ and -1 eigenspaces of the (representation space or $\mathbb{C}G$-module) endomorphism $\tau: U\otimes U \to U\otimes U$ by $u\otimes v \mapsto v\otimes u$, with $\tau^2=1$. If $\dim U=n$, $\dim S^2(U)=\frac{1}{2}n(n+1)$ and $\dim \Lambda^2(U)=\frac{1}{2}n(n-1)$. The action of G on these spaces is given by

$$u\otimes v \pm v\otimes u \longmapsto u\rho_g\otimes v\rho_g \pm v\rho_g\otimes u\rho_g$$

We aim to show that the characters are $\frac{1}{2}(\chi(g)^2 \pm \chi(g^2))$ at $g\in G$ where χ is the character for U.

We already know that the characters of the sum and tensor product of two representations are respectively the sum and product of the individual characters. It therefore remains only to prove that (in an obvious notation) $\chi_{S^2}(g)-\chi_{\Lambda^2}(g) = \chi(g^2)$. This being linear in the characters involved, it will suffice to prove the result for irreducible representations.

Let us turn first to $g\in G$. Since we are concerned only with $g, g^2\in G$ it suffices to prove the formulae for the cyclic (sub)group $\langle g\rangle \leq G$ for each $g\in G$, so wlog G is Abelian and all the irreducible representations have dimension 1 and may be considered simply as scalars. In other words the representation ρ_g and the character $\chi(g)$ coincide. But clearly if $\dim U=1$, $S^2(U)\cong U\otimes U\cong U$, $\Lambda^2(U)\cong 0$ and $\chi_{S^2}(g)=\chi_{U\otimes U}(g)=\rho_{U\otimes U,g}=\rho_{U,g}{}^2=\rho_{g^2}=\chi_U(g^2)$ and $\chi_{\Lambda^2}(g)=0$ as required.

These formulae make little contribution to the abstract theory of KG-modules, but they are of enormous practical advantage for calculating character tables. In the next section we shall see some more tricks for doing this and in a later section some group-theoretical deductions which can be made by inspection of the character table.

10. Group and field automorphisms.

Given a $\mathbb{C}G$-module U, how does G act on its dual (quâ vector space), $U^*=V$? For $u\in U$, $v\in V$, $g\in G$ we want $(ug)(vg) = uv$, so it is natural to put $vg=g^{-1}v$ [exercise: what does this mean?]. Then $v(gh)=(gh)^{-1}v=h^{-1}g^{-1}v=(vg)h$, as required. In this way U^* becomes a right $\mathbb{C}G$-module, or, alternatively, a left $\mathbb{C}G^{op}$-module, where G^{op} is the group (and $\mathbb{C}G^{op}$ the ring) with the same elements as G (or $\mathbb{C}G$) but the opposite multiplication. We have $G \cong G^{op}$ by $g \mapsto g^{-1}$. If U is irreducible, so is U^*; hence if $g \mapsto \chi(g)$ is an irreducible character so is $g \mapsto \chi(g^{-1})$. But we saw in §5 that $\chi(g^{-1}) = \overline{\chi(g)}$. Notice that if $C \subseteq G$ is a conjugacy class of G then so is $\bar{C} = \{g^{-1} : g \in C\}$.

This is an example of two important phaenomena in character theory, namely the effect of automorphisms of the field and the group (strictly speaking $g \mapsto g^{-1}$ is not an automorphism). Let us deal with the field first.

It is clear that for a particular finite group G we need by no means the whole of $\mathbb{C}$ to secure complete reducibility. (Notice that if the field is not algebraically closed we may fail to decompose representation spaces completely and so the theory will break down; it is usual to distinguish between <u>complete reducibility</u> over a particular field K and <u>absolute (ir)reducibility</u>, which says that the result is the same even over an extension field). In particular, if we require only the character table, we need only adjoin* a primitive $|G|^{th}$ root of unity to $\mathbb{Q}$.

Suppose then that $K:\mathbb{Q}$ is a finite normal extension such that KG is completely reducible, and let $\varphi: K \to K$ be an automorphism of K (which must, of course, fix $\mathbb{Q}$). Then since $K^{\varphi}G$ is isomorphic to KG it must have the same characters, so φ acts on the character table by permutation. In other words, applying φ to an irreducible character yields another.

Now consider group automorphisms. Inner automorphisms are of no interest since they merely permute the elements of particular conjugacy classes. Outer automorphisms, on the other hand, permute the classes and again may yield new irreducible characters.

If G is a group and H a group of (outer) automorphisms of G (so $h \in H$ acts as $g \mapsto g^h$), we may form the <u>extension</u> $E = G \rtimes H$ of G by H. This has elements (g,h) and composition $(g_1,h_1)(g_2,h_2) = (g_1 g_2^{h_1}, h_1 h_2)$. Thus $1 \to G \to E \to H \to 1$ is an exact sequence. The characters on G are found by restriction of those on E and the ones which are permuted by elements of H are found added together in the character table of E, since each conjugacy class of E which intersects G is a union of classes of G.

11. The Orthogonality relations proved.

We are now in a position to give a "clean" (ie non-matrix) proof of the orthogonality relations. Careful examination (exercise) of the foregoing sections will reveal that the argument is not, in fact, circular. This proof is essentially that in Adams, pp 49-50

We wish to show that $\Sigma_{g\in G}\overline{\chi_V(g)}\chi_W(g)/|G| = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}G}(V,W)$, so it's natural to consider the space $U = V^* \otimes_{\mathbb{C}} W$, which, by §§9,10 has character $\overline{\chi_V(g)}\chi_W(g)$ at $g\in G$. Now this space is naturally isomorphic to $\mathrm{Hom}_{\mathbb{C}}(V,W)$, the space of $\mathbb{C}$-linear maps between (the vector spaces) V and W, of which $U_G = \mathrm{Hom}_{\mathbb{C}G}(V,W)$ is the subspace on which G acts trivially (exercise).

Put $\chi(g) = \overline{\chi_V(g)}\chi_W(g) = \mathrm{tr}\,\rho_g$ for the character of U; we want $\Sigma_{g\in G}\chi(g)/|G|$. Now, $\mathrm{tr}: \mathrm{End}_{\mathbb{C}}(U)\to\mathbb{C}$ is linear, so we may write this as $1/|G|\,\mathrm{tr}\,\Sigma_{g\in G}\rho_g = \mathrm{tr}\,\theta$, say. We shall show that $\theta: U\to U$ is idempotent (ie $\theta^2=\theta$), so $U \cong U_1 \oplus U_0$ with $\theta|_{U_1} = 1$ and $\theta|_{U_0} = 0$. Then $\mathrm{tr}\,\theta = \mathrm{tr}\,\theta_{U_1} = \dim U_1 = \dim \mathrm{im}\,\theta$

Thus we wish to show $\theta^2=\theta$ and $\mathrm{im}\,\theta = U_G$. First, $\theta^2 = \Sigma_g\Sigma_h \rho_g\rho_h/|G|^2 = \Sigma_g\Sigma_h\rho_{gh}/|G|^2 = \Sigma_k\Sigma_\ell g_k/|G|^2 = \Sigma_k g_k/|G| = \theta$. Then $\mathrm{im}\,\theta \subseteq U_G$. For let $u\theta\in\mathrm{im}\,\theta$; then for $g\in G$, $u\theta\rho_g = u(\Sigma_h\rho_h/|G|)\rho_g = \Sigma_h u\rho_h\rho_g/|G| = \Sigma u\rho_k/|G| = u\theta$, so $u\theta\in U_G$. Finally $U_G\subseteq\mathrm{im}\,\theta$. For let $u\rho_g = u$ for all $g\in G$; then $u\theta = u\Sigma_g\rho_g/|G| = \Sigma_g(u\rho_g)/|G| = \Sigma_g u/|G| = u$ so $u\in\mathrm{im}\,\theta$.

Now that we have properly identified $\langle\chi,\psi\rangle$ with $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}G}(V,W)$, it is immediately apparent where Schur's lemma comes in: it gives the orthogonality relations directly.

Let $c_i = |C_i|$ be the size of the i^{th} conjugacy class for $1\le i\le h$. Then for $1\le j,k\le h$, the orthogonality relations between the irreducible characters χ_j, χ_k give $\Sigma_i {c_i}/{|G|}\,\overline{\chi_j(g_i)}\chi_k(g_i) = \delta_{jk}$. However, as is well known for orthonormal systems, the scalar product of two columns, $\Sigma_k\overline{\chi_k(g_i)}\chi_k(g_j)$, is also nonzero iff $i=j$. This provides a useful way of determining the last irreducible character when the others are known. [Exercise: what is the actual value of $\Sigma_k|\chi_k(g_i)|^2$?]

12. <u>Deducing the normal-subgroup structure from the character table.</u>

Recall some of the elementary facts about normal subgroups: they are exactly the kernels of homomorphisms out of the group; each normal subgroup is a union of conjugacy classes; and the intersection of two normal subgroups is normal. Also recall from §8 that, given any group homomorphism $\theta: G \to H$ (such as the natural quotient map $G \to G/N$ for $N \trianglelefteq G$) and a representation $\rho: H \to GL(V)$ for H, there is a representation $\theta_\rho: G \to GL(V)$ given by the composite; V (or rather $\mathrm{Res}_\theta(V)$) is then a KG-module.

These facts, together with the triangle-law in $\mathbb{C}$ (as a metric space in the usual way), are sufficient to deduce the normal subgroup structure of a group from its character table. It happens also to be true that a finite simple group is determined by its character table, so we may also determine the composition factors. Unfortunately, one usually knows these things about a group long before one knows its character table!

Let $\rho: G \to GL(V)$ be a representation with degree f, character χ and kernel $N = \ker\rho \trianglelefteq G$. Then $\rho_g = 1 \Leftrightarrow g \in N$ so it's clear that $\chi(g) = f$ for $g \in N$. But the converse also holds, for $\chi(g)$ is a sum of f $|G|^{th}$ roots of unity, each of which have norm 1, so $|\chi(g)| = f$ iff they are colinear in $\mathbb{C} \cong \mathbb{R}^2$, ie g acts as a scalar; in particular, $\chi(g) = f$ iff $\rho_g = 1$. If V is reducible, $\chi(g) = f$ iff g acts as 1 on each of the irreducible components.

Thus the kernel of a representation is the union of the conjugacy classes on which its character takes the same value as it does at the identity, and this is the intersection of the kernels of the irreducible components, which may be determined literally by inspection of the character table.

Conversely, given any normal subgroup $N \trianglelefteq G$, consider the regular representation $K(G/N)$ of the quotient, as a representation of G. This has kernel exactly N so every normal subgroup arises in this way.

Clearly G is simple iff no nontrivial character takes a value equal to its degree other than at the identity.

The centre of G consists (by elementary methods) of the singleton conjugacy classes. The derived group (generated by $x^{-1}y^{-1}xy$ for $x, y \in G$) is the intersection of the kernels of the linear representations (betrayed by a block of 1's in the character table) these are the representations of the Abelianisation, G/G', of G, which is the group obtained by imposing on G the extra relations $xy=yx$ for each $x,y \in G$; the number of them is of course $|G/G'|$.

13. Algebraic Integers and Character Degrees

We shall conclude the course by proving Burnside's "$p^\alpha q^\beta$" theorem, that a group of order $p^\alpha q^\beta$ (where p,q are prime) is not simple, using Representation Theory. This result will be used in the Part II Group Theory course without proof in order to prove Hall's theorem that a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is soluble iff it has subgroups of index $p_1^{\alpha_1}, \dots, p_k^{\alpha_k}$.

First we shall need to introduce the concept of an Algebraic Integer, which is a root of a monic polynomial over $\mathbb{Z}$ (ie of the form $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$). This will quickly enable us to prove that the degree of an irreducible representation divides the order of the group. A rational integer is an algebraic integer which is also (an ordinary) rational, ie (exercise) just an (ordinary) integer.

We have a lemma proved (exercise) by linear algebra: (i) if $y_1, \dots, y_n \in \mathbb{C}$ (not all zero) and $\gamma \in \mathbb{C}$ satisfies $\gamma y_i = \Sigma_1^n a_{ij} y_j$ for $i = 1,2,\dots,n$ and $a_{ij} \in \mathbb{Z}$ for $i,j = 1,\dots,n$ then γ is an algebraic integer; (ii) for any polynomial function $f \in \mathbb{Z}[X,Y]$ in two variables over the (rational) integers and algebraic integers α, β, then $f(\alpha,\beta)$ is also an algebraic integer; (iii) the algebraic integers form a ring and characters take algebraic integer values.

It is clear that an extension of $\mathbb{Q}$ generated by roots of unity (ie by character values) has Abelian Galois group. The converse is also true (but more difficult): any algebraic integer with Abelian Galois group is a sum of roots of unity.

Let χ be the character of an irreducible representation $\rho: G \to GL(V)$ of degree f, where $G = C_1 \cup C_2 \cup \dots \cup C_h$ is the decomposition of G into conjugacy classes. It will be clear that $f \mid |G|$ if we can prove that $|C_i| \chi(g_i)/f$ is an algebraic integer for $g_i \in C_i$, since $|G|/f = (|G|/f)\langle \chi,\chi \rangle = \Sigma_1^h (|C_i| \chi(g_i)/f)\chi(g_i^{-1})$ is also rational.

Put $c_i = \Sigma(g \in C_i) \in \mathbb{C}G$; indeed c_i is in the centre of $\mathbb{C}G$. Then (fixing i) $c_i c_k = \Sigma a_{k\ell} c_\ell$ with $a_{k\ell} \in \mathbb{Z}$ [why?]

Now $\rho: G \to \mathrm{Aut}(V)$ extends uniquely to a ring homomorphism $\rho: \mathbb{C}G \to \mathrm{End}(V)$ and $c_k\rho$ commutes with each $g\rho$ $(= \rho_g)$ so $c_k\rho$ is actually a $\mathbb{C}G$-module endomorphism of the irreducible module V. Hence by Schur, $c_k\rho = \gamma_k \in \mathbb{C}$. Thus $\gamma_i \gamma_k = \Sigma_1^h a_{k\ell}\gamma_\ell$ wherein not all the γ_k are zero (for $\gamma_1 = 1$), so by the lemma (with $\gamma = \gamma_i$ and $y_k = \gamma_k$ $k=1,\dots,h$) γ_i is an algebraic integer. Taking traces, $f\gamma_i = \mathrm{tr}(c_i\rho) = \Sigma_{g\in C_i} \mathrm{tr}\,\rho_g = \Sigma_{g\in C_i} \chi(g) = |C_i|\chi(g_i)$ so $|C_i|\chi(g_i)/f$ is an algebraic integer.

14. Burnside's $p^\alpha q^\beta$ theorem.

We aim to prove that a group G of order $p^\alpha q^\beta$ (where p,q are prime and $\alpha,\beta \geq 1$) is not simple. Recall that a homomorphism from a simple group is either injective or trivial, that the centre of a nonAbelian simple group is trivial, that a group of order q^β (q prime and $\beta \geq 1$) has nontrivial centre and that the size of the conjugacy class of an element of a group is equal to the index of its centraliser.

We need to use Sylow's theorem, that G has a subgroup Q of order q^β (and index p^α). This is the first part of the Part II Groups course; the present result is used without proof later in that course.

Recall from IB Rings & Modules or II Galois theory that a polynomial irreducible over $\mathbb{Z}$ is irreducible over $\mathbb{Q}$ and that its roots are permuted ~~cyclic~~ transitively by the Galois group of the polynomial.

Suppose $\alpha = \frac{1}{n}\Sigma_1^n \omega_i$ is an algebraic integer, where the ω_i are roots of unity; then either $\alpha = 0$ or $\omega_1 = \omega_2 = \dots = \omega_n = \alpha$. For there is a root ω of unity of which the ω_i are powers; let f be the minimal monic polynomial of α over $\mathbb{Z}$, so also its minimal polynomial over $\mathbb{Q}$; the roots α_i of f are the images of α under the action of the Galois group of $\mathbb{Q}(\omega):\mathbb{Q}$. Hence each α_i is an "average" of roots of unity and so $|\alpha_i| \leq 1$, with equality iff $\omega_1 = \omega_2 = \dots = \omega_n$; otherwise $|\Pi\alpha_i| < 1$, but $\pm\Pi\alpha_i$ is the constant coefficient in f so an integer, so $\Pi\alpha_i = 0$ and f has a root $\alpha_i = 0$.

Write $\chi_1,\dots,\chi_h$ for the irreducible characters of G (with χ_1 the trivial character) and $\rho_1,\dots,\rho_h$ for their representations of degrees $f_1,\dots,f_h$; also let $g_1,\dots,g_h$ be representatives of the conjugacy classes $C_1,\dots,C_h$.

If f_i and $|C_j|$ are coprime then either $\chi_i(g_j) = 0$ or $g_j\rho_i = \omega \in \mathbb{C}$ and $\chi_i(g_j) = f_i\omega$. For let $a,b \in \mathbb{Z}$ with

$an_i + b|C_j| = 1$ and let $\omega_1, \ldots, \omega_{f_i}$ be the eigenvalues of $g_j\rho_i$ (which can be diagonalised). Then

$$\frac{1}{f_i}\sum_1^{f_i}\omega_k = \frac{1}{f_i}\chi_i(g_j) = a(\chi_i(g_j)) + b\left(\frac{|C_j|}{n_i}\chi_i(g_j)\right)$$

is an algebraic integer, whence either $\chi_i(g_j)=0$ or $\omega_1 = \omega_2 = \ldots = \omega_{f_i} = \omega$ in which case $g_j\rho_i = \omega$.

If $|C_j| = p^r$ for some $j \neq 1$ then G is not non-Abelian simple. For suppose it were, then if $i \neq 1$ and $p \nmid n_i$ then either $\chi_i(g_j)=0$ or $g_j\rho_i = \omega$ and $g_j\rho_i \in Z(G\rho_i)$. But the latter case is impossible since $\omega \neq 1$ (for $i \neq 1$) so $\rho_i : G \cong G\rho_i$ which has trivial centre. Thus for $i \neq 1$ we have either $p | n_i$ or $\chi_i(g_j)=0$. Then by orthogonality of the columns 1 and j,

$$0 = \sum_1^h \chi_i(1)\chi_i(g_j) = 1 + \sum_1^h f_i\chi_i(g_j)$$

so $-1/p = \sum_2^h (f_i/p)\chi_i(g_j)$, in which each summand is either 0 (if $p \nmid n_i$) or an algebraic integer (if $p | n_i$), so $-1/p$ is an algebraic integer, which is nonsense.

Now we can prove the theorem. Let G be a group of order $p^\alpha q^\beta$ and let Q be a subgroup of order q^β. Q has nontrivial centre : choose z in it, so $z \neq 1$ and $G \geqslant C_G(z) \geqslant Q$. Then $|G:C| = p^r$ for $0 \leq r \leq \beta$. Then the conjugacy class of z has order p^r, so G is not nonAbelian simple.